Math 216
Differential Equations

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

September 10, 2008

Outline

ConcepTest

Question. Consider the following equations (where M and k are some constant values).

$$
\begin{aligned}
(1) \quad \frac{dx}{dt} &= kx(M - x) \\
(2) \quad \frac{dy}{dt} &= ky(y - M)
\end{aligned}
$$

Which of the equations best models each situation.

(a) This is a model for the spread of a disease through a population in which the disease is spread from infected to uninfected.

(b) This is a model of a closed population which mates through random encounters between males and females.

(c) This is a model of a closed population living in an environment which has a capacity to support a maximum population of M.

ConcepTest

Question. Consider the following equations (where M and k are some constant values).

$$(1) \quad \frac{dx}{dt} = kx(M - x)$$

$$(2) \quad \frac{dy}{dt} = ky(y - M)$$

Which of the equations best models each situation.

(a) This is a model for the spread of a disease through a population in which the disease is spread from infected to uninfected.

(b) This is a model of a closed population which mates through random encounters between males and females.

(c) This is a model of a closed population living in an environment which has a capacity to support a maximum population of M.

Answer. (1) : (a), (c) (2): (b)

ConcepTest

Question. Solve the differential equations, where r is a constant.

$$\begin{aligned} (1) \quad \frac{dx}{dt} &= \mu x(1-x), \qquad x(0) = x_0 \\ (2) \quad \frac{dy}{dt} &= \mu y(y-1), \qquad y(0) = y_0 \end{aligned}$$

ConcepTest

Question. Solve the differential equations, where r is a constant.

$$(1)\quad \frac{dx}{dt} = \mu x(1-x), \qquad x(0) = x_0$$

$$(2)\quad \frac{dy}{dt} = \mu y(y-1), \qquad y(0) = y_0$$

Answer.

$$(1)\quad x(t) = \frac{x_0}{x_0 + (1-x_0)e^{-\mu t}}$$

$$(2)\quad y(t) = \frac{y_0}{y_0 + (1-y_0)e^{\mu t}}$$

Simplified form for logistic equation

The logistic equation

$$(a) \quad \frac{dx}{dt} = kx(M - x)$$

can be always re-expressed in the following simplified form

$$(b) \quad \frac{dy}{dt} = \mu y(1 - y)$$

Let $\mu = Mk$, then

- If $x = x(t)$ is a solution to (a), then $y(t) = \frac{x(t)}{M}$ is a solution to (b).
- If $y = y(t)$ is a solution to (b), then $x(t) = My(t)$ is a solution to (a).

Outline

Elementary integrals

Problem. Solve the following autonomous equation

$$\frac{dx}{dt} = \sqrt{1 - x^3}.$$

Elementary integrals

Problem. Solve the following autonomous equation

$$\frac{dx}{dt} = \sqrt{1 - x^3}.$$

Answer. It is not possible to solve this integral using elementary functions.

An elementary function is a function you studied in your calculus course. For example,

(a) polynomials, exponentials, logarithms, trigonometric and inverse trigonometric functions.

(b) All functions obtained from the previous by addition, subtraction, multiplication, division, extraction of roots, raising to powers.

(c) Replacing an elementary function into the argument of another elementary function. (For example, $\sin(\cos x)$).

Nonelementary integrals

In solving differential equations we frequently encounter the need for integrating an expression which is **not** the differential of an elementary function. For example,

$$\int \exp -x^2\,dx \qquad \int \frac{e^{-x}}{x}\,dx \qquad \int x\tan x\,dx$$

$$\int \sin x^2\,dx \qquad \int \frac{\sin x}{x}\,dx \qquad \int \frac{dx}{\ln x}$$

$$\int \cos x^2\,dx \qquad \int \frac{\cos x}{x}\,dx \qquad \int \frac{dx}{\sqrt{1-x^3}}$$

(It is rare that the root of a polynomial of degree greater than 2 can be integrated.)

See also Example 3 of Section 1.5 for an appearance of an unsolvable integral.

