MATH 417
HOMEWORK 4
Winter, 2008

Computing Bases

1. Test for linear dependence and extract a linearly independent subset which spans the same subspace of $\mathbb{R}^6$ as the following vectors:

$$\begin{bmatrix} 2 \\ 4 \\ 3 \\ -1 \\ -2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -4 \\ -8 \\ -6 \\ 2 \\ 4 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 6 \\ 2 \\ -4 \\ -10 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 0 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

In reduced row-echelon form:

$$\begin{bmatrix} 1 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, a linearly independent set of vectors spanning the same subspace is

$$\begin{bmatrix} 2 \\ 4 \\ 3 \\ -1 \\ -2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 0 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

2. Find a basis for the image space and a basis for the kernel of the following matrix:

$$\begin{bmatrix} 1 & 2 & 4 & 5 & 7 \\ 1 & 2 & 3 & 4 & 5 \\ -1 & -2 & 0 & 2 & 1 \end{bmatrix}$$

In reduced row-echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

So, the image has rank of three, and has as a basis the column vectors

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

The kernel has dimension two and consists of the vectors

$$\left\{ \begin{bmatrix} -2k+\ell \\ k \\ -2\ell \\ 0 \\ \ell \end{bmatrix} : k, \ell \in \mathbb{R} \right\}$$

So, a basis for the kernel consists of the vectors (setting $k = 1, \ell = 0$ and $k = 0, \ell = 1$)

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

3. Find a matrix whose image space contains the vectors:

$$\begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 2 \\ -3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}$$

and whose kernel contains the vector

$$\begin{bmatrix} 7 \\ -1 \\ -7 \\ 1 \end{bmatrix}$$

We must solve the following matrix equation for a, b, c, d:

$$\begin{bmatrix} 1 & 2 & 1 & a \\ 3 & 2 & 3 & b \\ -1 & -3 & 0 & c \\ 0 & 4 & -1 & d \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiplying out we get

$$\begin{bmatrix} -2+a \\ -2+b \\ -4+c \\ 3+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{bmatrix}$$

4. Find a basis for the solution space of the following linear systems of homogeneous equations.

$$\begin{aligned} x + y + z + t &= 0 \\ 2x + 3y - z + t &= 0 \\ 3x + 4y + 2t &= 0 \end{aligned}$$

This problem can be put in the form of a matrix equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & 1 \\ 3 & 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This amounts to finding a basis for the kernel of the above matrix. This matrix in reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the subspace of all solutions is

$$\{ \begin{bmatrix} -4k - 2\ell \\ 3k + \ell \\ k \\ \ell \end{bmatrix} : k, \ell \in \mathbb{R} \}$$

So, a basis for the solution space consists of the vectors (setting $k = 1, \ell = 0$ and $k = 0, \ell = 1$)

$$\begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Section 3.1

38.

a. We always have $\ker(A^n) \subseteq \ker(A^{n+1})$ for any natural number n. Suppose $\vec{x} \in \ker(A^n)$. Then

$$\begin{aligned} A^{n+1}\vec{x} &= A\big(A^n\vec{x}\big) \\ &= A\vec{0} \\ &= \vec{0} \end{aligned}$$

so, $\vec{x} \in \ker(A^{n+1})$. We need **not** have $\ker(A^n) = \ker(A^{n+1})$. For example, let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \ker(A^2)$:

$$A^2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = A\big(A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\big) = A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But, as you can see $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin \ker(A)$.

b. We always have $\operatorname{im}(A^{n+1}) \subseteq \operatorname{im}(A^n)$ for any natural number n. Let $\vec{y} \in \ker(A^{n+1})$ and $\vec{x}$ any vector for which $A^{n+1}\vec{x} = \vec{y}$ (there must be at least one such vector). Then

$$\begin{aligned} \vec{y} &= A^{n+1}\vec{x} \\ &= A^n\big(A\vec{x}\big) \end{aligned}$$

so, $\vec{y} \in \text{im}(A^n)$. We need **not** have $\text{im}(A^n) = \text{in}(A^{n+1})$. For example, let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \text{im}(A)$ but $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin \text{im}(A^2)$

