MATH 417
HOMEWORK 10
Winter, 2008

Section 7.3

20. Consider the matrix:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}.$$

There are two eigenvalues: 1 (algebraic multiplicity 2) and 2 (algebraic multiplicity 1). The geometric multiplicity of 2 is always 1; there is an eigenbasis exactly when the geometric multiplicity of 1 is two.

$$\text{geometric multiplicity of } 1 = \text{ the dimension of } \ker \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

- If $a = 0$ then the geometric multiplicity of the eigenvalue 1 is two, regardless of the values of b or c. There is an eigenbasis, and the matrix is diagonalizable.
- If $a \neq 0$ then the geometric multiplicity of the eigenvalue 1 is one, regardless of the values of b or c. There is no eigenbasis and the matrix is not diagonalizable.

34. Suppose $B = S^{-1}AS$ for some $n \times n$ matrices A, B, C.

(a). Suppose $\vec{x} \in \ker(B)$. Then

$$\vec{0} = B\vec{x} = S^{-1}AS\vec{x} = S^{-1}\big(AS\vec{x}\big);$$

and since S^{-1} is invertible (thus, $\ker(S^{-1}) = \{\vec{0}\}$), we must have $AS\vec{x} = \vec{0}$, or $S\vec{x} \in \ker(A)$.

(b). From (a), if $\vec{x} \in \ker(B)$ then $T(\vec{x}) = S\vec{x} \in \ker(A)$. So, $T : \ker(B) \to \ker(A)$. We will show that $\ker(T) = \{\vec{0}\}$ (that is, T is 1-1), and $\text{im}(T) = \ker(A)$. First, $T(\vec{x}) = S(\vec{x})$ and since S is invertible, $\ker(T) \subseteq \ker(S) = \{\vec{0}\}$.

Second, let $\vec{y} \in \ker(A)$; we will show that $S^{-1}\vec{y} \in \ker(B)$, then $T(S^{-1}(\vec{y})) = SS^{-1}\vec{y} = \vec{y}$ – so that $\operatorname{im}(T) = \ker(A)$. Show $S^{-1}\vec{y} \in \ker(B)$ when $\vec{y} \in \ker(A)$:

$$BS^{-1}\vec{y} = S^{-1}AS\big(S^{-1}\vec{y}\big) = S^{-1}A\vec{y} = S^{-1}\vec{0} = \vec{0}.$$

(c). We have the nullity of A is equal to the nullity of B by (b), since there is an invertible transformation $T : \ker(B) \to \ker(A)$: the rank of T is the dimension of $\ker(A)$ and this is equal to the dimension of the domain of T, which is $\ker(B)$. A and B are $n \times n$ matrices so by the Rank-Nullity Theorem (Fact 3.3.7)

$$n = \operatorname{rank}(A) + \operatorname{nullity}(A) \qquad n = \operatorname{rank}(B) + \operatorname{nullity}(B),$$

and since $\operatorname{nullity}(A) = \operatorname{nullity}(B)$ we have $\operatorname{rank}(A) = \operatorname{rank}(B)$.

44. The dynamical system is given by

$$\vec{x}_{t+1} = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix} \vec{x}_t$$

(a). The River Inn carries away some of the pollutants from the upper lakes to lower lakes (so, 10% of Lake Silvaplana is carried down to Lake Sils and 20% of lake Sils is carried down to Lake St. Moritz); also, 20% of the pollutants in each lake breaks down naturally.