Why qualitative analysis

- Most nonlinear differential equations **cannot be solved** analytically.
- There are numerical techniques for approximating solutions to differential equations that work even when explicit solutions cannot be found. (see Sections 2.4-2.6).
- However, there are questions which we need to know that cannot be answered by numerical methods.
  - How do we know that there exist solutions?
  - If one method of approximation yields a solution, might we get a different solution using a different method?
  - What does the long-run behavior look like?
  - What if my data is only approximate; would I get a totally different solution by carrying out my approximation to another decimal place?

Qualitative Analysis

- Qualitative analysis provides us with information about the tendencies of how solutions will behave, and no quantitative information is given.
- Qualitative analysis can frequently be carried-out, even when no analytic solution is possible.
- Even when analytic solution is possible, qualitative analysis can provide qualitative information more efficiently then knowing explicit solutions.
- In many applications in the sciences, it is the qualitative properties of the equations that we care about.

What information does qualitative analysis provide?

You have already done some qualitative analysis: determining whether unique solutions exist for an equation. Here are some things we'll do today:

- Determine critical points and equilibrium solutions.
- Determine the stability of critical points.
- Measure the dependence upon parameters.
- Identify bifurcation points.

Other properties of solutions we can determine using qualitative analysis is about the periodicity properties of solutions, and chaotic behavior.

Outline

Autonomous first-order equations

An autonomous first-order equation is an equation of the form

$$\frac{dx}{dt} = f(x)$$

where f is a function of the dependent variable x alone. (We will assume f is nice – both f and f' are continuous. Why?)

If $f(b) = 0$, then the constant function, $x \equiv b$, is a solution to the autonomous equation $\frac{dx}{dt} = f(x)$. In this case,

- b is called a critical point of the autonomous equation.
- The solution, $x \equiv b$, is called an equilibrium solution for the autonomous equation.

ConcepTest

Problem. Find all critical points and equilibrium solutions for the logistic equation

$$\frac{dx}{dt} = \mu x(1 - x)$$

ConcepTest

Problem. Find all critical points and equilibrium solutions for the logistic equation

$$\frac{dx}{dt} = \mu x(1 - x)$$

Answer.

- Critical points. 0, 1
- Equillibrium solutions. The constant functions $x \equiv 0$ and $x \equiv 1$.

Outline

Stability

Definition

Suppose c is a critical point of the autonomous equation

$$\frac{dx}{dy} = f(x).$$

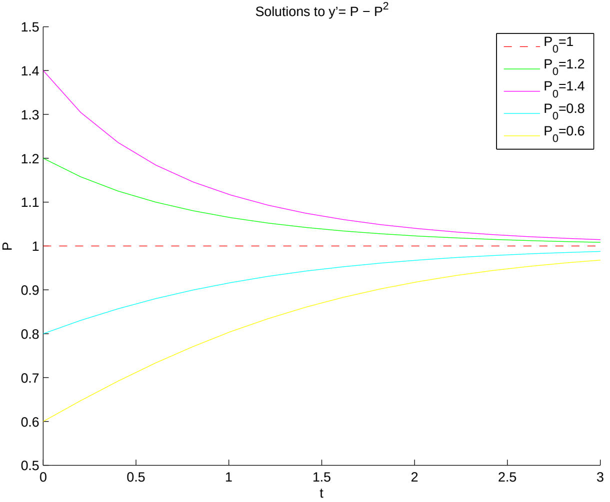
Then c is stable if whenever a solution $x = x(t)$ starts close to c (at time $t = 0$), the solution $x(t)$ will remain close to c for all time $t \geq 0$. More precisely,

> A critical point c is stable if for every (small quantitiy) $\epsilon > 0$, there is a quantity $\delta > 0$ such that
>
> $$|x(0) - c| < \delta \quad \text{implies that} \quad |x(t) - c| < \epsilon \quad \text{for all } t$$

A critical point is unstable if it is not stable.