44. Let $B = \text{rref}(A)$. The key fact is that Gauss-Jordan elimination preserves **solutions**; however, it does not preserve **equations**. Given a fixed vector $\vec{b}$, you want solutions $\vec{x}$ to the equation:

$$A\vec{x} = \vec{b}$$

You apply Gauss-Jordan elimination to the augmented matrix:

$$\begin{bmatrix} A & \vec{b} \end{bmatrix} \xrightarrow[\textbf{elimination}]{\textbf{Gauss-Jordan}} \begin{bmatrix} B & \vec{c} \end{bmatrix}$$

where you are guaranteed the solutions $\vec{x}$ to this **new equation** are **exactly the same**:

$$B\vec{x} = \vec{c}.$$

a. It is **always true** $\ker(A) = \ker(B)$. The reason is that if $\vec{b} = \vec{0}$ in the original equation, then $\vec{c} = \vec{0}$ in the new equation. Since the solutions to $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ provide the kernel of each matrix, and the solutions are the same, $\ker(A) = \ker(B)$.

b. It is **not** necessarily true that the **images** are the same. For example,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow[\textbf{elimination}]{\textbf{Gauss-Jordan}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = B$$

but the images of these two matrices are different:

$$\begin{aligned} \text{im}(A) &= \left\{ k \begin{bmatrix} 1 \\ 2 \end{bmatrix} : k \in \mathbb{R} \right\} \\ \text{im}(B) &= \left\{ k \begin{bmatrix} 1 \\ 0 \end{bmatrix} : k \in \mathbb{R} \right\} \end{aligned}$$

50. We will show that $\ker(A^3) = \ker(A^4)$. By problem **38**, we know $\ker(A^3) \subseteq \ker(A^4)$, so we need to show $\ker(A^4) \subseteq \ker(A^3)$. We are given that $\ker(A^2) = \ker(A^3)$. Let $\vec{x} \in \ker(A^4)$. Then

$$\begin{aligned} \vec{0} &= A^4\vec{x} \\ &= A^3\big(A\vec{x}\big) \end{aligned}$$

So, $A\vec{x} \in \ker(A^3) = \ker(A^2)$. Thus,

$$\begin{aligned} \vec{0} &= A^2\big(A\vec{x}\big) \\ &= A^3\vec{x} \end{aligned}$$

Therefore, $\vec{x} \in \ker(A^3)$.

Section 3.2

36. Let $T : \mathbb{R}^n \to \mathbb{R}^p$ be a linear transformation and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be linearly dependent vectors. Then the vectors $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_m)$ **must be linearly dependent**.

Suppose we have scalars $c_1, c_2, \ldots, c_m$ (not all zero) such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m = \vec{0}.$$

Then

$$\begin{aligned} c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \ldots + c_mT(\vec{v}_m) &= T\big(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m\big) \\ &= T(\vec{0}) \\ &= \vec{0}. \end{aligned}$$

40. Let A be an $n \times p$ matrix and B be a $p \times m$ matrix. Suppose the columns of A are linearly independent and the columns of B are linearly independent. Then the columns of AB **must be linearly independent**.

Recall that the columns of AB are linearly independent if and only if the only solution to the equation $AB\vec{x} = \vec{0}$ is the trivial solution $\vec{0}$; equivalently, $\ker(AB) = \{\vec{0}\}$. We are given that $\ker(A) = \{\vec{0}\} = \ker(B)$. So, suppose $AB\vec{x} = \vec{0}$ for some vector $\vec{x}$; then

$$\vec{0} = AB\vec{x} = A\big(B\vec{x}\big).$$

So, $B\vec{x} = \vec{0}$ since $B\vec{x} \in \ker(A)$; and, thus $\vec{x} = \vec{0}$, since $\vec{x} \in \ker(B)$.