Math 216
Differential Equations

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ConcepTest

Problem. A boat is traveling on a lake, when its engine cuts off. Its speed is $V\frac{ft}{s}$ when its engine dies. Suppose the resistance to the boat from the lake is given by

$$\frac{dv}{dt} = -kv$$

How far does the boat coast?

Answer. The equation for the distance the boat travels is given by

$$x(t) = -\frac{V}{k}e^{-kt} + \frac{V}{k}$$

So, the boat coasts $\frac{V}{k}$ feet.

ConcepTest

Problem. Why is there a unique solution on the interval $[0, 3]$ to the IVP

$$y' = y + x, \qquad y(0) = 1?$$

Find this solution.

Answer. This is a first-order linear equation,

$$y' - y = x, \qquad y(0) = 1?$$

whose coefficients, $P(x) = -1$ and $Q(x) = x$ are continuous on the interval $[0, 3]$, so the equation has a unique solution at the point $(0, 3)$. This solution is

$$y(x) = -x - 1 + 2e^x.$$

Matlab programs

I wrote two Matlab programs (based on the text examples in Sections 2.4 and 2.5).

- **euler1.m**: implements Euler's method.
- **euler2.m**: implements the improved Euler's method.

Nonelementary equations

- Most first-order differential equation initial value problems

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0$$

  cannot be solved explicitly. (See Lecture 4 for examples of integrals which are not solvable using elementary functions.)
- Euler's method is a simple technique for approximating solutions to an initial value problem. It is the numerical counterpart to the graphical technique of constructing a direction field from Section 1.3.
- This is a technique that **dfield** uses to construct a solution to an equation through a point (as well as the Runge-Kutta, which we will discuss Friday).

Euler's method

- Consider the general form for a first-order differential equation

$$\frac{dy}{dx} = f(x,y)$$

  This can be written in differential form as

$$dy = f(x,y)\,dx$$

- $\Delta y = |y(x+\Delta x) - y(x)|$ is the actual change in y. If y is differentiable at x, then dy can be made to approximate Δy to any desired accuracy, provided the value of Δx is made sufficiently small.

$$\Delta y \approx dy = f(x,y)\,dx$$

- So,

$$y + \Delta y = y(x+dx) \approx y + dy = y + f(x,y)\,dx.$$

  $(x+dx, y+dy)$ is the point on the line with slope $f(x,y)$ through (x,y).

Euler's method: first step

- We start with our initial value problem on some interval $[x_0 = a, b]$

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0$$

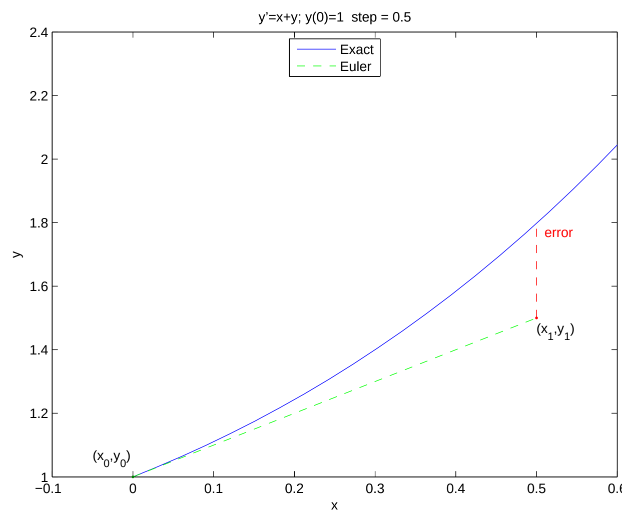
- This gives us two pieces of information
  (i) The value of the solution at a point, (x_0, y_0),
  (ii) The instantaneous slope $f(x_0, y_0)$ of this solution.
- Since we are assuming that the unknown solution is differentiable, the line through the point (x_0, y_0) with slope $f(x_0, y_0)$:

$$\ell_0(x) = y_0 + (x_0 - x)\cdot f(x_0, y_0)$$

  is a good approximation to the true solution, at least close to x_0.
- Fix a very small value h, and let $x_1 = x_0 + h$ and $y_1 = y_0 + h\cdot f(x_0, y_0)$. Then (x_1, y_1) should be close to the true unknown solution $y(x_1)$.

