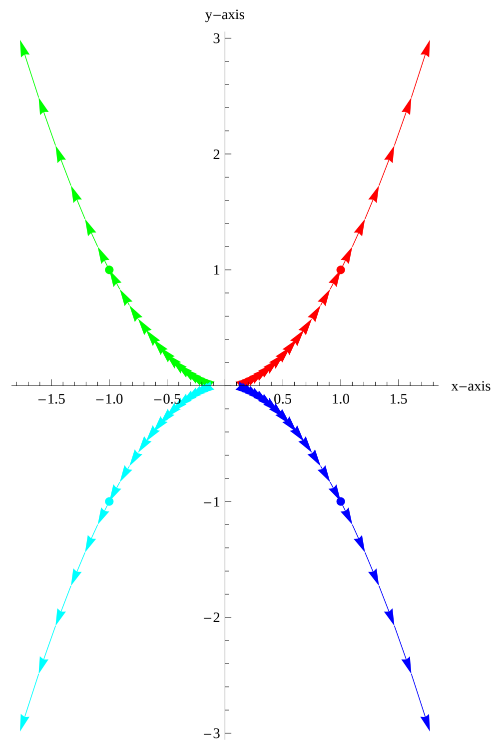
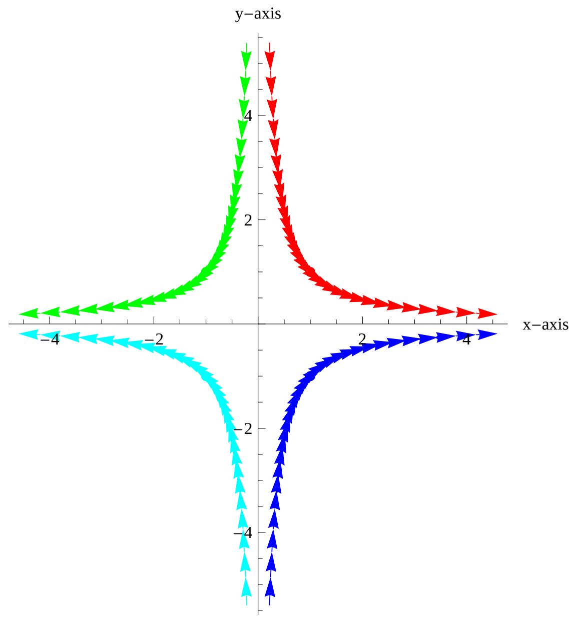
MATH 417
HOMEWORK 9
Winter, 2008

Section 7.1

30. I have plotted one trajectory from each quadrant:



32. I have plotted one trajectory from each quadrant:



54.

(a). The system of equations describing the evolution is

$$\begin{aligned} n(t+1) &= 2a(t) \\ a(t+1) &= n(t) + a(t) \end{aligned}$$

So, the matrix equation is given by

$$\begin{bmatrix} n(t+1) \\ a(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} n(t) \\ a(t) \end{bmatrix}$$

(b). The characteristic polynomial of A is $x^2 - x - 2$ so the eigenvalues are 2 and -1. The vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector associated with 2, and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is the eigenvector associated with -1.

(c). Let the initial state $\begin{bmatrix} n(0) \\ a(0) \end{bmatrix}$ have coordinates $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ in the eigenbasis from (b). Then the closed form equation is

$$\begin{bmatrix} n(t) \\ a(t) \end{bmatrix} = c_1 2^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(-1)^t \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Section 7.3

16. The characteristic polynomial of $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is $x^2 - (a+c)x + (ac - b^2)$. It will have two distinct (real values) when the discriminant $b^2 - 4ac$ of the quadratic equation is positive:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant here is

$$(a+c)^2 - 4(ac - b^2) = (a-c)^2 + 4b^2$$

So, we need $(a-c)^2 + 4b^2 > 0$; this holds exactly when $a - c \neq 0$ or when $b \neq 0$. That is, the only time when A has 1 eigenvalue is when $a = c$ and $b = 0$.

18. The characteristic polynomial of $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is $x^2 - 2ax + (a^2 - b^2)$. The roots are given by the quadratic formula

$$\frac{2a \pm \sqrt{4a^2 - 4(a^2 - b^2)}}{2} = a \pm b.$$

So, the eigenvalues of A are $a \pm b$.

24. The characteristic polynomial of $A = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$ is $x^2 - 1.25x + 1.00$. The eigenvalues are 1.0 and 0.25.

34. A 4×4 matrix with two distinct eigenvalues could have either (i) both eigenvalues with multiplicity 2 or (ii) one of the eigenvalues with algebraic multiplicity one and the other with algebraic multiplicity three:

$$(i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad (ii) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

40. Let A and B be $n \times n$ matrices. Compare the computations:

$$\begin{aligned} \operatorname{tr}(AB) &= \sum_{i=1}^{n} \big(\sum_{k=1}^{n} a_{ik} b_{ki} \big) \\ \operatorname{tr}(BA) &= \sum_{k=1}^{n} \big(\sum_{i=1}^{n} b_{ik} a_{ki} \big) \end{aligned}$$

Re-arrange the order of the terms of the first series to obtain the second series; that is

$$\operatorname{tr}(AB) = \sum_{i,k=1}^{n} a_{ik} bik = \operatorname{tr}(BA).$$

42. Let A and B be $n \times n$ matrices such that $BA = 0$. We will use two facts: (i) $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ (by (**40**)), and (ii) $\operatorname{tr}(X+Y) = \operatorname{tr}(X) + \operatorname{tr}(Y)$ (since addition is componentwise). Now,

$$\begin{aligned} \operatorname{tr}\big((A+B)^2\big) &= \operatorname{tr}\big(A^2 + AB + BA + B^2\big) \\ &= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) + \operatorname{tr}(AB) + \operatorname{tr}(BA) \\ &= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) + 2\operatorname{tr}(BA) \\ &= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) \end{aligned}$$

44. It is impossible to have two invertible $n \times n$ matrices with $AB - BA = A$. Suppose otherwise, that $AB - BA = A$. Then, multiplying by A^{-1}:

$$B - A^{-1}BA = I_n$$

We will use two facts: (i) $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ (by (**40**)), and (ii) $\operatorname{tr}(X+Y) = \operatorname{tr}(X) + \operatorname{tr}(Y)$ (since addition is componentwise). So,

$$\begin{aligned} n = \operatorname{tr}(I_n) &= \operatorname{tr}\big(B - A^{-1}BA\big) \\ &= \operatorname{tr}(B) - \operatorname{tr}\big(A^{-1}BA\big) \\ &= \operatorname{tr}(B) - \operatorname{tr}\big(A^{-1}AB\big) \\ &= \operatorname{tr}(B) - \operatorname{tr}(B) = 0 \end{aligned}$$

So, $n = 0$ which is a contradiction. Therefore, it is impossible to have invertible matrices A and B with $AB - BA = A$.