Math 216
Differential Equations

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Matlab programs

I wrote three Matlab programs (based on the text examples in Sections 2.4, 2.5 and 2.6).

- **euler1.m**: implements Euler's method.
- **euler2.m**: implements the improved Euler's method.
- **rk.m**: implements Runge-Kutta method.

Euler's method – algorithm

Definition (Euler's method)

Given an initial value problem on an interval $[x_0 = a, b]$

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

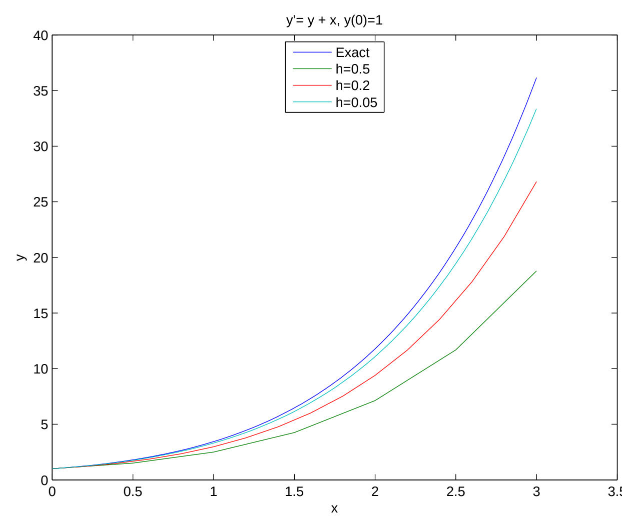
Euler's method with step size h consists in applying the iterative formula

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \qquad (n \geq 0)$$

to calculate successive approximations $y_0, y_1, y_2, \ldots$ to the true values $y(x_0), y(x_1), y(x_2), \ldots$ of the exact solution $y = y(x)$ at the points $x_0, x_1, x_2, \ldots$.

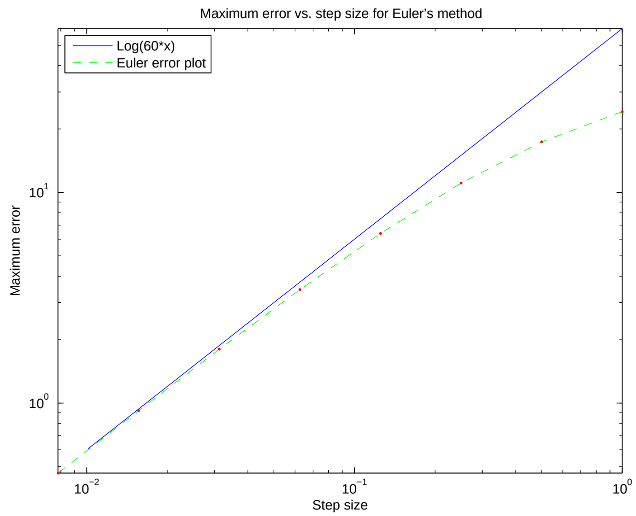
Euler's method: plot, side-by-side

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.5, 0.2, 0.05$



Euler's method: Logarithmic plot of error

Logarithm plot of step size vs. error for $y' = y + x, y(0) = 1$



Euler's method: Logarithmic plot of error

- The logarithmic plot suggests a linear relation between the logs of the step size and error:

$$\log(\text{error}) = A + B \cdot \log(\text{step})$$

- Raising to the power 10:

$$\text{error} = 10^A \cdot \text{step}^B$$

- The slope is $B \approx 1$ and $A \approx \log 60$, so it appears that

$$|y(x_n) - y_n| = 60 \cdot h \qquad \text{for all } n$$

  where h is the step size.

Error bound

- In many cases we can expect a linear relation between step size h and error:

$$\text{error}(n) = |y(x_n) - y_n| \le Ch \qquad \text{for all } n$$

  where the constant C is independent of step size.
- However, when $y' = f(x, y)$ fails to have a solution in the entire interval $[x_0 = a, b]$ through the point (x_0, y_0), the method can fail very badly.
- For example, $y' = x^2 + y^2$ and $y(0) = 1$ does not have a solution throughout the entire interval $[0, 1]$. See Example 5 of Section 2.4.