Euler's method: step 1

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.5$



Euler's method: second step

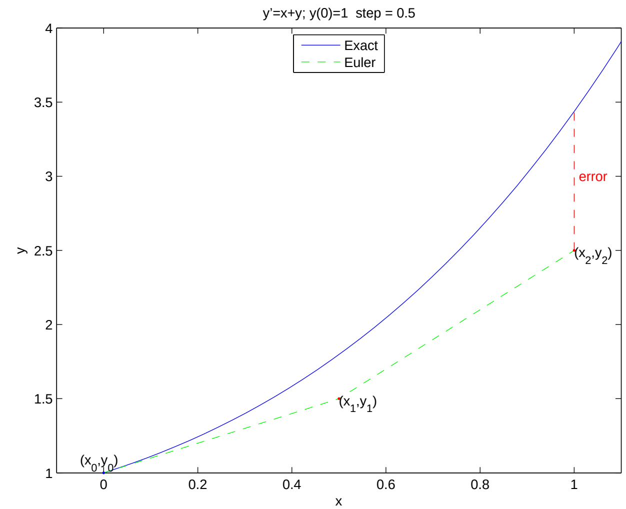
- We again have two pieces of information to work with
  - (i) The point (x_1, y_1), which may not be on the solution curve, but should be close.
  - (ii) The instantaneous slope of a solution through this point, $f(x_1, y_1)$.
- Since (x_1, y_1) is close to the true solution, $y(x_1)$, we expect that the line through this point with slope $f(x_1, y_1)$, should still be a good approximation to the true solution $y(x)$ near x_1.

$$\ell_1(x) = y_1 + (x_1 - x) \cdot f(x_1, y_1),$$

- Use our small step, h, and let $x_2 = x_1 + h$ and $y_2 = y_1 + h \cdot f(x_1, y_1)$. Then (x_2, y_2) should still be close to the true unknown solution $y(x_2)$.

Euler's method: step 2

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.5$



Euler's method: general strategy

- We iterate this process, choosing points: $x_0, x_1, x_2, \ldots$ close together, $|x_{n+1} - x_n| = h$,
- while computing successive approximations to the true solution, $y_0, y_1, y_2, \ldots$, so that they lie on the successive tangent lines

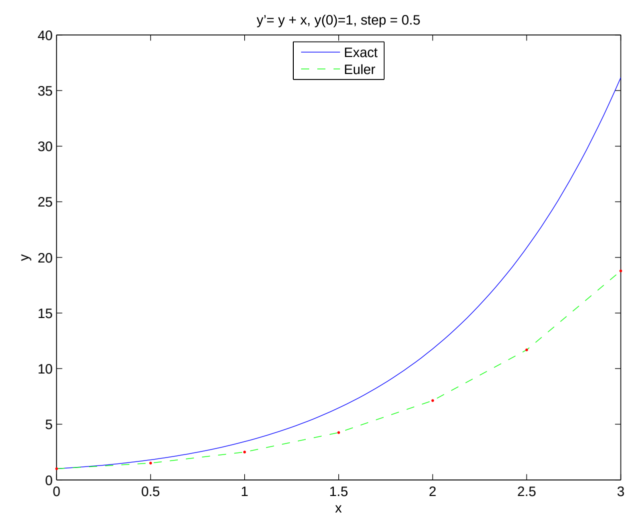
$$y_{n+1} = \ell_n(x_{n+1}) = y_n + h \cdot f(x_n, y_n).$$