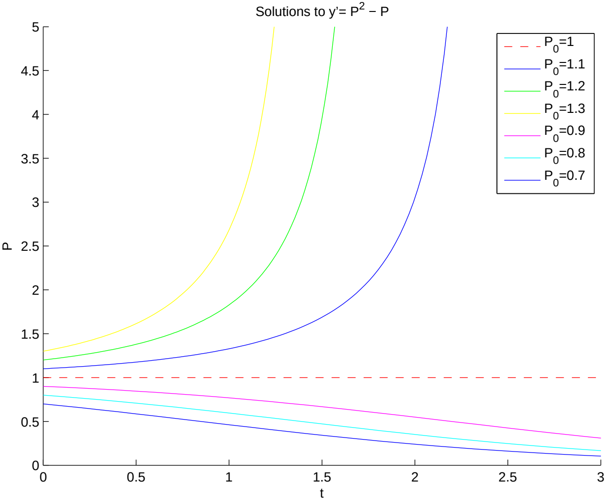
Example of a stability

The value 1 is stable critical point for $x' = x(1 - x)$



Example of a instability

The value 1 is an unstable critical point for $x' = x(x-1)$



Qualitative analysis of stability

- Let c is a critical point of the autonomous equation

$$\frac{dx}{dy} = f(x).$$

- Suppose $x_0 < c$ and there are no critical points between x_0 and c. Then

  If $f(x_0) > 0$, then the solution $x(0) = x_0$ must be increasing for all $t > 0$. So, if x_0 is close to c then the solution $x = x(t)$ will remain close to t.

- Suppose $x_0 > c$ and there are no critical points between x_0 and c. Then

  If $f(x_0) < 0$, then the solution $x(0) = x_0$ must be decreasing for all $t > 0$. So, if x_0 is close to c then the solution $x = x(t)$ will remain close to t.

Qualitative condition for stability

Theorem

Suppose f is nice, and that c is a critical point of the autonomous equation

$$\frac{dx}{dy} = f(x).$$

If there is a $\delta > 0$ such that

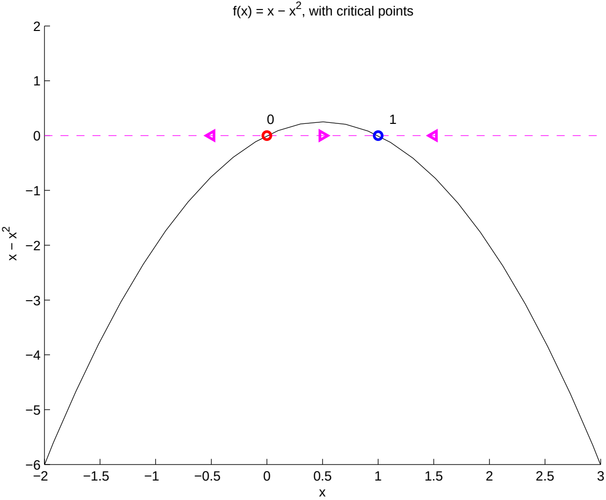
(i) $f(x_0) > 0$ *whenever* $c - \delta < x_0 < c$, *and*

(ii) $f(x_0) < 0$ *whenever* $c < x_0 < c + \delta$;

then c is stable. (In fact all solutions $x(0) = x_0$ where $|c - x_0| < \delta$ will converge to c in the limit: $x(t) \to c$ as $t \to \infty$.)

