Hw 8

Section 6.1

№ 18

$$\det \begin{bmatrix} 0 & 1 & k \\ 3 & 2k & 5 \\ 9 & 7 & 5 \end{bmatrix} = 0 + 45 + 21k - 18k^2 - 0 - 15 =$$

$$= -18k^2 + 21k + 30 = -3(k-2)(6k+5)$$

№ 20

$$\det \begin{bmatrix} 1 & k & 1 \\ 1 & k+1 & k+2 \\ 1 & k+2 & 2k+4 \end{bmatrix} = 1$$

№ 30

$$\det \begin{bmatrix} 4 & 2 & 0 \\ 4 & 6 & 0 \\ 5 & 2 & 3 \end{bmatrix} = 3 \det \begin{bmatrix} 4 & 2 \\ 4 & 6 \end{bmatrix} = 3 \cdot 2^3 \det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = 48$$

№ 56

(a) $d_n = (-1)^{n+1} d_{n-1}$

(b) $d_1 = 1$, $d_2 = -1$, $d_3 = -1$, $d_4 = 1$, $d_5 = 1$,

$d_6 = -1$, $d_7 = -1$, $d_8 = 8$

(c) $d_{100} = 1$

Section 6.2

№ 16

$$\det \begin{bmatrix} 6\vec{v}_1 + 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix} \begin{matrix} -2\,\text{IV} \\ \\ \\ \cdot \end{matrix} = \det \begin{bmatrix} \vec{0} \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix} = 0$$

№ 40

$$A A^T = I_n$$

$$\det(A A^T) = 1 \Rightarrow \det(A) \cdot \det(A^T) = 1$$

But $\det(A^T) = \det(A) \Rightarrow (\det A)^2 = 1 \Rightarrow$

$$\Rightarrow \det A = \pm 1.$$

№ 42

$$A = QR \Rightarrow A^T = R^T Q^T \quad \text{and}$$

$$A^T A = R^T Q^T Q R = \left\{ \begin{matrix} Q^T Q = I_n \text{ since} \\ Q \text{ is formed by} \\ \text{orthonormal columns} \end{matrix} \right\} = R^T R$$

Since both R^T and R are upper triangular

$$\det(A^T A) = \det(R^T R) = (\det R) \cdot (\det R^T) =$$

$$= r_{11} \cdots r_{mm} \cdot r_{11} \cdots r_{mm} = (r_{11} \cdots r_{mm})^2$$

№ 46

$$A = \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} \quad \text{and } B = \begin{bmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{bmatrix}$$

$\det A = 7$ and $\det B = 11$.

(a)

$$\det \begin{bmatrix} a & 3 & d \\ b & 3 & e \\ c & 3 & f \end{bmatrix} = 3 \det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 21$$

(B) $$\det \begin{bmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{bmatrix} = -3 \det \begin{bmatrix} b & e \\ c & f \end{bmatrix} + 5 \det \begin{bmatrix} a & d \\ c & f \end{bmatrix} -$$

$$-7 \det \begin{bmatrix} a & d \\ b & e \end{bmatrix}.$$

Now $\det A = 7 = -\det \begin{bmatrix} b & e \\ c & f \end{bmatrix} + \det \begin{bmatrix} a & d \\ c & f \end{bmatrix} - \det \begin{bmatrix} a & d \\ b & e \end{bmatrix}$

and

$$\det B = 11 = -\det \begin{bmatrix} b & e \\ c & f \end{bmatrix} + 2 \det \begin{bmatrix} a & d \\ c & f \end{bmatrix} - 3 \det \begin{bmatrix} a & d \\ b & e \end{bmatrix}.$$

Hence

$$\det \begin{bmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{bmatrix} = \det A + 2 \det B = 7 + 22 = 29$$

№ 48

Let $\vec{v}_1 \in \mathbb{R}^n$ be a vector such that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ form a basis of $\mathbb{R}^n$. Then

$$\underbrace{\det[\vec{v}_1\ \vec{v}_2 \dots \vec{v}_n]}_{= \cancel{\det}\, T(\vec{v}_1)} \neq 0.$$ Hence $\operatorname{Im} T = \mathbb{R}$.

Now $\dim(\operatorname{Im} T) + \dim(\ker T) = n \Rightarrow$

$\Rightarrow \dim(\ker T) = n-1.$

It is easy to see that $T(\vec{v}_i) = 0$ for $i = 2, 3, \dots, n$.
So $\operatorname{span}\{\vec{v}_2, \dots, \vec{v}_n\} \subset \ker T$ and hence since $\dim(\ker T) = n-1$ and $\vec{v}_2, \dots, \vec{v}_n$ are lin. ind we have $\ker T = \operatorname{span}\{\vec{v}_2, \dots, \vec{v}_n\}$