Homework 6

Section 5.3

30. (a) If $\vec{x} \in \ker(L)$, then $L\vec{x} = 0$ and $\|L(\vec{x})\| = \|\vec{x}\| = 0$. Hence $\ker L = \{\vec{0}\}$

(b) Since $\dim(\text{Im}(L)) + \dim(\ker(L)) = m$ and $\ker(L) = \{\vec{0}\}$, we have $\dim(\text{Im}(L)) = m$.

(c) Since $\dim(\text{Im}(L)) = \text{rank}(L)$ and by (b), we have ~~Rank(L)~~ $\text{rank}(L) = m$. By a well-known fact $\text{rank}(L) \leq n$ and hence

$$m \leq n$$

(d) Let $A = [L(\vec{e}_1) \dots L(\vec{e}_m)]$ be the matrix of L. Then since L preserves the length of a vector, we have $L(\vec{e}_1), \dots, L(\vec{e}_m)$ are unit vectors. Now, arguing as in the proof of Fact 5.3.2, we can show that

$$L(\vec{e}_i) \cdot L(\vec{e}_j) = 0 \quad \text{if } i \neq j.$$

Thus, the vectors $L(\vec{e}_1), \dots, L(\vec{e}_m)$ are orthonormal

(e)

let $A^T = \begin{bmatrix} - L(\vec{e}_1)^T - \\ - L(\vec{e}_2)^T - \\ \vdots \\ - L(\vec{e}_m)^T - \end{bmatrix}$ be the transpose of A.

Then $A^T A = \begin{bmatrix} L(\vec{e}_1)\cdot L(\vec{e}_1) & L(\vec{e}_1)\cdot L(\vec{e}_2) & \dots & L(\vec{e}_1)\cdot L(\vec{e}_m) \\ L(\vec{e}_2)\cdot L(\vec{e}_1) & L(\vec{e}_2)\cdot L(\vec{e}_2) & \dots & L(\vec{e}_2)\cdot L(\vec{e}_m) \\ \vdots & & & \\ L(\vec{e}_m)\cdot L(\vec{e}_1) & L(\vec{e}_m)\cdot L(\vec{e}_2) & \dots & L(\vec{e}_m)\cdot L(\vec{e}_m) \end{bmatrix}$.

Now, using (d) we get $A^T A = I_m$.

(f) From fact 5.3.10, we get that $A A^T$ is the matrix of the orthogonal projection onto $Im(A)$.

32. (a) If $m=n$, then $A^T \cdot A = I_m$ implies that A is orthogonal and $A^T = A^{-1}$.

Hence $A A^T = I_m = I_n$.

But If $m \neq n$, then It Is not necessarily true. let $m=2$ and $n=3$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $A^T A = I_2$ but $AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) see part (a).

36.

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & a \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & b \\ \frac{1}{3} & 0 & c \end{bmatrix}$$

Let $\quad \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$. Then $\|\vec{v}_1\| = \|\vec{v}_2\| = 1$ and $\vec{v}_1 \cdot \vec{v}_2 = 0$.

From $\vec{v}_1 \cdot \vec{v}_3 = 0$ and $\vec{v}_2 \cdot \vec{v}_3$, we get

$$\left| \begin{matrix} 2a + 2b + c = 0 \\ a - b \qquad = 0 \end{matrix} \right| \Rightarrow \left| \begin{matrix} b = a \\ c = -4a \end{matrix} \right| .$$

So $\vec{v}_3 = \begin{bmatrix} a \\ a \\ -4a \end{bmatrix}$. Now using $\|\vec{v}_3\| = 1$, we obtain $\sqrt{18a^2} = 1 \Rightarrow a = \pm \frac{1}{3\sqrt{2}}$

40. Using the Gram-schmidt process, we get an orthonormal basis of W.

$$\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{u}_2 = \frac{1}{10} \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix}.$$

Let $Q = [\vec{u}_1\ \vec{u}_2]$. Then the matrix of the orthogonal projection is

$$QQ^T = \frac{1}{100}\begin{bmatrix} 26 & 18 & 32 & 24 \\ 18 & 74 & -24 & 32 \\ 32 & -24 & 74 & 18 \\ 24 & 32 & 18 & 26 \end{bmatrix}$$

44. Let A be an $n \times m$ matrix. Then A^T is an $m \times n$ matrix and $\dim(\operatorname{Im} A^T) + \dim(\ker A^T) = n$.

Now $\dim(\operatorname{Im} A^T) = (\operatorname{Rank} A^T) = (\text{by Fact 5.3.9 c}) =$

$= \operatorname{Rank}(A) = \dim(\operatorname{Im} A)$.

Hence $\dim(\operatorname{Im} A) + \dim(\ker A^T) = n$. $\blacksquare$

Section 3.3

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To find ~~the row~~ a basis of the row space of A, we need to find a basis of the column space of A^T.

$$\operatorname{rref} A^T = \operatorname{rref}\left(\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the row space has a basis consisting of

$$\vec{v}_1 = [1 \; 1 \; 1 \; 1] \quad \text{and} \quad \vec{v}_2 = [1 \; 2 \; 3 \; 4].$$