(b) The eigenvalues and eigenvectors for the matrix are

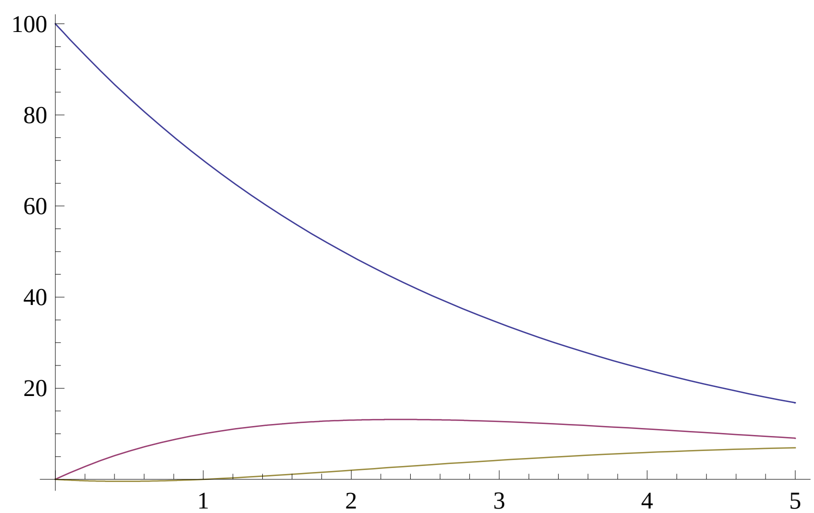
- eigenvalue: 0.8; eigenvector: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- eigenvalue: 0.7; eigenvector: $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$
- eigenvalue: 0.6; eigenvector: $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

The initial state of pollution is $\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$, whose coordinates in the eigenbasis is $\begin{bmatrix} 100 \\ 100 \\ -100 \end{bmatrix}$. The closed formulas for the polution in each of the three lakes is

$$\begin{aligned} x_1(t) &= (0.7)^t(100) \\ x_2(t) &= (0.7)^t(100) - (0.6)^t(100) \\ x_3(t) &= (0.8)^t(100) - (0.7)^t(200) + (0.6)^t(100) \end{aligned}$$

Pollution reaches a maximum in Lake Sils of 13 kgs in week 2. (More precisely, a maximum of 13.1437 kgs at 2.33 weeks.)

A 5 week plot is as follows (blue: Lake Silvaplana, mauve: Lake Sils, gold: Lake St. Moritz):



Section 7.4

20. Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The eigenvalues and associated eigenvectors is as follow

- eigenvalue: 2; eigenvector: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
- eigenvalue: 1; eigenvector: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
- eigenvalue: 0; eigenvector: $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

This matrix can be diagonalized:

$$S^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{where } S = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

26. Consider the following matrix

$$\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues are 1 (algebraic multiplicity of two) and 2 (algebraic multiplicity of one); so, the matrix is only diagonalizable when the geometric multiplicity of the eigenvalue 1 is two. That happens when

$$\text{nullity} \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix} = 2$$

The nullity is two exactly when $b = ac$. So, the matrix is diagonalizable when $b = ac$.

30. Consider the following matrix

$$\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial is $-x^3+3x+a$. We know that there is at least one real root; suppose b is a root, then factoring

$$x^3 - 3x - a = (x - b)(x^2 + bx + (b^2 - 3)) \qquad a = b^3 - 3b$$

If the matrix is to be diagonalizable then $x^2 + bx + (b^2 - 3)$ must have only real roots: the discriminant of the quadratic equation must be at least zero

$$0 \leq b^2 - 4(b^2 - 3) \qquad \text{so } |b| \leq 2$$

What values does a take when $-2 \leq b \leq 2$? The local maximum and minimum in the interval $|b| < 2$ can be computed by differentiating $x^3 - 3x$: $3x^2 - 3$, and finding the zeros of the derivative: $x = \pm 1$. Thus, the maximum and minimum values of a occur at either the endpoints or at the local maximum and minimum. You can verify that $a = 2$ at $b = -1, 2$ and $a = -2$ at $b = 1, -2$. So, $a \leq |2|$ when b is a root of $x^3 - 3x - a$ and $|b| \leq 2$.