- It is probably only true that (x_0, y_0) is on the true solution, however if all is well, the successive approximations are not too far away:

$$\text{error}(n) = |y(x_n) - y_n| \qquad \text{remains small.}$$

Euler's method: plot, step = 0.5

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.5$



Euler's method – algorithm

Definition (Euler's method)

Given an initial value problem on an interval $[x_0 = a, b]$

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

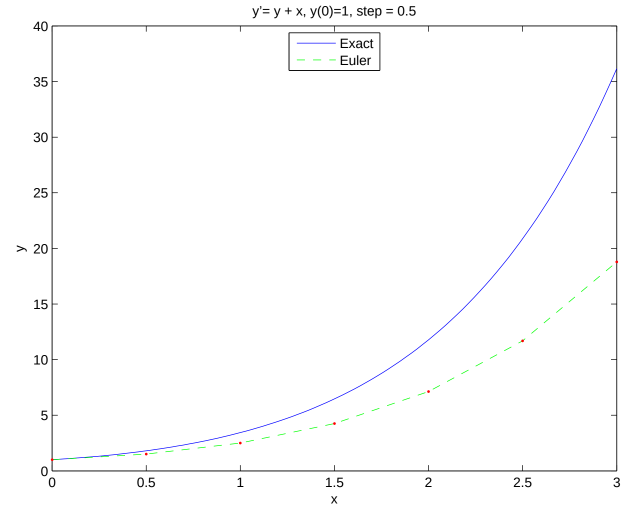
Euler's method with step size h consists in applying the iterative formula

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \qquad (n \geq 0)$$

to calculate successive approximations $y_0, y_1, y_2, \ldots$ to the true values $y(x_0), y(x_1), y(x_2), \ldots$ of the exact solution $y = y(x)$ at the points $x_0, x_1, x_2, \ldots$.

