Math 216
Differential Equations

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

September 12, 2008

Interchanging variables

Consider a differential equation with dependent variable $y = y(x)$

$$\frac{dy}{dx} = F(x, y).$$

It might be easier to solve the “inverse problem":

$$\frac{dx}{dy} = \frac{1}{F(x, y)}.$$

where the dependent variable is now $x = x(y)$.

The justification is the relation

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

This is a form of the Inverse Function Theorem.

ConcepTest

Problem. Solve the following equation for $x = x(y)$:

$$(x + ye^y)\frac{dy}{dx} = 1.$$

Answer.

$$x = (\frac{y^2}{2} + C)e^y.$$

Qualitative condition for stability

Theorem

Suppose f is nice, and that c is a critical point of the autonomous equation

$$\frac{dx}{dy} = f(x).$$

If there is a $\delta > 0$ such that

(i) *$f(x_0) > 0$ whenever $c - \delta < x_0 < c$, and*
(ii) *$f(x_0) < 0$ whenever $c < x_0 < c + \delta$;*

then c is stable. (In fact all solutions $x(0) = x_0$ where $|c - x_0| < \delta$ will converge to c in the limit: $x(t) \to c$ as $t \to \infty$.)

Parameter values

We will analyze the qualitative behavior of the equation

$$\frac{dx}{dt} = \mu x - x^3$$

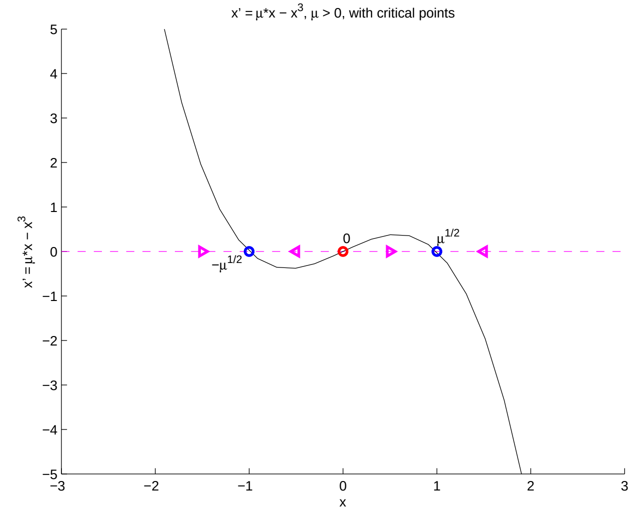
as the parameter μ is varied

0, and $\sqrt{\mu}$ and $-\sqrt{\mu}$ are roots when $\mu > 0$; only 0 is a root when $\mu \leq 0$. are roots for the right-hand side, $f(x) = \mu x - x^2$.

So, this equation has three critical points when $\mu >$ and one critical point when $\mu \leq 0$.

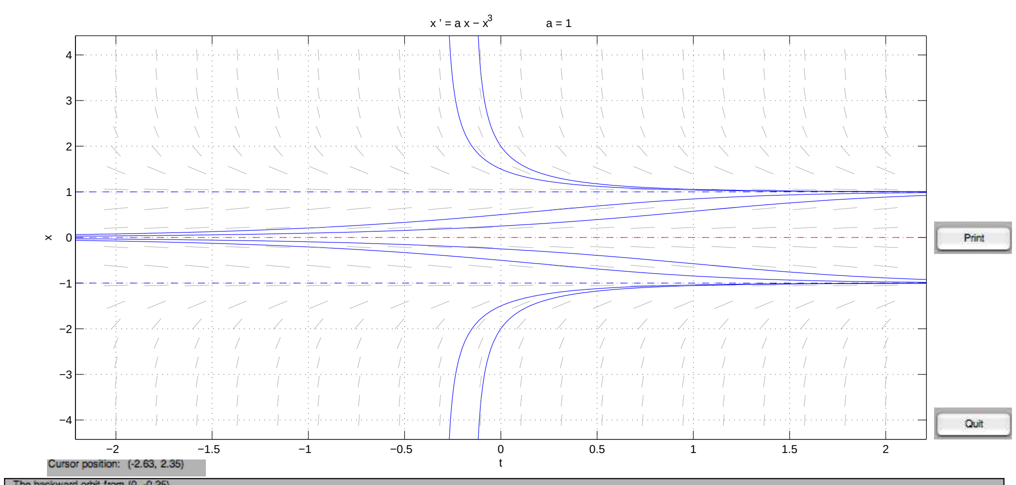
Phase plot for $\mu > 0$

Plot of $f(x) = \mu x - x^3$ when $\mu > 0$.