Thus, a has real roots only when it takes values on the interval $-2 < a < 2$. It may not be diagonalizable though, if it does not contain three distinct roots. This can happen only when either (i) the discriminant of the quadratic is zero ($-3b^2 + 12 = 0$, or $b = \pm 2$) or (ii) when b itself is a root with algebraic multiplicity at least two ($(x - b)^2$ divides the polynomial $x^3 - 3x - a$, or $b = \pm 1$.) In either of the four possibilities in (i) and (ii) we have $a = \pm 2$. When $a = \pm 2$ the matrix is not diagonalizable since the geometric multiplicity of the multiple root is one in each case.

Therefore, for all values of a with $-2 < a < 2$ the matrix

$$\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

is diagonalizable.

34. Consider the matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

This matrix has the following eigenvalues and associated eigenvectors:

- eigenvalue: 1, eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- eigenvalue: $\frac{1}{4}$, eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So, this matrix is diagonalizable and

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} = S \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} S^{-1} \qquad \text{where } S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

We can now easily compute powers:

$$\begin{aligned}\begin{bmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{2} & \frac{1}{4}\end{bmatrix}^t &= S\begin{bmatrix}1^t & 0\\ 0 & \left(\frac{1}{4}\right)^t\end{bmatrix}S^{-1}\\ &= \begin{bmatrix}1+2\left(\frac{1}{4}\right)^t & 1-\left(\frac{1}{4}\right)^t\\ 2-2\left(\frac{1}{4}\right)^t & 2+\left(\frac{1}{4}\right)^t\end{bmatrix}\end{aligned}$$

Notice that

$$\lim_{t\to\infty} A^t = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ \\ \frac{2}{3} & \frac{2}{3}\end{bmatrix}$$

38. A and B are 2×2 matrices with $\det(A) = \det(B) = 4$ and $\operatorname{tr}(A) = \operatorname{tr}(B) = 4$, so they have the same characteristic equation, $x^2 - 4x + 4$ and thus have the same eigenvalues, 2 (with algebraic multiplicity of two.) Thus, A and B are similar to the same diagonal matrix, $\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$. Thus, A and B must be similar.

58. Let A be a diagonalizable $n\times n$ matrix with m distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Since A is diagonalizable, it has an eigenbasis, $\vec{v}_1, \ldots, \vec{v}_n$. We will show that for each eigenvector $\vec{v}_i$ in the eigenbasis

$$(A - \lambda_1 I_n)\cdots(A - \lambda_m I_n)\vec{v}_i = \vec{0},$$

it will then follow that this holds for every vector $\vec{v}$, since $\vec{v}$ can be written as a linear combination of the eigenvectors:

$$\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n.$$

First, note that the terms in the above product commute: for any $i, j \le m$

$$(A - \lambda_i I_n)(A - \lambda_j I_n) = (A - \lambda_j I_n)(A - \lambda_i I_n)$$

since

$$\begin{aligned}(A - \lambda_i I_n)(A - \lambda_j I_n) &= A^2 - (\lambda_i + \lambda_j)A + \lambda_i\lambda_j I_n\\ &= A^2 - (\lambda_j + \lambda_i)A + \lambda_j\lambda_i I_n\\ &= (A - \lambda_j I_n)(A - \lambda_i I_n)\end{aligned}$$

Now, we can re-arrange the order of the terms in the matrix

$$(A - \lambda_1 I_n) \cdots (A - \lambda_m I_n)$$

to bring any product to the far right (in this case, the ith term):

$$(A - \lambda_1 I_n) \cdots (A - \lambda_{i-1} I_n)(A - \lambda_{i+1} I_n) \cdots (A - \lambda_m I_n)(A - \lambda_i I_n)$$

The product remains unchanged in this re-ordering. So, let $\vec{v}_k$ be an eigenvector with associated eigenvalue λ_i, then $(A - \lambda_i I_n)\vec{v}_i = \vec{0}$, so

$$(A - \lambda_1 I_n) \cdots (A - \lambda_{i-1} I_n)(A - \lambda_{i+1} I_n) \cdots (A - \lambda_m I_n)(A - \lambda_i I_n)\vec{v}_i = \vec{0}.$$

