Math 216
Differential Equations

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ConcepTest

Problem. Find all solutions to the equation

$$xy' + (2x - 3)y = 4x^4.$$

Answer. This is a linear first-order equation, so we can use the method of integrating factors. The equation, when written in proper form, is

$$y' + \frac{2x-3}{x}y = 4x^3.$$

ConcepTest – solution

$$y' + \frac{2x-3}{x}y = 4x^3.$$

The integrating factor is

$$\begin{aligned}\rho(x) &= \exp\Big(\int \frac{2x-3}{x}\,dx\Big)\\ &= \exp(2x - 3\ln x) = e^{2x}x^{-3}\end{aligned}$$

Multiplying the equation by $\rho(x)$

$$D_x\left(e^{2x}x^{-3}y\right) = e^{2x}x^{-3}(4x^3) = 4e^{2x}$$

Integrating

$$y = x^3e^{-2x}\Big(\int 4e^{2x}\,dx + C\Big) = 2x^3 + Cx^3e^{-2x}.$$

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Solve the IVP

$$2x\frac{dy}{dx} = y + 2x\cos x, \qquad y(1) = 0.$$

Recall, that the solution for a linear first-order equation and initial value of the form

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(x_0) = y_0.$$

is given by

$$\begin{aligned}\rho(x) &= \exp\Big(\int_{x_0}^{x} P(t)\,dt\Big)\\ y(x) &= \frac{1}{\rho(x)}\Big[y_0 + \int_{x_0}^{x}\rho(t)Q(t)\,dt\Big].\end{aligned}$$

ConcepTest – solution

The solution to

$$2x\frac{dy}{dx} = y + 2x\cos x, \qquad y(1) = 0.$$

is

$$y(x) = x^{\frac{1}{2}} \int_1^x t^{-\frac{1}{2}} \cos t \, dt$$

(the integral cannot be solved using elementary functions – see Lecture 4.)

The integrating fact was

$$\rho(x) = \exp(\int_0^x -\frac{1}{2t}\,dt) = \exp(-\frac{1}{2}\ln x) = x^{-\frac{1}{2}}.$$

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Problem. The time rate of change of a population P is proportional to the square root of P. Initially, the population is 400 and the rate of increase is 80. What is the population at time t=10?

ConcepTest solution

1. The equation modeling the population is

   $$\frac{dP}{dt} = k\sqrt{P}, \qquad P'(0) = 80, P(0) = 400$$

   where k is the constant of proportionality. Solving for k from the initial data, $k = 4$.
2. Use separation of variables to solve this equation:

   $$P(t) = (2t + C)^2$$

   Since $P(0) = 400$ we have $C = 20$.
3. Our population equation is

   $$P(t) = (2t + 20)^2$$

   At time $t = 10$ the population is

   $$P(10) = (2(10) + 20)^2 = 1600 \text{ souls.}$$

Falling body problems

Problem. We will be considering a falling body of mass m through the Earth's atmosphere. We will be considering only two forces, gravity (F_G) and air resistance (F_A).

We need to pay attention to the direction the forces are acting.
Convention. Motion will only be up (away from the surface of the Earth) and down. Our convention is as follows:

Up is **positive**;

Down is **negative**.

The forces in the problem

- **Gravity**. The force of gravity F_G acts downward on objects with a force
$$F_G = -mg$$
where m is the mass, and $g \approx 32 \frac{\text{ft}}{\text{sec}^2}$ (this is the constant of gravitational attraction, as measured on the Eath's surface.)
- **Air resistance**. Air resistance F_A will be assumed to be proportional to the velocity, but always applied in the opposite direction. Thus,
$$F_A = -kv$$
where k is the constant of proportionality and v is the object's velocity.
- **Note**. In a falling body problem (with our convention), the velocity v will be negative, so that $F_R = -kv$ is positive (so acts upward).

Newton's second law of motion

- Newton's second law of motion says
$$m\frac{dv}{dt} = F(t, v)$$
where $F(t, v)$ is the sum total of all forces.
- Applying Newton's law to the forces of gravity and air resistance,
$$m\frac{dv}{dt} = F_G + F_R = -mg - kv$$
- Dividing by mass m, our IVP becomes
$$\frac{dv}{dt} = -g - \rho v, \qquad v(0) = v_0$$
where $\rho = \frac{k}{m} > 0$ (the drag coefficient) and $v(0) = v_0$ is the initial velocity of the body.

Terminal velocity

- The solution for our falling body problem with initial velocity v_0 is
$$v(t) = \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}.$$
- A falling body has a limiting speed,
$$\lim_{t\to\infty} v(t) = -\frac{g}{\rho}.$$
(the velocity is negative since the body is falling.)
- Thus, a falling body does not increase indefinitely, but approaches a **finite** limiting speed, the terminal speed,
$$|v_\tau| = \frac{g}{\rho} = \frac{mg}{k}.$$
where $v_\tau = -\frac{g}{\rho}$ (note the direction is down).