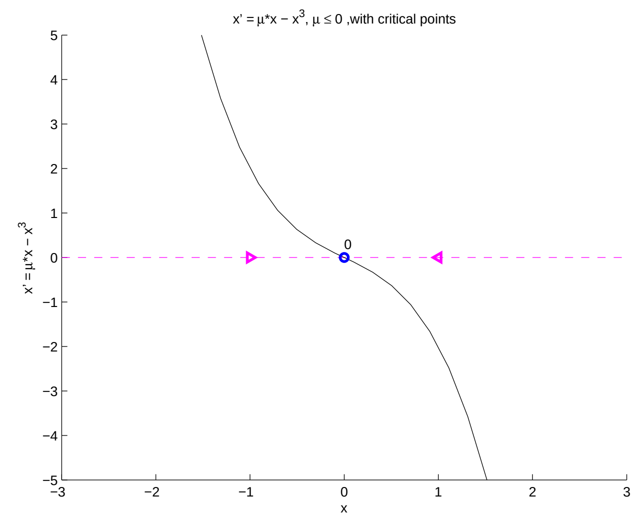
Difference field for $\mu = 1$

The solution field for $y' = x - x^3$ ($\mu = 1$). Note how the solutions confirm our qualitative analysis that ± 1 are stable, and 0 is unstable.



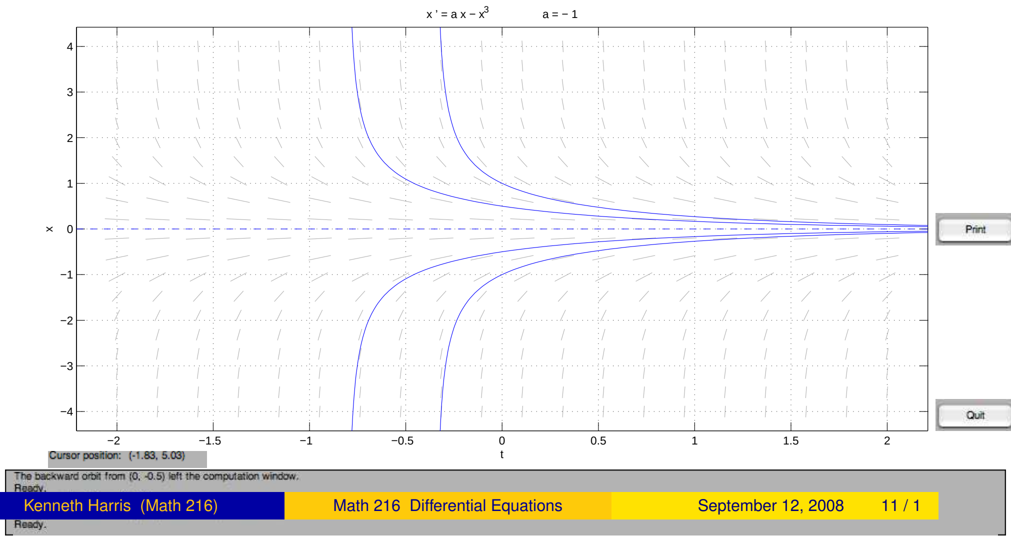
Phase plot for $\mu \leq 0$

Plot of $f(x) = \mu x - x^3$ when $\mu \leq 0$.



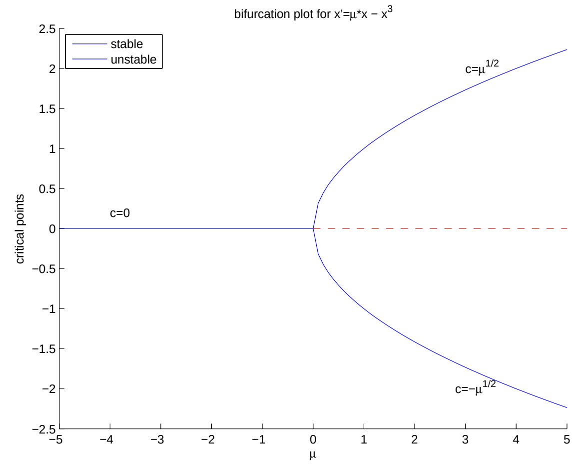
Difference field for $\mu = -1$

The solution field for $y' = -x - x^3$ ($\mu = -1$). Note that there is only one critical point, 0, and it is stable.