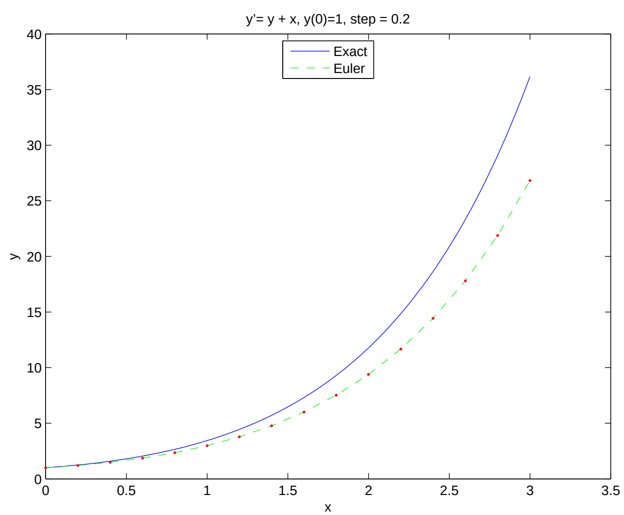
Euler's method: plot, step = 0.5

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.5$



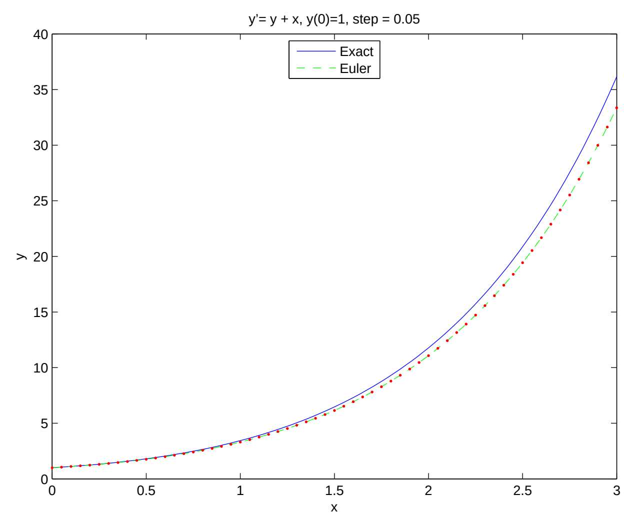
Euler's method: plot, step = 0.2

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.2$



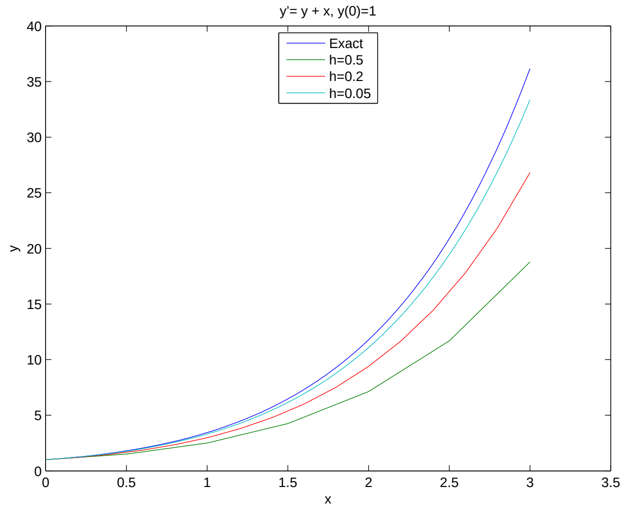
Euler's method: plot, step = 0.05

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.05$



Euler's method: plot, side-by-side

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.5, 0.2, 0.05$

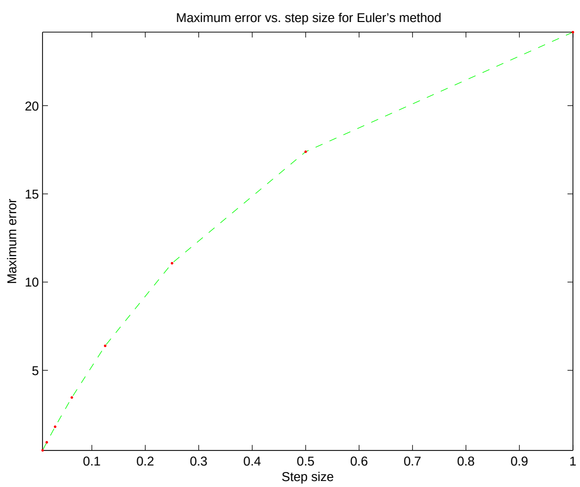


Error

- The error in Euler's method is the difference between our successive approximations $y_0, y_1, y_2, \ldots$ and the true values of the solution $y(x_0), y(x_1), y(x_2), \ldots$: $|y(x_n) - y_n|$.
- There are three sources of error:
  1. local error: Our guess of $y_{n+1} = y_n + dy$ was only approximate to begin with. In fact, for $n > 0$, each point (x_n, y_n) (for $n > 0$) lies on a different solution curve!! So, $f(x_n, y_n)$ is probably not the true value, $f(x_n, y(x_n))$.
  2. cumulative error: Errors compound. The successive values (x_n, y_n) drift away from the true values $(x_n, y(x_n))$ as $|x_n - x_0|$ grows.
  3. roundoff error: a computer is limited to the number of significant digits it can compute, which is especially pronounced for small choices of h.

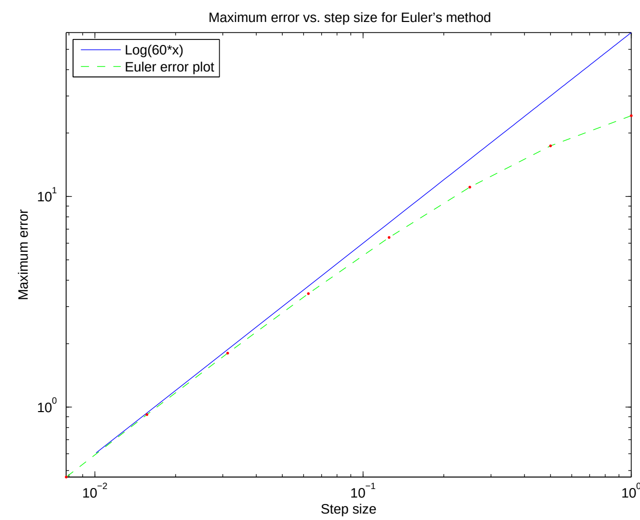
Euler's method: error

Plot of step size vs. error for $y' = y + x, y(0) = 1$.
Note. Error halves when step size is halved.



