MATH 517
HOMEWORK 6
Winter, 2008

1 Section 5.1

12. Prove $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

Compute the squares of both sides:

$$\begin{aligned} \|\vec{v}+\vec{w}\|^2 &= (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) \\ &= \vec{v}\cdot\vec{v} + 2\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w} \\ &= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\vec{v}\cdot\vec{w} \\ (\|\vec{v}\| + \|\vec{w}\|)^2 &= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\| \end{aligned}$$

By Cauchy Schwartz, (Fact 5.1.11), $2|\vec{v}\cdot\vec{w}| \leq 2\|\vec{v}\|\|\vec{w}\|$, which is the difference between the two computations:

$$\begin{aligned} \|\vec{v}+\vec{w}\|^2 &= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\vec{v}\cdot\vec{w} \\ &\leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\| \\ &= (\|\vec{v}\| + \|\vec{w}\|)^2 \end{aligned}$$

Now, take square roots: $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

16. Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

We want solutions $\vec{x}$ so that $\vec{u}_i \cdot \vec{x} = \vec{0}$ for each $i = 1, 2, 3$. Equivalently, $\vec{u}_i^T \vec{x} = 0$ for each $i = 1, 2, 3$. (see Fact 5.3.6). We can write this as a matrix equation

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

There are two solutions for a, b, c, d for which the vector is a unit vector:

$$\vec{u}_1 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

28. Let V be the subspace spanned by the orthonormal set of vectors:

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

The projection matrix for this set (using 5.3.10) is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}$$

So, the projection of $\vec{e}_1$ is

$$\frac{1}{4} \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

30. Let V be a vector subspace of $\mathbb{R}^n$. Fix an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$ for V. Let $\vec{y} = \text{proj}_V(\vec{x})$.

$$\begin{aligned} \vec{y} &= (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \ldots + (\vec{x} \cdot \vec{u}_m)\vec{u}_m \\ \vec{y} \cdot \vec{x} &= (\vec{x} \cdot \vec{u}_1)(\vec{u}_1 \cdot \vec{x}) + \ldots + (\vec{x} \cdot \vec{u}_m)(\vec{u}_m \cdot \vec{x}) \\ &= (\vec{x} \cdot \vec{u}_1)^2 + \ldots + (\vec{x} \cdot \vec{u}_1)^2 \\ \vec{y} \cdot \vec{y} &= \|\vec{y}\|^2 \\ &= \big((\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \ldots + (\vec{x} \cdot \vec{u}_m)\vec{u}_m\big) \cdot \big((\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \ldots + (\vec{x} \cdot \vec{u}_m)\vec{u}_m\big) \\ &= (\vec{x} \cdot \vec{u}_1)^2 + \ldots + (\vec{x} \cdot \vec{u}_1)^2 \end{aligned}$$

The last equality use distributivity of dot product and the fact that

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So, $\vec{y} \cdot \vec{x} = \|\vec{y}\|^2$.

32. Let $\vec{v}_1, \vec{v}_2$ be two vectors. When is the following matrix invertible:

$$\begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 \end{bmatrix}$$

By Exercise 13 in Section 2.1, we must have

$$(\vec{v}_1 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{v}_2) - (\vec{v}_1 \cdot \vec{v}_1)^2 \neq 0$$

Equivalently $\|\vec{v}_1\|\|\vec{v}_2\| \neq |\vec{v}_1 \cdot \vec{v}_2|$. By Cauchy's Theorem this is true **exactly** when $\vec{v}_1$ and $\vec{v}_2$ are **not parallel**. That is, for no $k \in \mathbb{R}$ do we have $\vec{v}_1 = k\vec{v}_2$.

34. We want to find the maximum value of $a \in \mathbb{R}$ satisfying the following two conditions for each vector $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$:

1. $x_1 + \ldots + x_n = a$, and
2. $x_1^2 + \ldots + x_n^2 = 1$ (that is, $\|\vec{x}\| = 1$.)

Let $\vec{u} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. Then for any $\vec{x} \in \mathbb{R}^n$

$$\vec{u} \cdot \vec{x} = x_1 + \ldots + x_n$$

By the Cauchy-Schwarz Inequality

$$|\vec{u} \cdot \vec{x}| \leq \|\vec{u}\|\|\vec{x}\| = \sqrt{n}\|\vec{x}\|$$

Since we are insisting $\|\vec{x}\| = 1$, we have

$$|\vec{u} \cdot \vec{x}| \leq \sqrt{n}.$$

So, $a \leq \sqrt{n}$. But, as $\begin{bmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix}$ is a unit vector (satisfying condition 2), and the sum of its components is $\sqrt{n}$, we have $a = \sqrt{n}$.

In the case of $\mathbb{R}^2$ the maximum value is $\sqrt{2}$; in the case of $\mathbb{R}^3$ the maximum value is $\sqrt{3}$.

Section 5.2

14. There are three steps in the Gram-Schmidt process. Step 1:

$$\frac{1}{10}\begin{bmatrix}1\\7\\1\\7\end{bmatrix}$$

Step 2.

$$\begin{bmatrix}0\\7\\2\\7\end{bmatrix} \cdot \frac{1}{10}\begin{bmatrix}1\\7\\1\\7\end{bmatrix} = 10$$

So the second vector (unnormalized) is

$$\begin{bmatrix}0\\7\\2\\7\end{bmatrix} - \begin{bmatrix}1\\7\\1\\7\end{bmatrix} = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

Step 3.

$$\begin{bmatrix}1\\8\\1\\6\end{bmatrix} \cdot \frac{1}{10}\begin{bmatrix}1\\7\\1\\7\end{bmatrix} = 10 \qquad \begin{bmatrix}1\\8\\1\\6\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} = 0$$

So the third vector (unnormalized) is

$$\begin{bmatrix}1\\8\\1\\6\end{bmatrix} - \begin{bmatrix}1\\7\\1\\7\end{bmatrix} = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$$

Therefore, the orthonormal basis from Gram-Schmidt is

$$\frac{1}{10}\begin{bmatrix}1\\7\\1\\7\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$$

32. Find an orthonormal basis for the plane $x + y + z = 0$. A basis for the plane is

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Apply Gram-Schmidt to get an orthonormal basis

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \sqrt{\frac{3}{2}} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

34. A basis for the kernel of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ is given by

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Applying Gram-Schmidt we obtain an orthornormal basis:

$$\frac{1}{2} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{6} \begin{bmatrix} 3 \\ -5 \\ 1 \\ 1 \end{bmatrix}$$