Bifurcation diagram

Bifurcation diagram for $x' = \mu x - x^3$. Here, $\mu = 0$ is called a (transcritical) bifurcation point – 0 changes its stability property to $\sqrt{\mu}$ and $-\sqrt{\mu}$. This is called a pitchfork diagram.



Parameter values

Now, lets analyze the qualitative behavior of the logistic equation,

$$\frac{dx}{dt} = \mu x - x^2$$

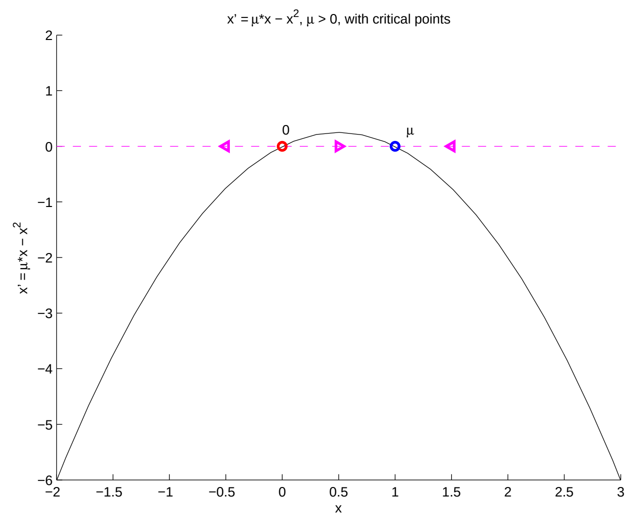
as the parameter μ is varied

0 and μ are roots for the right-hand side, $f(x) = \mu x - x^2$.

So, the logistic equation has two critical points (provided $\mu \neq 0$.

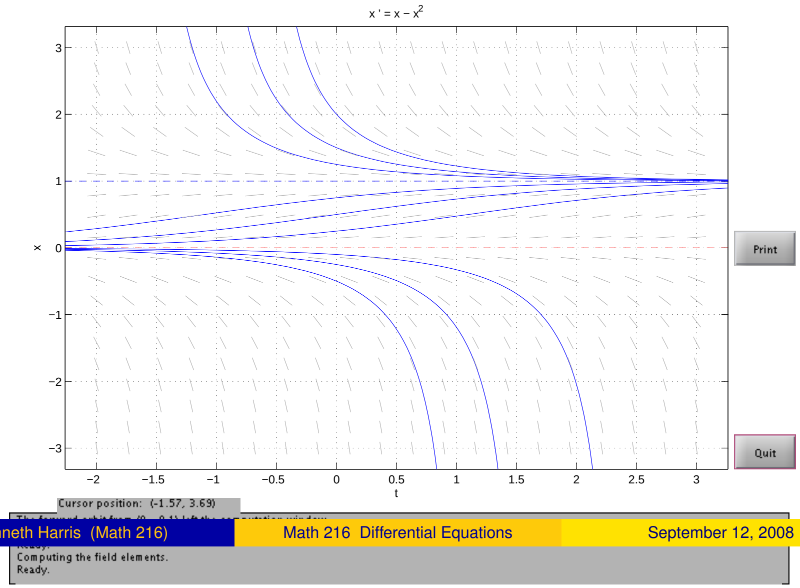
Phase plot for $\mu > 0$

Plot of $f(x) = \mu x - x^2$ when $\mu > 0$.



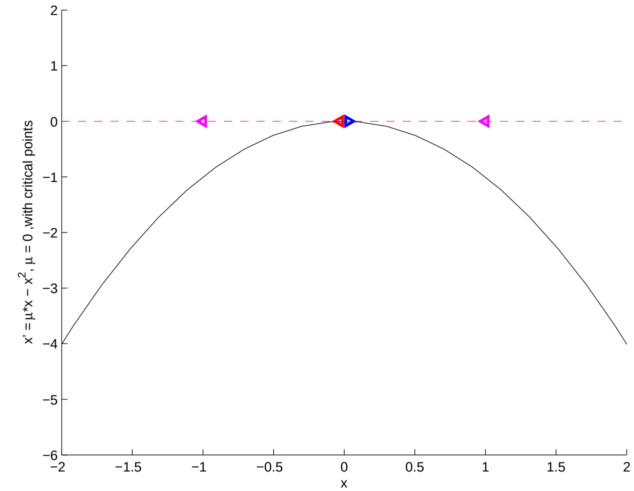
Direction field and solution plot

Solutions for $\frac{dx}{dy} = x(1-x)$. The stable critical point is 1, the unstable critical point is 0.