Euler's method: Logarithmic plot of error

Logarithm plot of step size vs. error for $y' = y + x, y(0) = 1$



Euler's method: Logarithmic plot of error

- The logarithmic plot suggests a linear relation between the logs of the step size and error:

$$\log(\text{error}) = A + B \cdot \log(\text{step})$$

- Raising to the power 10:

$$\text{error} = 10^A \cdot \text{step}^B$$

- The slope is $B \approx 1$ and $A \approx \log 60$, so it appears that

$$|y(x_n) - y_n| = 60 \cdot h \qquad \text{for all } n$$

  where h is the step size.

Error bound

Theorem

Suppose that f is a very nice function on some interval $[a, b]$, and that the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(a) = y_0$$

has a unique solution on this interval. Then there is a constant C such that the following holds.

> *For any sequence of approximations $y_0, y_1, y_2, \ldots$ using Euler's method with step size $h > 0$, the cumulative error is bounded:*

$$|y(x_n) - y_n| < Ch \qquad \text{for all } n$$

f is very nice means that f, f_x, f_y are all continuous.

Error bound

- The constant C depends only on the interval $[a, b]$ and f. By reducing the size of the step, h, we reduce the theoretical bound on the size of the error on each step:

$$|y(x_n) - y_n| < Ch$$

  This method is called an order one method because error is proportional to the first power of step size.
- In practice, there is a limit to how small we can make h before round-off error becomes a significant source of error.
- When f in $y' = f(x, y)$ is not very nice or fails to have a solution in the entire interval $[a, b]$ the Euler method can fail very badly. For example, $y' = x^2 + y^2$ and $y(0) = 1$ does not have a solution throughout the entire interval $[0, 1]$. See Example 5 of Section 2.4.
- Computing C is very hard, so knowing it exists is only of theoretical interest. You are still in the dark about how far off your error is.

Strategy for choosing h

- How should we choose h? Here is a strategy.
- Choose a desired number of significant digits of accuracy, within reason of what you can expect using your method of computation.
  1. Apply Euler's method using a small choice of step size h.
  2. Repeat the method, halving the step size each time: $\frac{h}{2}, \frac{h}{4}, \ldots$.
  3. Continue until you obtain successive step sizes which agree on their values, within the desired number of significant digits.
- The approximate values at the last step size will then be your approximation to the solution.

Computing e: Euclid

The number $e = y(1)$, where $y(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = y, \qquad y(0) = 1.$$

We use Euclid's method to approximate e to three decimal places, where we want agreement in two successive subintervals. The correct value is 2.71828.

Steps	Value
50	2.692
100	2.705
200	2.712
400	2.715
800	2.717
1600	2.717

Improved Euler's method

- We require 1600 steps (and a step size of .000625) just to compute three significant digits of e. This is not good.
- What is needed are numerical routines that will provide more accuracy without having to drastically reduce the step size and therefore increase the computation time, and the potential for rounding errors.
- The Runge-Kutta routines, which use predictor-corrector methods are designed provide orders of magnitude of accuracy in fewer steps.
- The simplest of these is the improved Euler method, also known as the Runge-Kutta method of order two.