Strategy for choosing h

- Computing C is very hard, so knowing it exists is only of theoretical interest. You are still in the dark about how far off your error is.
- A strategy for deciding the step size h.
  1. Apply Euler's method using a small choice of step size h.
  2. Repeat the method, halving the step size each time: $\frac{h}{2}, \frac{h}{4}, \ldots$.
  3. Continue until you obtain successive step sizes which agree on their values, within the desired number of significant digits.

Computing e: Euclid

The number $e = y(1)$, where $y(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = y, \qquad y(0) = 1.$$

We use Euclid's method to approximate e to three decimal places, where we want agreement in two successive subintervals. The correct value is 2.718.

Steps	Value
50	2.692
100	2.705
200	2.712
400	2.715
800	2.717
1600	2.717

Improved Euler's method

- We require 1600 steps (and a step size of .000625) just to compute three significant digits of e. This is not good.
- We would like better accuracy without having to drastically reduce step size (which increases computation time and round-off errors.
- The Improved Euler method (Section 2.5) and Runge-Kutta method (Section 2.6) use predictor-corrector methods, which provides greater accuracy in fewer steps.
- These methods provide a numerical approximation to an IVP

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

on an interval $[x_0 = a, b]$.

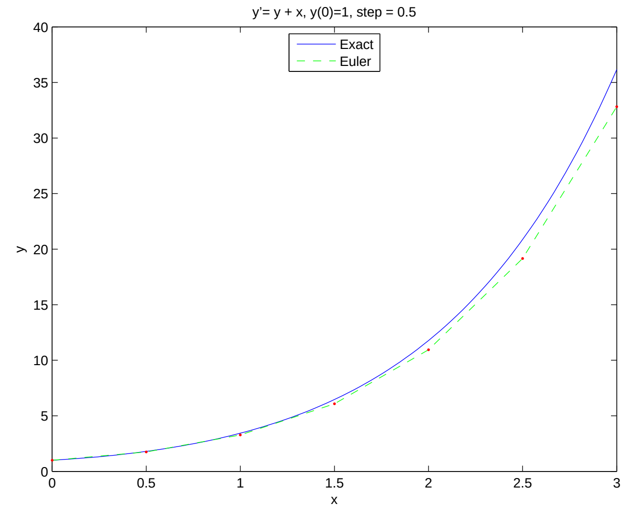
Improved Euler's method

Define successive approximations $y_0, y_1, y_2, \ldots$ to the solution at $x_0, x_1, x_2, \ldots$ where $x_{n+1} = x_n + h$.

1. Compute the slope at (x_n, y_n):
$$K_1 = f(x_n, y_n).$$
2. Predict the next value to be on the line with slope K through (x_n, y_n)
$$u_{n+1} = y_n + h \cdot K_1.$$
3. Compute the slope at the new point (x_{n+1}, u_{n+1}):
$$K_2 = f(x_{n+1}, u_{n+1}).$$
4. Average the two slope guesses
$$K = \frac{1}{2}(K_1 + K_2).$$
5. Correct the prediction by using the line with slope K through (x_n, y_n):
$$y_{n+1} = y_n + h \cdot K.$$

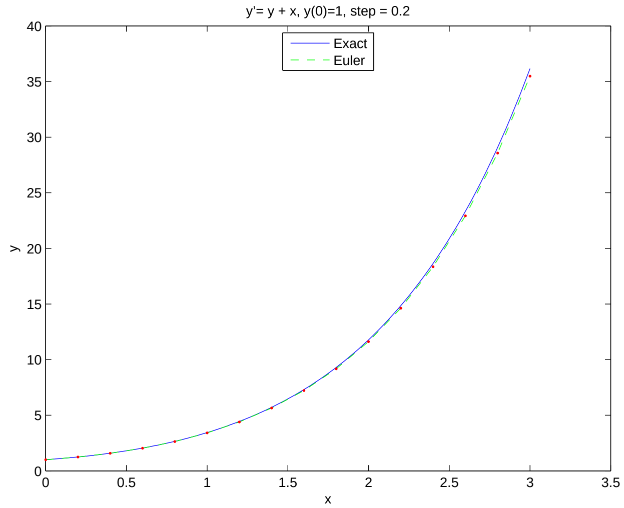
Improved Euler's method: plot, step = 0.5