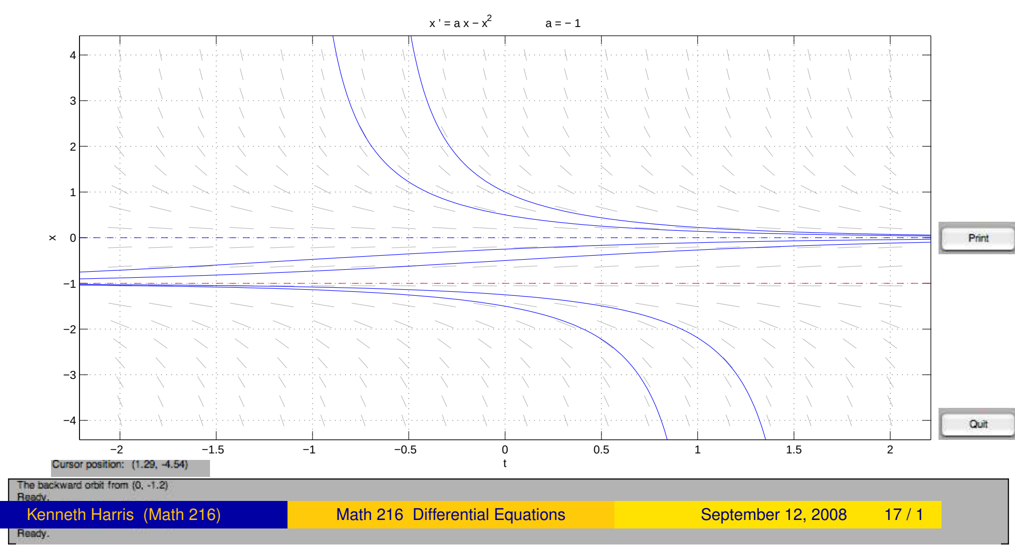
Phase plot for $\mu = 0$

Plot of $f(x) = \mu x - x^2$ when $\mu = 0$, a semistable critical point.



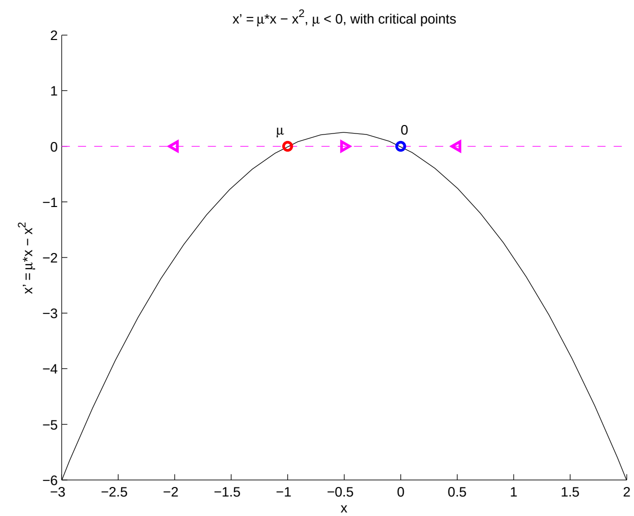
Difference field for $\mu = 0$

The solution field for $y' = - - x^2$ ($\mu = 0$). Note that 0 is is the only critical point, and is semistable.