Computing height

- To compute the height function of the falling body, we let the ground be at 0 feet, so that the body starts at x_0. Thus, we have an IVP
$$v(t) = \frac{dx}{dt} = \left(v_0 - v_\tau\right)e^{-\rho t} + v_\tau, \qquad x(0) = x_0.$$
(I replaced $\frac{g}{t}$ by v_τ – watch for the sign!)
- The solution of this IVP
$$x(t) = -\frac{1}{\rho}(v_0 - v_\tau)e^{-\rho t} + v_\tau t + \left(x_0 + \frac{1}{\rho}(v_0 - v_\tau)\right).$$
Gathering terms
$$x(t) = x_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t}).$$

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Problem. A parachutist bails out of an airplane at 10,000 feet, falls freely for 20 seconds, then opens her parachute.

Assumptions. Air resistance is proportional to speed. Assume for the drag coefficient, $\rho = 0.15$ without a parachute and $\rho = 1.5$ with a parachute.

Question. Find an IVP (equation and initial value) which models the problem (i) during the period of free fall, and (ii) after the parachute opens.

Answer.

$$\begin{aligned} (i)\quad \frac{dv}{dt} &= -g - 0.15v, \qquad v(0) = 0 \\ (ii)\quad \frac{dv}{dt} &= -g - 1.5v, \qquad v(20) = v_{20} \end{aligned}$$

Two problems

We have two different problems here,

1. Free fall for 20 seconds.
2. Parachute for the remainder of the time in the air.

They have overlapping data, but involve different equations (once we input the data).

Here are the general equations.

$$\begin{aligned} v(t) &= (v_0 - v_\tau)e^{-\rho t} + v_\tau \\ x(t) &= -\frac{1}{\rho}(v_0 - v_\tau)e^{-\rho t} + v_\tau t + \left(x_0 + \frac{1}{\rho}(v_0 - v_\tau)\right). \end{aligned}$$

For problem two, we need to solve for t knowing that $x(t) = 0$. We will need its initial values, $v(20)$ and $x(20)$, after freefalling 20 seconds. This must be determined from problem one.

Problem 1: free fall

$$\begin{aligned} v(t) &= (v_0 - v_\tau)e^{-\rho t} + v_\tau \\ x(t) &= -\frac{1}{\rho}(v_0 - v_\tau)e^{-\rho t} + v_\tau t + \left(x_0 + \frac{1}{\rho}(v_0 - v_\tau)\right). \end{aligned}$$

- We must determine $v(20)$ and $x(20)$. Our data is as follows:
  - $\rho = 0.15$, $v_\tau = -\frac{g}{\rho} = -\frac{32}{0.15} = -213.3$.
  - $v(0) = 0$, $x(0) = 10000$.
- So,
  $$v(t) = 213.3e^{-0.15t} - 213.3$$
  and $v(20) = -202.68\frac{\text{ft}}{\text{sec}}$;
- And
  $$x(t) = 10000 - 213.3t + \frac{1}{0.15}(213.3)(1 - e^{-0.15t})$$
  and $x(20) = 7084.7$ feet .

Problem 2: parachute

For the rest of the computation, the parachutist has opened her parachute. We have turned back time to $t = 0$ (since this was assumed in deriving the general equation).

- The initial data
  - $\rho = 1.5$, $v_\tau = -\frac{g}{\rho} = -\frac{32}{1.5} = -21.3$.
  - $v(0) = -202.68$, $x(0) = 7084.7$ (as determined by Problem 1).
- We want to solve for t in the equation
  $$x(t) = 0 = x_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t}).$$
- After plugging in values, we get the equation
  $$6964.08 - 21.3t - 120.92e^{-1.5t}.$$
  I solved this using the Mathematica function **Solve**, and got $t \approx 327$ seconds. Together with the additional 20 seconds of freefall,

is the time elapsed before touch down.

Newton's law of gravitation

- We have been assuming the force of gravity was constant: $F_G = -mg$. This is approximately true.
- Newton's law of gravitation states that the gravitational force of attraction between two bodies of masses M and m located at a distance r apart (as measured by their centers) is given by

$$F_G = \frac{GMm}{r^2},$$

  where $G \approx 6.6726 \times 10^{-11} \mathrm{N} \cdot (\frac{\mathrm{m}}{\mathrm{kg}})^2$
- For a body of mass m near the surface of the earth, this reduces to mg.

Escape Velocity

- What is the initial velocity needed for a projectile fired from the surface of a planet to escape from the planet altogether?
- Assume the planet has mass M and radius R, and the mass of the projectile is m.
- The only force we will consider is the force of gravity, F_G, which is acting downward (by our convention, negative). By Newton's law of gravitation:

$$F_G = -\frac{GMm}{r^2}$$

- From Newton's second law,

$$m\frac{dv}{dt} = F_G = -\frac{GMm}{r^2}.$$

  where $r = r(t)$ is the changing distance of the projectile from the center of the planet.