59. Let A be an $n \times n$ diagonalizable matrix with characteristic polynomial

$$f_A(x) = (-x)^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

Since A is diagonalization, we can completely factor this polynomial: let $\lambda_1, \ldots, \lambda_m$ with algebraic multiplicities $k_1, \ldots, k_m$

$$f_A(x) = (x - \lambda_1)^{k_1} \cdots (x - \lambda_m)^{k_m}$$

Then, substitute A for x:

$$f_A(A) = (A - \lambda_1)^{k_1} \cdots (A - \lambda_m)^{k_m}$$

By Problem 58, for any vector $\vec{v}$,

$$(A - \lambda_1)^{k_1} \cdots (A - \lambda_m)^{k_m} \vec{v} = \vec{0};$$

and since

$$(A - \lambda_1)^{k_1} \cdots (A - \lambda_m)^{k_m} = (-A)^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0$$

so,

$$(-A)^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 = 0.$$

60. Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

The eigenvalues and associated eigenvectors are

- eigenvalue: $\lambda_1 = 5$, eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- eigenvalue: $\lambda_2 = -1$, eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(a). We compute $A - \lambda_1 I_2$ and $A - \lambda_2 I_2$:

$$A - \lambda_1 I_2 = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \quad A - \lambda_2 I_2 = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

The columns of each matrix are eigenvectors of A.

Let A be any diagonalizable 2×2 matrix, with eigenvalues λ_1, λ_2. By problem 58,

$$(A - \lambda_1 I_2)(A - \lambda_2 I_2) = 0$$

Let $A - \lambda_2 I_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$; then

$$0 = (A - \lambda_1 I_2)(A - \lambda_2 I_2) \quad = \quad (A - \lambda_1 I_2) \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$$
$$\begin{bmatrix} (A - \lambda_1 I_2)\vec{v}_1 & (A - \lambda_1 I_2)\vec{v}_2 \end{bmatrix}.$$

So,

$$(A - \lambda_1 I_2)\vec{v}_1 = 0 \qquad (A - \lambda_1 I_2)\vec{v}_2 = 0$$

and thus, $\vec{v}_1$ and $\vec{v}_2$ (the columns of $A - \lambda_2 I_2$) are eigenvectors of $A - \lambda_1 I_2$. Similarly, the columns of $A - \lambda_1 I_2$ are eigenvectors of $A - \lambda_2 I_2$.

The argument works for any diagonalizable $n \times n$ matrix A with two distinct distinct eigenvalues λ_1, λ_2 since by Problem 58 we have

$$(A - \lambda_1 I_2)(A - \lambda_2 I_2) = 0$$

so that

$$0 = (A - \lambda_1 I_2) \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} (A - \lambda_1 I_2)\vec{v}_1 & \dots & (A - \lambda_1 I_2)\vec{v}_n \end{bmatrix}$$

where $\vec{v}_1, \dots, \vec{v}_n$ are the columns of $A - \lambda_2 I_2$. Thus, each $(A - \lambda_1 I_2)\vec{v}_i = 0$, so that the columns of $A - \lambda_2 I_2$ are eigenvectors for $A - \lambda_1 I_2$.

(b). Let A be the original 2×2 matrix, then

$$A - \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & 4 \end{bmatrix}$$

and again the columns are eigenvectors.

This property holds generally. Let A be any diagonalizable 2×2 matrix with distinct eigenvalues λ_1 and λ_2; then the columns

$$A - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

are also eigenvectors of A, since the first column of this matrix is the first column of $A - \lambda_1 I_2$ which is an eigenvector of A by (a); and similarly, the second column of this matrix is the second column of $A - \lambda_2 I_2$ which is an eigenvector of A by (a). This property generalizes to any diagonalizable $n \times n$ matrix with two distinct eigenvalues.