MATH 417
HOMEWORK 5
Winter, 2008

16. To compute coordinates, solve the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$$

The unique solution is $\begin{bmatrix} 21 \\ -22 \\ 8 \end{bmatrix}$.

18. To compute coordinates, solve the matrix equation

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}$$

There is no solution to this equation, so no coordinates.

28. Let S be the transformation from coordinates in $\mathcal{B}$ to coordinates standard coordinates. Then

$$S = \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \qquad S^{-1} = \frac{1}{9} \begin{bmatrix} 2 & 2 & 1 \\ 5 & -4 & 2 \\ 1 & 1 & -4 \end{bmatrix}$$

So, $B = [T]_{\mathcal{B}}$ is given by

$$\begin{aligned} B = S^{-1}AS &= \frac{1}{9} \begin{bmatrix} 2 & 2 & 1 \\ 5 & -4 & 2 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \end{aligned}$$

30. Let S be the transformation from coordinates in $\mathcal{B}$ to coordinates standard coordinates. Then

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

So, $B = [T]_{\mathcal{B}}$ is given by

$$\begin{aligned} B = S^{-1}AS &= \begin{bmatrix} 0 & 2 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

56. We want vectors $\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} \right)$ such that

$$\begin{aligned} 3\begin{bmatrix} a \\ c \end{bmatrix} + 5\begin{bmatrix} b \\ d \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ 2\begin{bmatrix} a \\ c \end{bmatrix} + 3\begin{bmatrix} b \\ d \end{bmatrix} &= \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{aligned}$$

We can write these conditions as a matrix equation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

So the solution is

$$\begin{bmatrix} 12 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} -7 \\ -8 \end{bmatrix}$$

60. We want an invertible matrix satisfying the following matrix equation:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We find as a solution (not necessarily invertible)

$$\begin{bmatrix} a & a \\ c & -c \end{bmatrix}$$

Let $a = 1$ and $c = 1$; so the following matrix is invertible

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(You can verify this by applying the condition in Exercise 13 of Section 2.1.)

62. The problem is finding an invertible matrix such that the following matrix equation holds:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

We find as a solution (not necessarily invertible)

$$\begin{bmatrix} a & b \\ 2a & -b \end{bmatrix}$$

Let $a = 1$ and $b = 1$; so the following matrix is invertible

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

(You can verify this by applying the condition in Exercise 13 of Section 2.1.) This corresponds to the following basis (given in standard coordinates)

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

64. The values of a, b, c, d are given to us (so fixed.) We want an invertible matrix satisfying the matrix equation

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

(the values x, y, z, w are ours to choose to satisfy the equation.)

This reduces to four equations determined by the matrix equation

$$\begin{bmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{bmatrix} = \begin{bmatrix} ax + cz & ay + cw \\ bx + dz & by + dw \end{bmatrix}$$

It follows that $y = z$ must hold and that $bx = cw + (a - d)y$ must hold. There are several cases to consider:

$b, c = 0$. Let $y = z = 0$, $x = w = 1$, so the following invertible matrix works

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$b, c \neq 0$. Let $y = z = 0$, $x = 1$, $w = \frac{b}{c}$, so the following invertible matrix works

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{c}{b} \end{bmatrix}$$

$b \neq 0, c = 0$. Let $y = z = 1$, $x = \frac{a-d}{b}$, $w = 0$, so the following invertible matrix works

$$\begin{bmatrix} \frac{a-d}{b} & 1 \\ 1 & 0 \end{bmatrix}$$

$b = 0, c \neq 0$. Let $y = z = 1$, $x = 0$, $w = \frac{d-a}{c}$, so the following invertible matrix works

$$\begin{bmatrix} 0 & 1 \\ 1 & \frac{d-a}{c} \end{bmatrix}$$

Therefore, the original matrices **are similar** for all values of a, b, c, d.

71. Suppose $B = S^{-1}AS$.

(a). Let $\vec{x} \in \ker(B)$. Then $B\vec{x} = \vec{0}$. So,

$$\begin{aligned} \vec{0} &= \left(S^{-1}AS\right)\vec{x} \\ &= S^{-1}\left(A(S^{-1}\vec{x})\right) \end{aligned}$$

Since S^{-1} is invertible, it must be that $A(S\vec{x}) = \vec{0}$. So, $S\vec{x} \in \ker(A)$.

We also have $A = SBS^{-1}$, so a similar argument shows that if $\vec{x} \in \ker(A)$ then $S^{-1}\vec{x} \in \ker(B)$.

(b). Suppose nullity$(A) = k$ and let $\vec{v}_1, \dots, \vec{v}_k$ be a basis for $\ker(A)$. I will show that $S^{-1}\vec{v}_1, \dots, S^{-1}\vec{v}_k$ is a linearly independent set from $\ker(B)$. This implies that any basis for $\ker(B)$ must have at least k vectors (Fact

3.3.1), so that $\text{nullity}(A) \leq \text{nullity}(B)$.

Let $c_1, \ldots, c_k$ be scalars such that

$$\begin{aligned} \vec{0} &= c_1 S^{-1}\vec{v}_1 + \ldots + c_k S^{-1}\vec{v}_k \\ &= S^{-1}\big(c_1\vec{v}_1 + \ldots + c_k\vec{v}_k\big) \end{aligned}$$

Since S^{-1} is invertible this means $\vec{0} = c_1\vec{v}_1 + \ldots + c_k\vec{v}_k$. But, $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent, so $c_1 = \ldots = c_k = 0$. Thus, $S^{-1}\vec{v}_1, \ldots, S^{-1}\vec{v}_k$ are linearly independent. These vectors are in $\ker(B)$ by part (a).

We can also show that $\text{nullity}(B) \leq \text{nullity}(A)$: let $\vec{w}_1, \ldots, \vec{w}_\ell$ be a basis for $\ker(B)$, then $S\vec{v}_1, \ldots, S\vec{v}_\ell$ is a linearly independent set of vectors for $\ker(A)$. The argument is the same as above (these vectors are in $\ker(A)$ from part (a).)

72. If A is similar to B then $\text{rank}(A) = \text{rank}(B)$. From Exercise 71(b), $\text{nullity}(A) = \text{nullity}(B)$. Since the number of columns of A and B are the same (that is, they have the same domain), by the Rank-Nullity Theorem (Fact 3.3.7)

$$\text{rank}(A) + \text{nullity}(A) = \text{rank}(B) + \text{nullity}(B).$$

Thus, $\text{rank}(A) = \text{rank}(B)$.