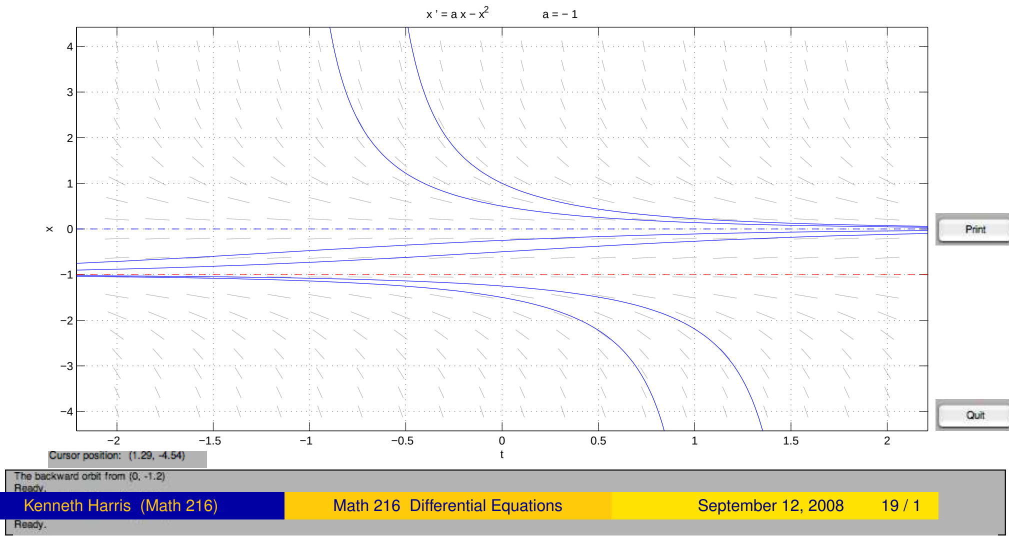
Phase plot for $\mu < 0$

Plot of $f(x) = \mu x - x^2$ when $\mu > 0$.



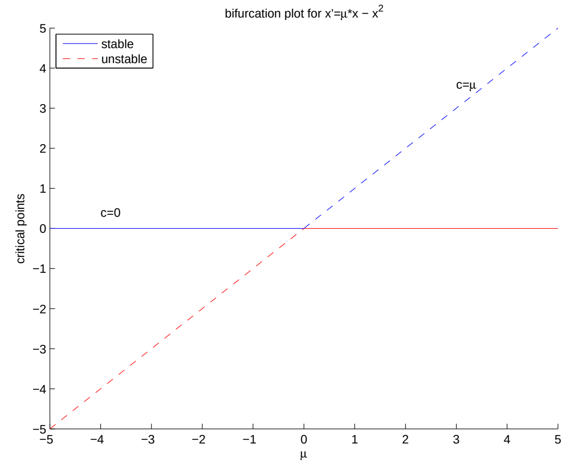
Difference field for $\mu = -1$

The solution field for $y' = -x - x^2$ ($\mu = -1$). Note that 0 is stable and -1 is unstable.



Bifurcation diagram

Bifurcation diagram for $x' = \mu x - x^2$. Here, $\mu = 0$ is called a (transcritical) bifurcation point – 0 passes its stability property to μ – as μ moves from negative to positive.



Logistic equation

We have seen the logistic equation in the form

$$\frac{dx}{dt} = ax - bx^2$$

where $b > 0$ and a is arbitrary. In this case $\mu = \frac{a}{b}$.

- When $a > 0$ the critical point $\frac{a}{b}$ is stable, and 0 is unstable.
- When $a = 0$ there is one critical point, 0, and it is semistable.
- When $a < 0$ the critical point $\frac{a}{b}$ is unstable, and 0 is stable.

Lasers and the laser threshold

- A laser is a device that emits light through stimulated emission.
- Light Amplification by Stimulated Emission of Radiation.
- Laser light (unlike light produced by a lightbulb) has three properties
  - It contains one specific wavelength (frequency).
  - It is in phase (each photon in the light moves in step with each other).
  - It is directional – it has a strong tight beam.

The set-up

- The laser consists of a medium of atoms inside a highly reflective cavity.
- The cavity has two mirrors on either end for reflecting photons back through the medium, to ensure the light makes many passes through the medium.
- One of the mirrors is half-silvered, so it lets some light pass through – which will be the laser light.
- Power is supplied to the medium – a process called pumping.

How it works

- A stimulated atom is one with an electron in a high energy state.
- When the electron falls back to its ground state it emits a photon. The frequency and phase of the photon depends only upon the energy state of the electron.
- When a photon encounters an atom with an electron in the same excited state it leads to stimulated emission of a photon from the atom.
- In stimulated emission a new photon is produced by the atom (causing the electron to fall back to its ground state). The new photon has the same frequency, phase and direction as the old photon.
- Most importantly, the old photon is not absorbed by the atom. There are now two photons with the same frequency, phase and direction.