$$y' = y + x, \qquad y(0) = 1 \qquad h = 0.5$$



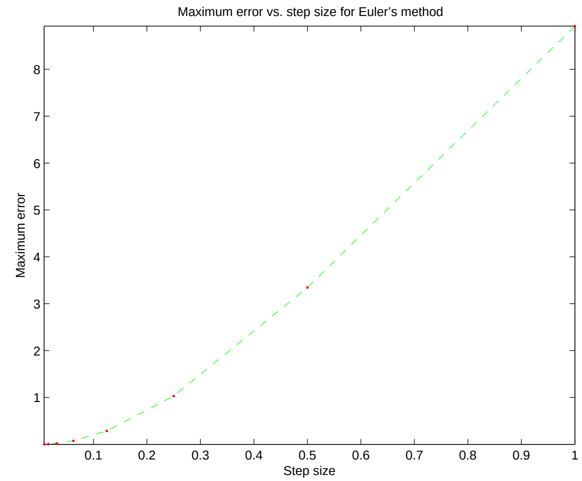
Improved Euler's method: plot, step = 0.2

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.2$



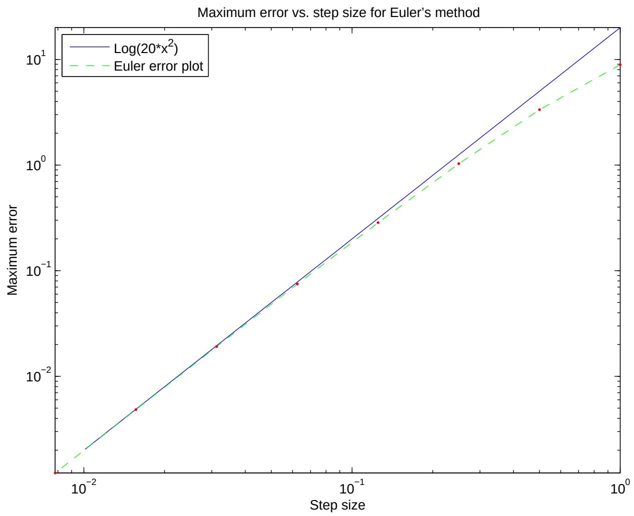
Improved Euler's method: error

Plot of step size vs. error for $y' = y + x, y(0) = 1$.
Note. Error is quartered when step size is halved.



Improved Euler's method: Logarithmic plot of error

Logarithm plot of step size vs. error for $y' = y + x, y(0) = 1$



Improved Euler's method: Logarithmic plot of error

- The logarithmic plot suggests a linear relation between the logs of the step size and error:

$$\log(\text{error}) = A + B \cdot \log(\text{step})$$

- Raising to the power 10:

$$\text{error} = 10^A \cdot \text{step}^B$$

- The slope is $B \approx 2$ and $A \approx \log 20$, so it appears that

$$|y(x_n) - y_n| < 20 \cdot h^2 \qquad \text{for all } n$$

where h is the step size

Computational cost of Improved Euler

- In many cases we can expect a quadratic relation between step size h and error:

$$\text{error}(n) = |y(x_n) - y_n| \leq Ch^2 \qquad \text{for all } n$$

  where the constant C is independent of step size.
- Each improved Euler step requires two computations of $f(x, y)$ for each such computation in the Euler method.
- However, the improved Euler method provides better accuracy with significantly fewer steps required:

  doubling the number of steps decreases the error fourfold (as opposed to a twofold decrease with Euler's method.)