Improved Euler's method

- We start with an IVP on an interval $[x_0 = a, b]$

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

  and choose some small step size h. We inductively define the approximation $(x_0, y_0), (x_1, y_1), \ldots$ as follows.
- Predict the next value (as in Euler's method):

$$u_{n+1} = y_n + h \cdot f(x_n, y_n).$$

- Correct the prediction by computing the slope at the new point, $f(x_{n+1}, u_{n+1}$ (recall, $x_{n+1} = x_n + h$);
- Redetermine the true slope by averaging the two guesses

$$m = \frac{1}{2}\left(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right)$$

- Predict the new value by using the averaged slope

$$y_{n+1} = y_n + h \cdot m.$$

Improved Euler's method – algorithm

Definition (Euler's Improved method)

Given an initial value problem on an interval $[x_0 = a, b]$

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

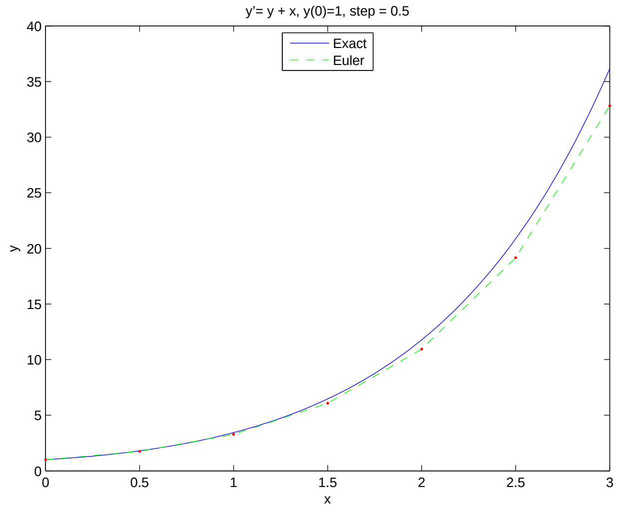
the improved Euler's method with step size h consists in applying the iterative formula

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ u_{n+1} &= y_n + h \cdot k_1 \\ k_2 &= f(x_{n+1}, u_{n+1}) \\ y_{n+1} &= y_n + h \cdot \frac{k_1 + k_2}{2} \end{aligned}$$

to calculate successive approximations $y_0, y_1, y_2, \ldots$ to the true values $y(x_0), y(x_1), y(x_2), \ldots$ of the exact solution $y = y(x)$ at the points $x_0, x_1, x_2, \ldots$.

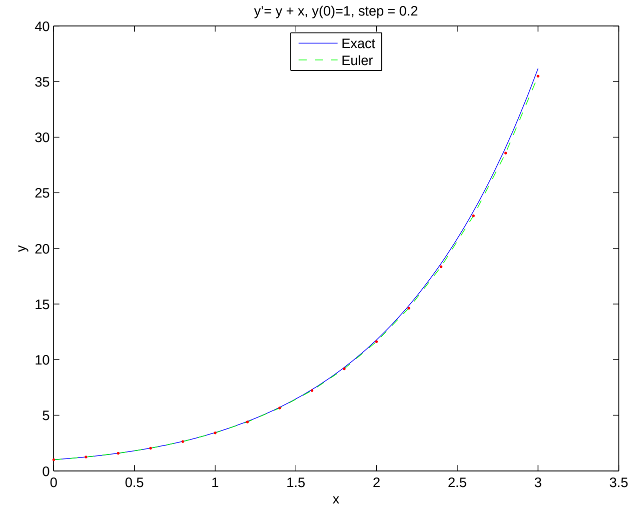
Improved Euler's method: plot, step = 0.5

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.5$



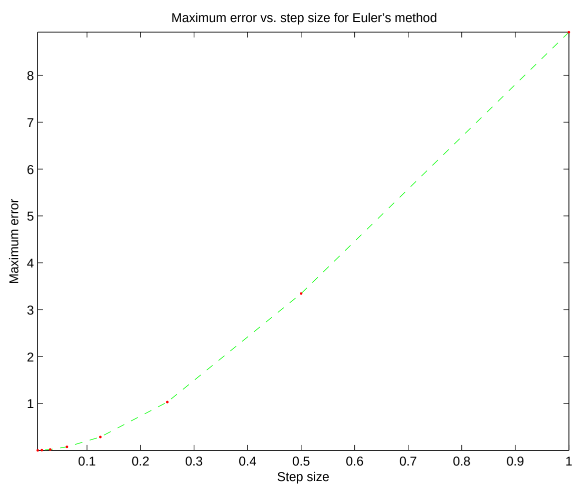
Improved Euler's method: plot, step = 0.2

