Math 216
Differential Equations

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Second-order linear equation

- A second-order differential equation in the unknown $y = y(x)$ is one which can be put into the form

$$G(x, y, y', y'') = 0.$$

- A second-order linear differential equation is one that can be put into the form

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

  The equation is homogeneous if $F(x) = 0$, and otherwise nonhomogeneous.

- Given a second-order linear equation (as above), its associated homogeneous equation is

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

ConcepTest

Problem. Which of the following equations are linear.

(a) $27y'' + (\cos x)y' + e^{xy} = x^3.$

(b) $x^2y'' = 5 - xy'$

(c) $(y'')^2 + y' + y = 0.$

(d) $27y'' + 5y' - 3(\tan x)y = 5e^x$

Answer. (b), (d)

ConcepTest

Problem. Find a general solution to the second-order linear equation

$$y'' + y' = 0$$

Hint: substitute $u = y'$.

Answer. First, solve $u' + u = 0$, to get

$$y'(x) = u(x) = C_1e^{-x} + C_2.$$

Solve for y.

$$y(x) = C_1e^{-x} + C_2.$$

General method

- If the dependent variable y is missing from a second-order equation, then the equation takes the form

$$G(x, y', y'') = 0$$

- Make the substitution

$$p = y' = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx}$$

to reduce the second-order equation to a first-order equation

$$G(x, p, p'') = 0$$

- Solve this equation for a general solution $p(x, C_1)$ involving the parameter C_1, then solve for y

$$y(x) = \int y'(x)\, dx = \int p(x, C_1)\, dx + C_2.$$

Superposition of solutions

Theorem (Superposition of Solutions I)

Let y_1 and y_2 be two solutions on an interval I of the homogeneous linear equation

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

Then for any constants c_1 and c_2, the linear combination

$$c_1 y_1 + c_2 y_2$$

is also a solution on I.

Proof of superposition of solutions

Proof.

Let $y = c_1 y_1 + c_2 y_2$ where y_1 and y_2 are solutions to the homogeneous equation. Then by the linearity properties of differentiation:

$$y' = c_1 y_1' + c_2 y_2' \quad \text{and} \quad y'' = c_1 y_1'' + c_2 y_2''$$

So,

$$\begin{aligned} Ay'' + By' + Cy &= A(c_1 y_1'' + c_2 y_2'') + B(c_1 y_1' + c_2 y_2') + C(c_1 y_1 + c_2 y_2) \\ &= c_1 (Ay_1'' + By_1' + Cy_1) + c_2 (Ay_2'' + By_2' + Cy_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0. \end{aligned}$$

□

Superposition of solutions

Theorem (Superposition of Solutions II)

Let y_p be a solution on an interval I of the nonhomogeneous linear equation

$$A(x)y'' + B(x)y' + C(x)y = F(x),$$

and let y_c be any solution on I of the associated homogeneous equation.

Then $y_p + y_c$ is also a solution to the nonhomogeneous equation on I.

Proof.

See the practice exercise 27 in Section 3.1. □

Existence and Uniqueness Theorem

Theorem

Suppose that A, B, C, F are continuous functions on the open interval I containing the point a, and $A(x) \neq 0$ for any x in I. Then, given any two numbers b_0 and b_1, the equation

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

has a unique solution on the entire interval I satisfying the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

Existence and Uniqueness Theorem

- **Assumption.** From now on we will assume that the functions A, B, C, F of the linear equation

  $$A(x)y'' + B(x)y' + C(x)y = F(x)$$

  are continuous on some open interval I, and that $A(x) \neq 0$ for all x in I.
- We will write the general second-order linear equation (under this assumption) as

  $$y'' + p(x)y' + q(x)y = f(x)$$

  where p, q, f are continuous on I.