Computing e: Euler

The number $e = y(1)$, where $y(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = y, \qquad y(0) = 1.$$

We use Euclid's method to approximate e to five decimal places, where we want agreement in two successive subintervals. The correct value is 2.71828.

Euler		Improved Euler	
Steps	**Value**	**Steps**	**Value**
50	2.692	10	2.71408
100	2.705	20	2.71719
200	2.712	40	2.71800
400	2.715	80	2.71821
800	2.717	160	2.71826
1600	2.717	320	2.71828
		640	2.71828

Runge-Kutta method

- The Runge-Kutta method is another iterative predictor-corrector method, like the improved Euler method. The method adds several corrections to compute the slope, which dramatically improves the accuracy.
- We want a numeric solution to the equation

$$\frac{y}{x} = f(x, y), \qquad y(x_0) = y_0.$$

  in an interval $[x_0 = a, b]$.
- We choose a step size h, and approximate the solution y by iteratively calculating successive approximations $y_0, y_1, y_2, \ldots$ at the points $x_0, x_1, x_2, \ldots$, where $x_{n+1} = x_n + h$.

Simpson's Rule

- Simpson's rule is the following approximation to the integral:

$$\int_x^{x+h} y'(t)\,dt \approx \frac{h}{6}\left[y'(x) + 4y'(x + \frac{h}{2}) + y'(x+h)\right]$$

- By the first fundamental theorem of calculus,

$$\int_x^{x+h} y'(t)\,dt = y(x+h) - y(x).$$

- To approximate $y(x+h)$ then,

$$y(x+h) \approx y(x) + \frac{h}{6}\left[y'(x) + 4y'(x + \frac{h}{2}) + y'(x+h)\right]$$

Runge-Kutta: 1-step

Suppose we have already computed an approximation (x_n, y_n) to

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0.$$

1 Compute the slope at (x_n, y_n):

$$K_1 = f(x_n, y_n).$$

The line with slope K_1 through (x_n, y_n) is: $\ell_1(x) = y_n + K_1(x - x_n)$.

2 Compute the midpoint value $\ell_1(x_n + \frac{h}{2}) = y_n + \frac{h}{2}K_1$; and use this value to compute the slope at this midpoint (first time):

$$K_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1).$$

The line with slope K_2 through (x_n, y_n) is: $\ell_2(x) = y_n + K_2(x - x_n)$.

Runge-Kutta: 1-step

3 Re-compute the midpoint value on $\ell_2(x_n + \frac{h}{2}) = y_n + \frac{h}{2}K_2$; and use this value to re-compute the slope at this midpoint (second time):

$$K_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_2).$$

The line with slope K_3 through (x_n, y_n) is: $\ell_3(x) = y_n + K_3(x - x_n)$.

4 Compute the endpoint on $\ell_3(x_n + h) = y_n + hK_3$, and use this value to compute the slope at this endpoint:

$$K_4 = f(x_n + h, y_n + hK_3).$$

5 Take the weighted mean of the four slopes (using Simpson's rule)

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

The line with slope K through (x_n, y_n) is: $\ell(x) = y_n + K(x - x_n)$.

6 Compute the next guess to be on the line ℓ:

$$y_{n+1} = y_n + hK.$$

Review: Runge-Kutta algorithm

Given (x_n, y_n), let $x_{n+1} = x_n + h$ and compute y_{n+1} as follows:

- K_1 is the slope at (x_n, y_n).
- K_2 is the slope at the midpoint of $[x_n, x_n + h]$ using the midpoint value on the line $y_n + K_1(x_n - x)$.
- K_3 is the slope at the midpoint, but now using the midpoint value on the line $y_n + K_2(x_n - x)$.
- K_4 is the slope at the end of the interval $x = x_n + h$ on the line $y_n + K_3(x_n - x)$.
- K is weighted average of all four slopes, with greater weight given to the slopes at the midpoint:

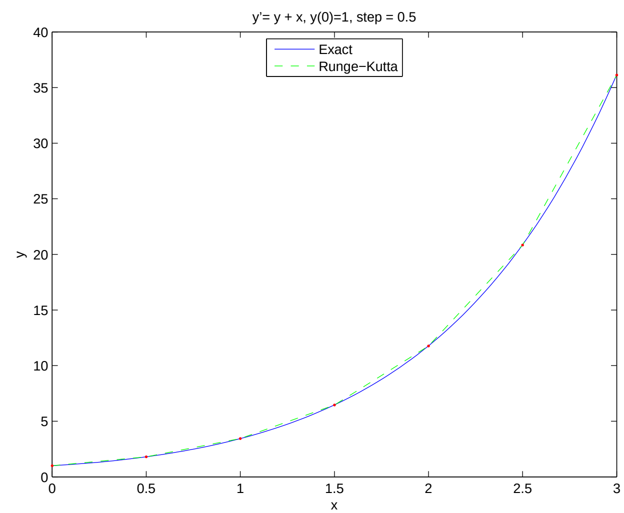
$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Finally, we compute y_{n+1} using the line $y_n + K(x_n - x)$

$$y_{n+1} = y_n + hK.$$

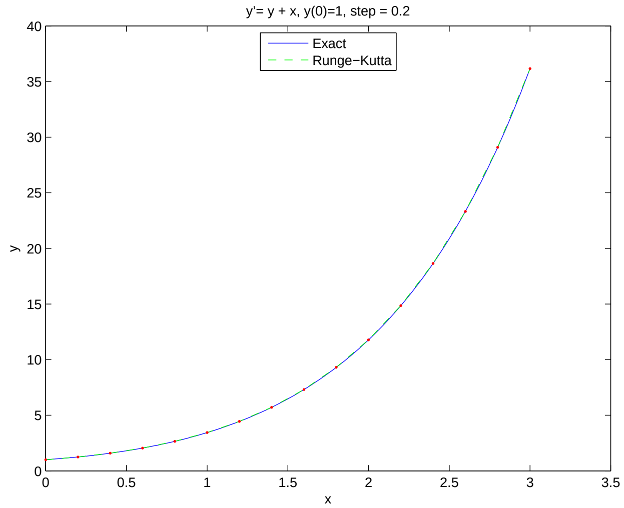
Runge-Kutta: plot, step = 0.5

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.5$



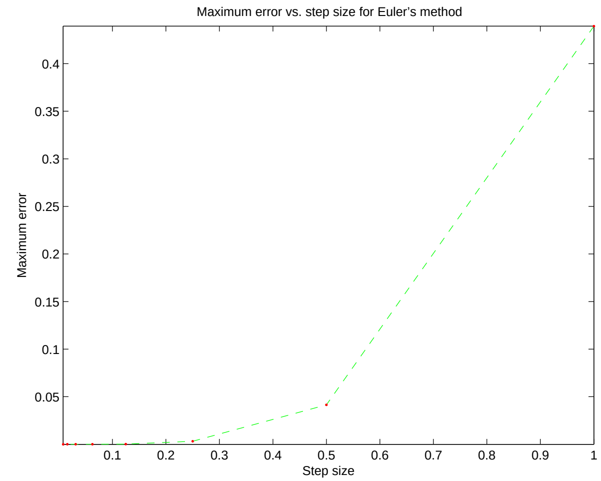
Runge-Kutta: plot, step = 0.2

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.2$



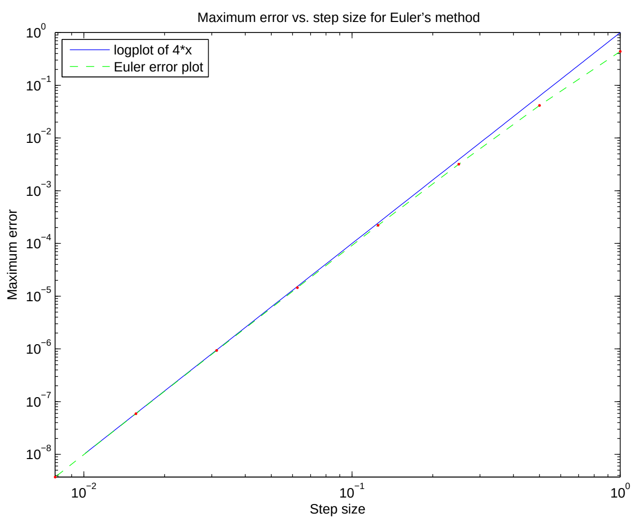
Runge-Kutta: error

Plot of step size vs. error for $y' = y + x, y(0) = 1$.
Note. Error is is cut sixteenfold when step size is halved.



Runge-Kutta: Logarithmic plot of error

Logarithm plot of step size vs. error for $y' = y + x, y(0) = 1$



Runge-Kutta method: Logarithmic plot of error

- The logarithmic plot suggests a linear relation between the logs of the step size and error:

$$\log(\text{error}) = A + B \cdot \log(\text{step})$$

- Raising to the power 10:

$$\text{error} = 10^A \cdot \text{step}^B$$

- The slope is $B \approx 4$ and $A \approx \log 1$, so it appears that

$$|y(x_n) - y_n| = 1 \cdot h^4 \qquad \text{for all } n.$$

Comparison of error

On an initial value problem on an interval $[x_0 = a, b]$

$$\frac{y}{x} = f(x, y), \qquad y(x_0) = y_0.$$

- Euler's method is a first-order method, meaning that we can expect the cumulative error to be bounded by step size h as