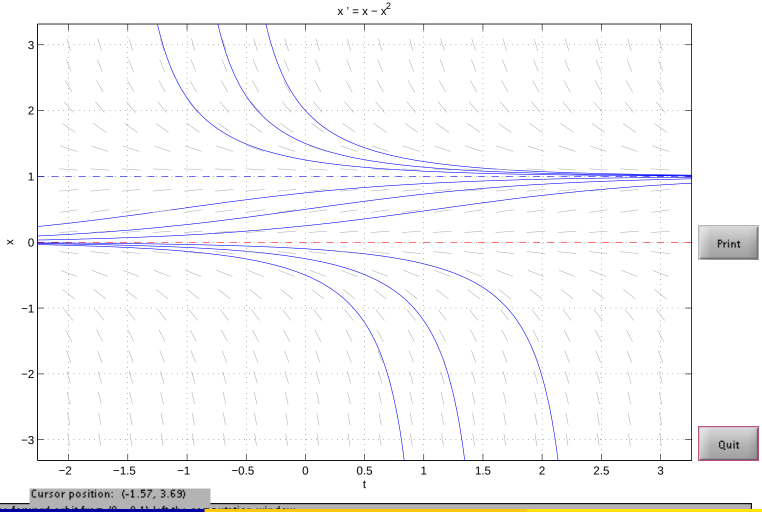
Phase diagram

Plot of $f(x) = x(1 - x)$. The stable critical point is 1, the unstable critical point is 0.



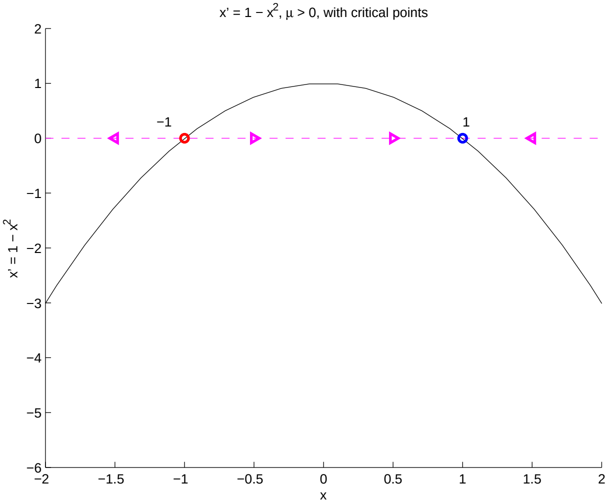
Direction field and solution plot

Solutions for $\frac{dx}{dy} = x(1-x)$. The stable critical point is 1, the unstable critical point is 0.



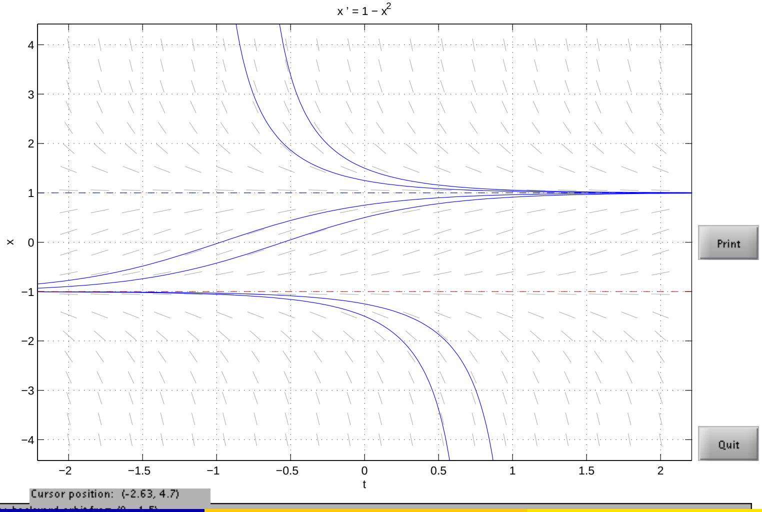
Phase diagram

Plot of $f(x) = 1 - x^2$. The stable critical point is 1, the unstable critical point is -1.



Direction field and solution plot

Solutions for $\frac{dx}{dy} = 1 - x^2$. The stable critical point is 1, the unstable critical point is 0.



Outline

Parameter values

Consider the autonomous first-order equation

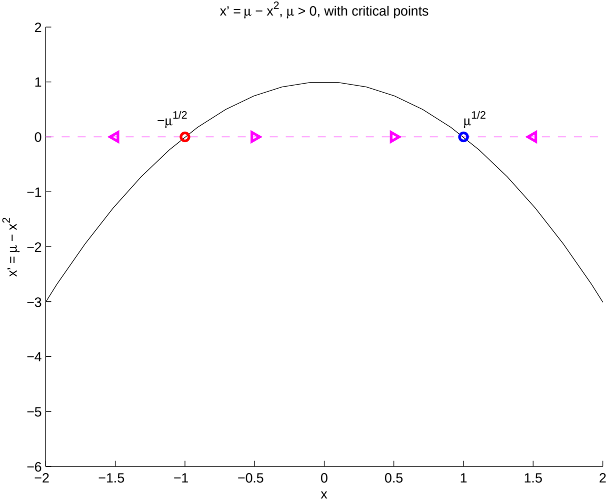
$$\frac{dx}{dt} = 1 - x^2$$

where μ is some constant.