Example

Problem. Find the unique solution to the second-order IVP

$$y'' + 9y = 0, \quad y(0) = -1, \quad y'(0) = 3.$$

Answer. The general solution to this equation is

$$y(x) = c_1 \cos 3x + c_2 \sin 3x.$$

Substituting,

$$\begin{aligned} y(0) &= -1 = c_1 \cos 0 + c_2 \sin 0 = c_1 \\ y'(0) &= 3 = -3c_1 \sin 0 + 3c_2 \cos 0 = 3c_2 \end{aligned}$$

So, $c_1 = -1$ and $c_2 = 1$. Thus,

$$y(x) = \sin 3x - \cos 3x.$$

ConcepTest

Problem. Find the unique solution to the second-order initial value problem

$$y'' + y' = 0, \quad y(0) = -2, y'(0) = 8$$

How do we know there is a unique solution on the real line?

Answer. The general solution was

$$y = C_1 e^{-x} + C_2.$$

So, we have two equations

$$\begin{aligned} y(0) = -2 &= C_1 + C_2 \\ y'(0) = 8 &= -C_1. \end{aligned}$$

Solving for $C_1 = -8$ and $C_2 = 6$ provides the unique solution

$$y = -8e^{-x} + 6.$$

Example: Changing parameters

Problem. What values of the parameters c_1 and c_2 to the solution

$$c_1 \cos x + c_2 \sin x$$

satisfy the initial value problem

$$y'' + y = 0, \quad y(0) = 1, y'(0) = b?$$

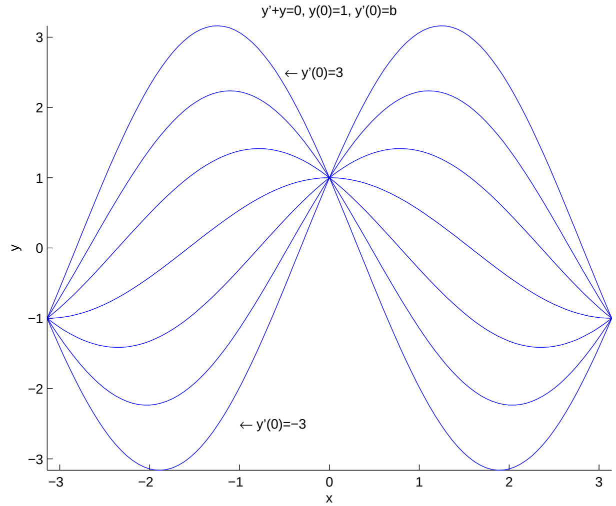
Solution. Substitute the initial conditions into the general solution:

$$\begin{aligned} y(0) &= 1 = c_1 \cos 0 + c_2 \sin 0 = c_1 \\ y'(0) &= b = -c_1 \sin 0 + c_2 \cos 0 = c_2 \end{aligned}$$

So, the solution is $\cos x + b \sin x$.

Graphing changing parameters

Plotting: $\cos x + b \sin x$ for some values of b.



Example: Changing parameters

Problem. What values of the parameters c_1 and c_2 to the solution

$$c_1 \cos x + c_2 \sin x$$

satisfy the initial value problem

$$y'' + y = 0, \quad y(0) = a, y'(0) = 1?$$

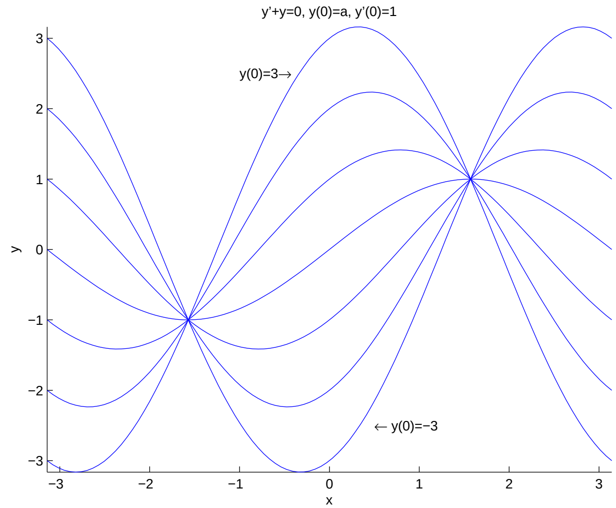
Solution. Substitute the initial conditions into the general solution:

$$\begin{aligned} y(0) &= a = c_1 \cos 0 + c_2 \sin 0 = c_1 \\ y'(0) &= 1 = -c_1 \sin 0 + c_2 \cos 0 = c_2 \end{aligned}$$

So, the solution is $a \cos x + \sin x$.