$$|y(x_n) - y_n| < C_1 \cdot h \qquad \text{for all } n.$$

- The improved Euler's method is a second-order method, meaning that we can expect the cumulative error to be bounded by bounded by step size h as

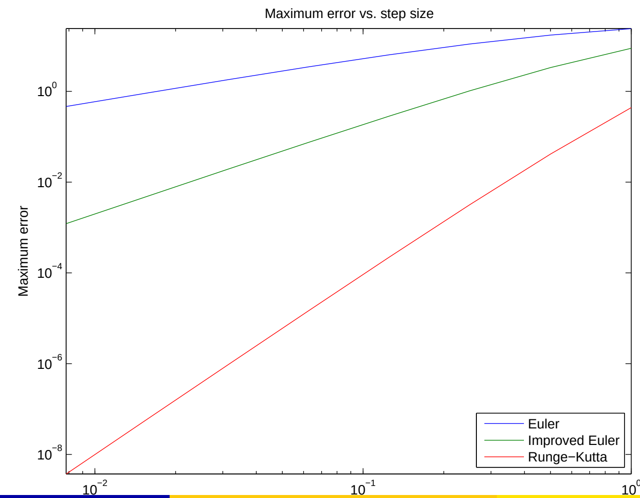
$$|y(x_n) - y_n| < C_2 \cdot h^2 \qquad \text{for all } n.$$

- The Runge-Kutta method is a fourth-order method, meaning that we can expect the cumulative error to be bounded by bounded by step size h as

$$|y(x_n) - y_n| < C_3 \cdot h^4 \qquad \text{for all } n.$$

Comparison of error

Logarithm plot of step size vs. error for $y' = y + x, y(0) = 1$. A comparison of all three methods.



Computing e: Euler

The number $e = y(1)$, where $y(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = y, \qquad y(0) = 1.$$

A comparison of all three methods: Euler's to 3 significant places, Improved Euler's to 5 significant places, and Runge-Kutta to 9 significant places. The correct value is 2.718281828.

Euler		Improved Euler		Runge-Kutta	
Steps	**Value**	**Steps**	**Value**	**Steps**	**Value**
50	2.692	10	2.71408	10	2.781279744
100	2.705	20	2.71719	20	2.718281693
200	2.712	40	2.71800	40	2.718281820
400	2.715	80	2.71821	80	2.718281828
800	2.717	160	2.71826	160	2.718281828
1600	2.717	320	2.71828		
		640	2.71828		