Lasers and the laser threshold

- The pump creates a certain number of excited atoms. The excited atoms produce photons.
- Some photons are lost through the cavity.
- As the pump strength is increased the system goes through a period in which the excited atoms oscillate independently of each other, emitting randomly phased light (as in a light bulb).
- When the pumping exceeds the laser threshold, a beam of radiation that is more coherent and intense is produced. The beam is sent through the medium many times, intensifying its strength as more photons of the same frequency, phase and direction are added.
- A laser is produced.

Determining the equation

Let $n(t)$ be the number of photons and $N(t)$ be the number of excited atoms in the system. Then,

$$\frac{dn}{dt} = \text{Gain} - \text{Loss}$$

where

- Gain occurs when photons stimulate excited atoms to emit more photons. So, Gain is proportional to the product $n(t)N(t)$ (the chance of a random interaction).
- Loss occurs when photons escape the system, and is proportional to the number of photons.

Thus,

$$\frac{dn}{dt} = \text{Gain} - \text{Loss} = GnN - Ln$$

where $G, L > 0$.

Determining the number of excited atoms

We want to express $N(t)$ using $n(t)$.

- When an excited atom drops back to its ground state it emits a photon.
- The job of the external pump is to keep the number of excited atoms at a constant level, N_0 before the laser is produced.
- The actual number of excited atoms is determine by

$$N(t) = N_0 - \alpha n(t)$$

where $\alpha > 0$ is the rate at which the atoms drop back to their ground state (emitting a photon).

Our equation can be finalized,

$$\frac{dn}{dt} = \text{Gain} - \text{Loss} = GnN - Ln = (GN_0 - L)n - \alpha G n^2.$$

The equation

The differential equation

$$\frac{dn}{dt} = (GN_0 - L)n - \alpha G n^2.$$

has the form of the logistic equation,

$$\frac{dn}{dt} = an - bn^2$$

where

- $a = (GN_0 - L)$
- $b = \alpha G$

The scientist can control the value of the parameter N_0 using the external pump.

Determining the number of excited atoms

The differential equation

$$\frac{dn}{dt} = an - bn^2$$

has two critical points, 0 and $\frac{a}{b}$. Their qualitative properties is as follows:

- When $a = (GN_0 - L) < 0$, that is $N_0 < \frac{L}{G}$, the only stable critical point is 0, and the tendancy of the system is to fall back to no photons. No laser is produced.
- When $a = (GN_0 - L) \geq 0$, that is $N_0 \geq \frac{L}{G}$, 0 stops being stable, and passes this on to the stable critical point $\frac{a}{b} = \frac{GN_0 - L}{\alpha G} > 0$. The number of photons stabilizes at this value. A laser is produced.

Falling body problems

Problem. We will be considering a falling body of mass m through the Earth's atmosphere. We will be considering only two forces, gravity (F_G) and air resistance (F_A).

We need to pay attention to the direction the forces are acting. **Convention**. Motion will only be up (away from the surface of the Earth) and down. Our convention is as follows:

Up is **positive**;

down is **negative**.

The forces in the problem

Gravity. The force of gravity F_G acts downward on objects with a force

$$F_G = -mg$$

where m is the mass, and $g \approx 32\frac{\text{ft}}{\text{sec}^2}$ (this is the constant of gravitational attraction, as measured on the Eath's surface.)

Air resistance. Air resistance F_A will be assumed to be proportional to the velocity, but always applied in the opposite direction. Thus,

$$F_A = -kv$$

where k is the constant of proportionality and v is the object's velocity. **Note**. In a falling body problem (with our convention), the velocity v will be negative, so that $F_R = -kv$ is positive (so acts upward).

Newton's second law of motion

Newton's second law of motion says

$$m\frac{dv}{dt} = F(t, v)$$

where $F(t, v)$ is the sum total of all forces.

Applying Newton's law to the forces of gravity and air resistance,

$$m\frac{dv}{dt} = F_G + F_R = -mg - kv$$

Dividing by mass m, our IVP becomes

$$\frac{dv}{dt} = -g - \rho v, \qquad v(0) = v_0$$

where $\rho = \frac{k}{m} > 0$ (the drag coefficient) and $v(0) = v_0$ is the initial velocity of the body.