Graphing changing parameters

Plotting: $a \cos x + \sin x$ for some values of a.



Linear Independence

Definition

Given functions $y_1, \ldots, y_n$ defined on an interval I, if there exist constants $c_1, \ldots, c_n$ (not all zero) such that

$$c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0$$

for all x in some interval I, then the functions $y_1, \ldots, y_n$ are said to be linearly dependent on I. If there are no such constants, then the functions are said to be linearly independent on I.

Linear Independence

Problem. Show that the functions $x^2, 1, 3x^2 - 5$ are linearly dependent. (Where 1 is the constant function with value 1.)

Solution. Let $c_1 = 3, c_2 = -5, c_3 = -1$, then

$$3(x^2) - 5(1) - (3x^2 - 5) = 0.$$

ConcepTest

Problem. Show that the functions $\sin^2 x, \cos^2 x, 1$ are linearly dependent on $I = (-\pi, \pi)$.

Answer. Let $c_1 = c_2 = 1$ and $c_3 = -1$. Then

$$\sin^2 x + \cos^2 x - 1 = 0.$$

ConcepTest

Problem. Show that the functions $y_1, \ldots, y_n, y \equiv 0$ are linearly dependent on any interval I for any functions $y_1, \ldots, y_n$ defined on I.

Answer. Let $c_1 = \ldots = c_n = 0$ and $c_{n+1} = 5$. Then

$$5 \cdot 0 + 0 \cdot y_1 + \ldots + 0 \cdot y_n = 0.$$

Testing for linear independence

Problem. Show that e^x, e^{2x} are linearly independent on the real line.

Suppose e^x and e^{2x} were linearly dependent. Then, there would be constants c_1, c_2, not both zero, such that

$$c_1 e^x + c_2 e^{2x} = 0.$$

for every real x. But then one of the following would have to be true:

$$\begin{aligned} e^x &= -\frac{c_2}{c_1} e^{2x} \qquad \text{if } c_1 \neq 0 \\ e^{2x} &= -\frac{c_1}{c_2} e^{x} \qquad \text{if } c_2 \neq 0 \end{aligned}$$

for every real x. This is not the case.

Therefore, e^x and e^{2x} are linearly independent.

Testing for linear independence

Problem. How to show that a pair of non-zero functions f_1, f_2 are linearly independent on an interval I. (A function is non-zero on an interval I if takes a value other than zero on the interval.)

Solution. Suppose f_1 and f_2 are linearly dependent on I. Then there would be two constants c_1 and c_2, not both zero, satisfying

$$c_1 f_1 + c_2 f_2 = 0.$$

for every x in I. Then, we must have both c_1 and c_2 nonzero (why?), so

$$f_1 = k f_2 \qquad \text{where } k = -\frac{c_2}{c_1}$$

If we can show this is impossible, then f_1 and f_2 are linearly independent on I.

ConcepTest

Problem. Show that $\sin x, \cos x$ are linearly independent on the interval $(-\pi, \pi)$.

Answer. It is not true that $\sin x = k \cos x$ or that $\cos x = k \sin x$ for some constant k in the interval $(-\pi, \pi)$:

- $x = 0$: $\cos x = 1$ and $\sin x = 0$, so there is no k with $\cos x = k \sin x$ on $(-\pi, \pi)$.
- $x = \frac{\pi}{2}$: $\cos x = 0$ and $\sin x = 1$, so there is no k with $\sin x = k \cos x$ on $(-\pi, \pi)$.

Therefore, $\sin x$ and $\cos x$are linearly independent.