$y' = y + x, \qquad y(0) = 1 \qquad h = 0.2$



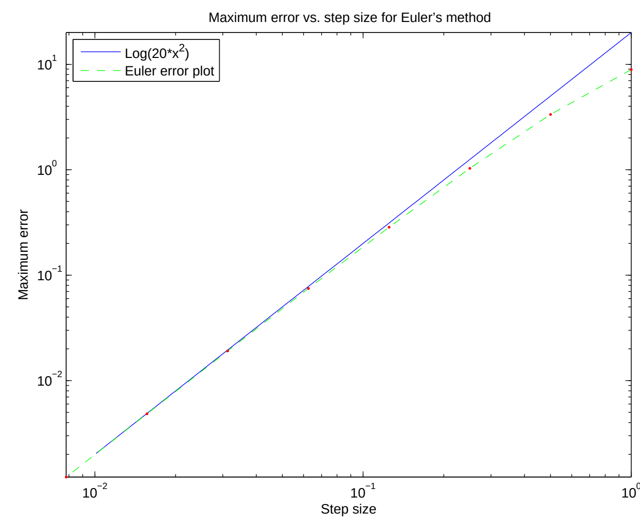
Improved Euler's method: error

Plot of step size vs. error for $y' = y + x, y(0) = 1$.
Note. Error is quartered when step size is halved.



Improved Euler's method: Logarithmic plot of error

Logarithm plot of step size vs. error for $y' = y + x, y(0) = 1$



Improved Euler's method: Logarithmic plot of error

- The logarithmic plot suggests a linear relation between the logs of the step size and error:

$$\log(\text{error}) = A + B \cdot \log(\text{step})$$

- Raising to the power 10:

$$\text{error} = 10^A \cdot \text{step}^B$$

- The slope is $B \approx 2$ and $A \approx \log 20$, so it appears that

$$|y(x_n) - y_n| = 20 \cdot h^2 \qquad \text{for all } n$$

where h is the step size and the error at step n is

$$\text{error}(n) = |y(x_n) - y_n|.$$

Error bound

Theorem

Suppose that f is a very, very nice function on some interval $[a, b]$, and that the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(a) = y_0$$

has a unique solution on this interval. Then there is a constant C such that the following holds.

For any sequence of approximations $y_0, y_1, y_2, \ldots$ using Euler's method with step size $h > 0$, the cumulative error is bounded:

$$|y(x_n) - y_n| < Ch^2 \qquad \text{for all } n$$

f is very, very nice means that f, f_x, f_y are all continuous, and the solution y has a continuous third derivative.

Computational cost of Improved Euler

- The improved Euler method does provide improved accuracy, but it does it do so at a greater computational cost? Each improved Euler step requires two computations of $f(x, y)$ for each such computation in the Euler method.
- However, the improved Euler method provides better accuracy with significantly fewer steps required: doubling the number of steps decreases the error fourfold (as opposed to a twofold decrease with Euler's method.
- The reason is that there is a quadratic relation between step size h and error:

$$\text{error}(n) = |y(x_n) - y_n| \leq Ch^2$$

where the constant C depends only on the step size.

Computing e: Euclid

The number $e = y(1)$, where $y(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = y, \qquad y(0) = 1.$$

We use Euclid's method to approximate e to three decimal places, where we want agreement in two successive subintervals. The correct value is 2.71828.

Euler Steps	Euler Value	Improved Euler Steps	Improved Euler Value
50	2.692	10	2.71408
100	2.705	20	2.71719
200	2.712	40	2.71800
400	2.715	80	2.71821
800	2.717	160	2.71826
1600	2.717	320	2.71828
		640	2.71828