ConcepTest

Solve the IVP

$$\frac{dv}{dt} = -g - \rho v, \qquad v(0) = v_0$$

Answer. Use separation of variables,

$$v(t) = (v_0 + \frac{g}{\rho})e^{-\rho t} - \frac{g}{\rho}.$$

Terminal velocity

The solution for our falling body problem with initial velocity v_0 is

$$v(t) = (v_0 + \frac{g}{\rho})e^{-\rho t} - \frac{g}{\rho}.$$

Notice that

$$v(t) \to -\frac{g}{\rho} \quad \text{as } t \to \infty.$$

(body is falling, so the velocity is negative!)

Thus, a falling body does not increase indefinitely, but approaches a **finite** limiting speed, the terminal speed,

$$|v_\tau| = \frac{g}{\rho} = \frac{mg}{k}.$$

where $v_\tau = -\frac{g}{\rho}$ (note the direction is down).

Computing height

Let's compute the height function for a falling body. The ground is 0 feet, so the body starts at height x_0. Thus, we have an IVP

$$v(t) = \frac{dx}{dt} = (v_0 - v_\tau)e^{-\rho t} + v_\tau, \qquad x(0) = x_0.$$

(I replaced $\frac{g}{t}$ by v_τ – watch for the sign!)

Solve this IVT.
Answer.

$$x(t) = -\frac{1}{\rho}(v_0 - v_\tau)e^{-\rho t} + v_\tau t + \left(x_0 + \frac{1}{\rho}(v_0 - v_\tau)\right).$$

Gathering terms

$$x(t) = x_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t}).$$

Example

Problem. A parachutist bails out of an airplane at 10,000 feet, falls freely for 20 seconds, then opens her parachute.

Assumptions. Air resistance is proportional to speed. Assume for the drag coefficient, $\rho = 0.15$ without a parachute and $\rho = 1.5$ with a parachute.

Question. Find an IVP (equation and initial value) which models the problem (i) during the period of free fall, and (ii) after the parachute opens.

Answer.

$$\begin{array}{lcll} (i) \ \frac{dv}{dt} & = & -g - 0.15v, & v(0) = 0 \\ (i) \ \frac{dv}{dt} & = & -g - 1.5v, & v(20) = v_{20} \end{array}$$

Two problems

We have two different problems here,

1. Free fall for 20 seconds.
2. Parachute for the remainder of the time in the air.

They have overlapping data, but involve different equations (once we input the data).

Here are the general equations.

$$\begin{array}{rcl} v(t) & = & (v_0 - v_\tau)e^{-\rho t} + v_\tau \\ x(t) & = & -\frac{1}{\rho}(v_0 - v_\tau)e^{-\rho t} + v_\tau t + \left(x_0 + \frac{1}{\rho}(v_0 - v_\tau)\right). \end{array}$$

For problem two, the initial values needed are $v(20)$ and $x(20)$ after freefalling 20 seconds. This must be determined from problem one.

Problem 1: free fall

We must determine $v(20)$ and $x(20)$. Our data is as follows:

$\rho = 0.15$, $v_\tau = -\frac{g}{\rho} = -\frac{32}{0.15} = -213.3$.
$v(0) = 0$ and $x(0) = 10000$.

So,

$$v(t) = 213.3e^{-0.15t} - 213.3$$

and $v(20) = -202.68\frac{\text{ft}}{\text{sec}}$.

$$x(t) = 10000 - 213.3t + \frac{1}{0.15}(213.3)(1 - e^{-0.15t})$$

and $x(20) = 7084.7$ feet .

Problem 2: parachute

For the rest of the computation, the parachutist has opened her parachute. We have turned back time to $t = 0$ (since this was assumed in deriving the general equation).

$\rho = 1.5$, $v_\tau = -\frac{g}{\rho} = -\frac{32}{1.5} = -21.3$.
$v(0) = -202.68$, $x(0) = 7084.7$ (as determined by Problem 1).

We want to solve for t in the equation

$$x(t) = 0 = x_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t}).$$

After plugging in values, we get the equation

$$6964.08 - 21.3t - 120.92e^{-1.5t}.$$

I solved this using the Mathematica function **Solve**, and got $t \approx 327$ seconds. Together with the additional 20 seconds of freefall,

$$t \approx 347 \text{ seconds.}$$

is the time elapsed before touch down.