Multiple solutions

The homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

must have at least two linearly independent solutions.

We know there are solutions y_1 and y_2 for the two conditions

$$\begin{aligned} y_1(a) = 0 \quad &, \quad y_1'(a) = 1 \\ y_2(a) = 1 \quad &, \quad y_2'(a) = 0 \end{aligned}$$

Reason:

- We cannot have $y_1 = k y_2$, since otherwise $y_1'(x) = k y_2'(x)$ for all x, which is impossible at $x = a$: $y_2'(a) = 0 \neq y_1'(a)$.
- We cannot have $y_2(x) = k y_1(x)$ for all x, since $y_1(a) = 0 \neq y_2(a)$.

General solutions

Theorem

Let y_1 and y_2 be linearly independent solutions on some open interval I to the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

with p, q continuous in I.

Then, for any solution y on I, there exists constants c_1 and c_2 such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all x in I.

ConcepTest

Problem. Find all solutions in the interval $(0, \infty)$ to the linear equation

$$xy'' = y'$$

How do you know you have all of them?

Answer. All solutions are included in

$$y(x) = C_1 x^2 + C_2.$$

The solutions $y_1(x) = x^2$ and $y_2(x) \equiv 1$ are linearly independent (since x^2 is not constant).

Linear equations with constant coefficients

Let a, b, c be constants coefficients in the linear equation

$$ay'' + by' + cy = 0.$$

Consider the substitution $y = e^{rx}$:

$$\begin{aligned} 0 &= a(e^{rx})'' + b(e^{rx})' + ce^{rx} \\ &= ar^2 e^{rx} + bre^{rx} + ce^{rx} \\ &= (ar^2 + br + c)e^{rx}. \end{aligned}$$

If r satisfies the quadratic equation

$$ar^2 + br + c = 0,$$

then e^{rx} is a solution to the linear differential equation.

Example

Problem. Find a general solution to the equation:

$$2y'' - 6y' + 4y = 0.$$

Answer. We guess a solution is of the form e^{rx}, so r must satisfy

$$0 = 2r^2 - 6r + 4 = (2r - 2)(r - 2).$$

Thus, if $r = 2$ and $r = 1$, then e^{rx} is a solution. The general solution is

$$c_1 e^{2x} + c_2 e^x.$$

Question. How do we know that all solutions can be obtained by choosing values for c_1 and c_2?

ConcepTest

Problem. Find the unique solution to the IVP

$$2y'' - 6y' + 4y = 0, \quad y(0) = 2, y'(0) = -1$$

Answer. The general solution is $c_1 e^{2x} + c_2 e^x$. Substituting for the initial conditions,

$$\begin{aligned} c_1 + c_2 &= 2 \\ 2c_1 + c_2 &= -1. \end{aligned}$$

So, $y(x) = -3e^{2x} + 5e^x$ is the solution to the IVP.

ConcepTest

Problem. Find the general solution to the equation

$$10y'' + 5y' = 0$$

Answer. We want the roots r of the polynomial
$10r^2 + 5r = 5r(2r + 1) = 0$.
The general solution is

$$y(x) = c_1 e^{-\frac{x}{2}} + c_2 e^{0 \cdot x} = c_1 e^{-\frac{x}{2}} + c_2$$

ConcepTest

Problem. Solve the IVP

$$2y'' - 3y' - 2y = 0, \quad y(0) = 4, y'(0) = 1$$

Answer. We want the roots r of the polynomial
$2r^2 - 3r - 2 = (r - 2)(2r + 1) = 0$.
The general solution is

$$y(x) = c_1 e^{-\frac{x}{2}} + c_2 e^{2x}.$$

Substituting for the initial values:

$$\begin{aligned} c_1 + c_2 &= 4 \\ -\frac{1}{2}c_1 + 2c_2 &= 1. \end{aligned}$$

So, $y(x) = 2e^{-\frac{x}{2}} + 2e^{2x}$ is the solution to the IVP.