Escape Velocity

- We have the following initial value problem (dividing by m):

$$\frac{dv}{dt} = -\frac{GM}{r^2}, \qquad v(0) = v_0, r(0) = R$$

  where v_0 is the unknown escape velocity.
- We cannot solve this equation until we express $\frac{dv}{dt}$ in terms of the changing distance r, not the changing time t. Use the chain rule,

$$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = v\frac{dv}{dr}$$

  (Recall, $v = \frac{dr}{dt}$).
- Our new initial value problem is

$$v\frac{dv}{dr} = -\frac{GM}{r^2}, \qquad v(0) = v_0, r(0) = R$$

  where v_0 is the unknown escape velocity.

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Solve the IVP,

$$v\frac{dv}{dr} = -\frac{GM}{r^2}, \qquad v(0) = v_0, r(0) = R.$$

Answer. Separate the variables and integrate both sides:

$$\int v\, dv = \int -\frac{GM}{r^2}\, dr.$$

So,

$$\frac{v^2}{2} = \frac{GM}{r} + C$$

At $t = 0$ we have $v(0) = v_0$ and $r(0) = R$, so

$$C = \frac{v_0^2}{2} - \frac{GM}{R}$$

Projectile equation

The solution of the IVP

$$v\frac{dv}{dr} = -\frac{GM}{r^2}, \qquad v(0) = v_0, r(0) = R.$$

is

$$\frac{v^2}{2} = \frac{GM}{r} + \frac{v_0^2}{2} - \frac{GM}{R}$$

The velocity $v = v(t)$ will remain positive provided $\frac{v_0^2}{2} - \frac{GM}{R} \geq 0$. So, the escape velocity from the planet is

$$v_0 = \sqrt{\frac{2GM}{R}},$$

where M is the planet's mass and R is the planet's radius.

Some escape velocities

Escape velocity:

$$v_0 = \sqrt{\frac{2GM}{R}},$$

where $G \approx 6.6726 \times 10^{-11} \mathrm{N} \cdot (\frac{\mathrm{m}}{\mathrm{kg}})^2$.

- Earth: $M = 5.975 \times 10^{24}$kg and $R = 6.378 \times 10^6$m

$$V_{\text{earth}} \approx 11,800\frac{\mathrm{m}}{\mathrm{s}}, \quad \text{or } 25,000 \text{ mph}$$

- Moon: $M = 7.35 \times 10^{22}$kg and $R = 1.74 \times 10^6$m

$$V_{\text{moon}} \approx 2375\frac{\mathrm{m}}{\mathrm{s}}, \quad \text{or } 5000 \text{ mph}$$

- Sun: $M = 1.9718 \times 10^{30}$kg and $R = 6.952 \times 10^8$m

$$V_{\text{sun}} \approx 617,000\frac{\mathrm{m}}{\mathrm{s}}, \quad \text{or } 1,300,000 \text{ mph}$$

Black holes I

Question. How small would the Earth's radius have to shrink for it to become a black hole: that is, to where its escape velocity is $3 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}}$?

We want to solve for R in the equation

$$3 \times 10^8 = \sqrt{\frac{2GM}{R}},$$

where $M = 5.975 \times 10^{24}$kg and $G \approx 6.6726 \times 10^{-11} \mathrm{N} \cdot (\frac{\mathrm{m}}{\mathrm{kg}})^2$.

So,

$$R = \frac{2GM}{9 \times 10^{16}}$$

Answer. 0.88 cm.

Black holes II

Question. The sun is 330,000 times more massive and has a radius 109 times that of the earth. How small would the Sun's radius have to shrink for it to become a black hole: that is, to where its escape velocity is $3 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}}$?

We want to solve for R (M is the Earth's mass)

$$3 \times 10^8 = \sqrt{\frac{2G(333000M)}{R}}.$$

We solved $R_{\text{Earth}} = 0.88$cm on the previous slide,

$$\begin{aligned} R &= (333000)\frac{2GM}{9 \times 10^{16}} \\ &= 330000 R_{\text{Earth}}. \end{aligned}$$

Answer. 2.9 km.

Maximal height

Problem. Suppose the projectile's initial velocity v_0 is less than the escape velocity. What is the maximum distance from the planet's center that the projectile travels?

For what value of r is $v = 0$ in the equation:

$$\frac{v^2}{2} = \frac{GM}{r} + \frac{v_0^2}{2} - \frac{GM}{R}$$

given the assumption that $v_0^2 - \frac{GM}{R} < 0$?

Solving for r

$$r_{\max} = \frac{2GMR}{2GM - Rv_0^2}$$

Example

Problem. Suppose a projectile is launched at 50% of the escape velocity of the Earth. What is the maximum height achieved?

Answer. The projectile's speed is $v_0 = 0.5\sqrt{\frac{2GM}{R}}$, so plugging into our formula

$$r_{\max} = \frac{2GMR}{2GM - Rv_0^2} = \frac{2GMR}{0.75(2GM)} = R(\frac{1}{0.75}).$$

Since $R = 6.378 \times 10^6$m,

$$r_{\max} = 8500 \text{ km}.$$