MATH 517
HOMEWORK 3
Winter, 2008

Section 2.3

42. All *permutation matrices* are invertible. The easiest way to see this is by Gauss-Jordan elimination: let P be an $n \times n$ permutation matrix, then to compute the rref(P):

$$\begin{bmatrix} P & I_n \end{bmatrix} \xrightarrow[\textit{elimination}]{\textit{Gauss-Jordan}} \begin{bmatrix} \text{rref}(P) & Q \end{bmatrix}$$

we only need to permute the rows to put the 1's on the diagonals. There is exactly one row which has a 1 in the first column, so move this row to the first row; there is exactly one row which has a 1 in the second column, so move this row to the second row; and so on. Thus, $\text{rref}(P) = I_n$.

The inverse matrix will be Q, and is obtained from I_n by permutating rows. Since I_n is a permutation matrix – there is exactly one 1 in each row and in each column, with 0's in all other entries. Q is obtained from P by *transposing* the rows of P to become columns of Q – the entry $p_{i,j}$ in P becomes the entry $q_{j,i}$ in Q.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

54. We want values of λ such that the following matrix equation:

$$A\vec{x} = \lambda\vec{x}$$

has nontrivial solutions (i.e. where $\vec{x} \neq \vec{0}$.) By a little matrix algebra, our problem can be rephrased: Find all values of λ for which the following matrix equation

$$\big(A - \lambda I_2\big)\vec{x} = \vec{0}$$

has nontrivial solutions. This will be true when $A - \lambda I_2$ is noninvertible.

We can either compute the reduced row-echelon form of the matrix

$$\begin{bmatrix} 1-\lambda & 10 \\ -3 & 12-\lambda \end{bmatrix}$$

or apply Fact 2.3.6 (p. 75). Either way, we find the matrix will only be noninvertible when $\lambda = 6, 7$.

When $\lambda = 6$, solve the matrix equation

$$\begin{bmatrix} -5 & 10 \\ -3 & 6 \end{bmatrix} \vec{x} = \vec{0}$$

So, solutions are all vectors $(2t, t)$ for any real number t.

When $\lambda = 7$, solve the matrix equation

$$\begin{bmatrix} -6 & 10 \\ -3 & 5 \end{bmatrix} \vec{x} = \vec{0}$$

So, solutions are all vectors $(\frac{5t}{3}, t)$ for any real number t.

Section 2.4

20. This is **not necessarily true**. What is always true is

$$(A - B)(A + B) = A^2 - B^2 + AB - BA$$

So, any pair of invertible matrices which do not commute is sufficient. For example, the noncommutative matrices on p. 82 (which are invertible) is a counterexample to the equation in the text:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

28. We want values for a, b, c, d (not all zero) satisfying

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If either $b = 0$ or $c = 0$ then we must have $a = d = 0$. So, we must have one of $b \neq 0$ or $c \neq 0$. This is sufficient (I have chosen $b = 1$, but any nonzero value works)

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

More generally, any 2×2 matrix A of the form

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \qquad \text{where } a^2 = -bc$$

will satisfy $A^2 = Z$ (where Z is the 2×2 matrix of 0s.) So, a less trivial example is

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

36. We want to determine all values of a, b, c, d such that

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This reduces the following system of four equations in four unknowns:

$$\begin{aligned} a + 2c &= 0 \\ b + 2d &= 0 \\ 2a + 4c &= 0 \\ 2b + 4d &= 0 \end{aligned}$$

Solving these leads to the dependencies $a = -2c$ and $b = -2d$. So, the following matrix is a solution for any choice of c, d:

$$X = \begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix}$$

38. We want an invertible matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} S = S \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

(Equivalently, you can use Fact 2.3.6 to write S^{-1} in terms of the entries of S.) Multiplying out we get

$$\begin{bmatrix} a + 3c & b + 3d \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} 2a & b \\ 2c & d \end{bmatrix}$$

The invertible matrices which satisfy the equality are

$$\begin{bmatrix} 3c & b \\ c & 0 \end{bmatrix} \qquad \text{for any } b, c \neq 0 \ .$$

40. We use the equality $(AB)^{-1} = B^{-1}A^{-1}$. So, we can compute A^{-1} from the given data

$$B(AB)^{-1} = A^{-1}$$

Thus,

$$A^{-1} = \begin{bmatrix} -1 & -5 \\ 1 & 4 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix}$$

44. One way to solve this is to find all values of a, b, c, d such that the following matrix equation is satisfied:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

This gives the following 4 equations in 4 unknowns

$$\begin{aligned} a + 2b &= 2 \\ 2a + 5b &= 1 \\ c + 2d &= 1 \\ 2c + 5d &= 3 \end{aligned}$$

Solving this system gives the following answer

$$\begin{bmatrix} 8 & -3 \\ -1 & 1 \end{bmatrix}$$

A second way is by computing

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1}$$