Distinct real roots

Theorem

If r_1 and r_2 are distinct and real roots of the quadratic equation $ar^2 + br + c = 0$, then

$$c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is the general solution to the linear homogeneous equation

$$ay'' + by' + cy = 0$$

Example

Problem. Find a general solution to the equation:

$$y'' - 6y' + 9 = 0.$$

Answer. The quadratic equation $r^2 - 6r + 9 = (r-3)^2$ has only one real (repeated) root, $r = 3$. Thus, e^{3x} is one root. What is the other root?

Verify by substitution that xe^{3x} is a second solution

$$0 = (xe^{3x})'' - 6(xe^{3x})' + 9xe^{3x}$$

and is also linearly independent of e^{3x}. Thus, the general solution is

$$c_1 e^{3x} + c_2 xe^{3x}$$

Example

Problem. Find a general solution to the equation:

$$y'' - 2sy' + s^2 = 0.$$

where s is a real value.

Answer. The quadratic equation $r^2 - 2sr + s^2 = (r-s)^2$ has only one real (repeated) root, $r = s$. Thus, e^{sx} is one solution and xe^{sx} is another linearly independent solution. The general solution is

$$c_1 e^{sx} + c_2 xe^{sx}.$$

Repeated real roots

Theorem

If the quadratic equation $ar^2 + br + c = 0$ has a single real (repeated) root r_1, then

$$(c_1 + c_2 x) e^{r_1 x}$$

is the general solution to the linear homogeneous equation

$$ay'' + by' + cy = 0$$

ConcepTest

Problem. Find the unique solution to the IVP

$$ay'' + 2y' + y = 0, \quad y(0) = 1, y'(0) = b.$$

Answer. The general solution is $c_1 e^{-x} + c_2 xe^{-x}$. Substituting for the initial conditions, we have two equations

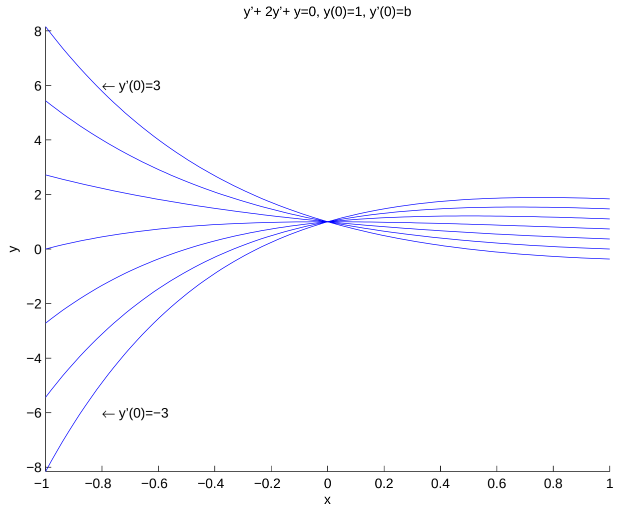
$$\begin{aligned} 1 &= c1 \\ b &= -c_1 + c_2 \end{aligned}$$

So, $c_1 = 1$ and $c_2 = b + 1$. Thus, the solution to the IVP is

$$e^{-x} + (b+1)xe^{-x}.$$

Graphing changing parameters

Plotting: $e^{-x} + (b+1)xe^{-x}$ for some values of b.



ConcepTest

Problem. Find the unique solution to the IVP

$$ay'' + 2y' + y = 0, \quad y(0) = a, y'(0) = 1.$$

Answer. The general solution is $c_1 e^{-x} + c_2 xe^{-x}$. Substituting for the initial conditions, we have two equations

$$\begin{aligned} a &= c1 \\ 1 &= -c_1 + c_2 \end{aligned}$$

So, $c_1 = a$ and $c_2 = a + 1$. Thus, the solution to the IVP is

$$ae^{-x} + (a+1)xe^{-x}.$$

Graphing changing parameters

Plotting: $ae^{-x} + (a+1)xe^{-x}$ for some values of a.