- We have seen how solutions behave when $\mu = 1$. This is useful if we have only an approximation to the initial value $x(0) = x_0$.
- However, suppose that we have only an approximation to μ. How do different values of μ change the qualitative behavior of solutions?

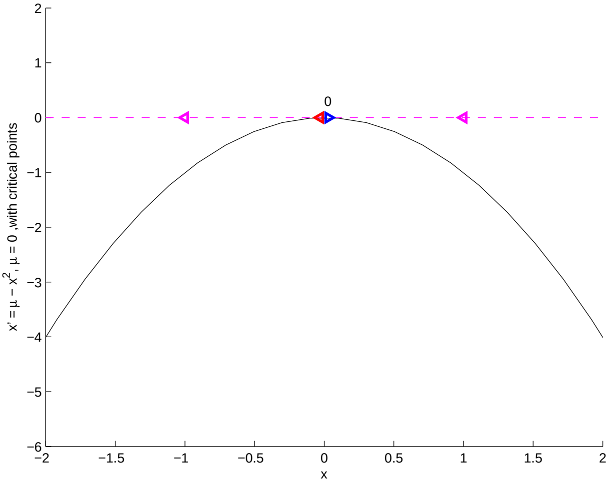
Phase plot for $\mu > 0$

Plot of $f(x) = \mu - x^2$ when $\mu > 0$.



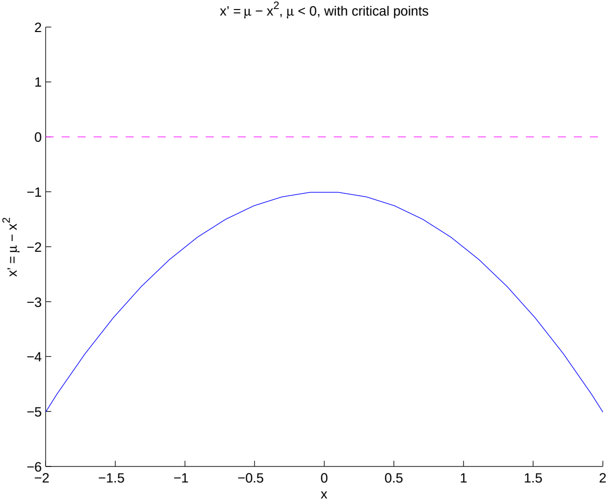
Phase plot for $\mu = 0$

Plot of $f(x) = \mu - x^2$ when $\mu = 0$. A semistable critical point at 0.



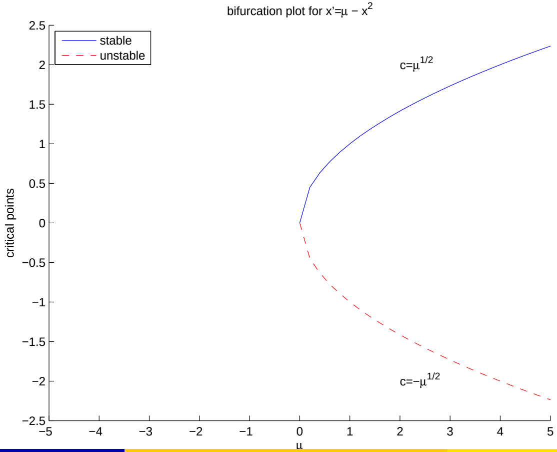
Phase plot for $\mu < 0$

Plot of $f(x) = \mu - x^2$ when $\mu < 0$.



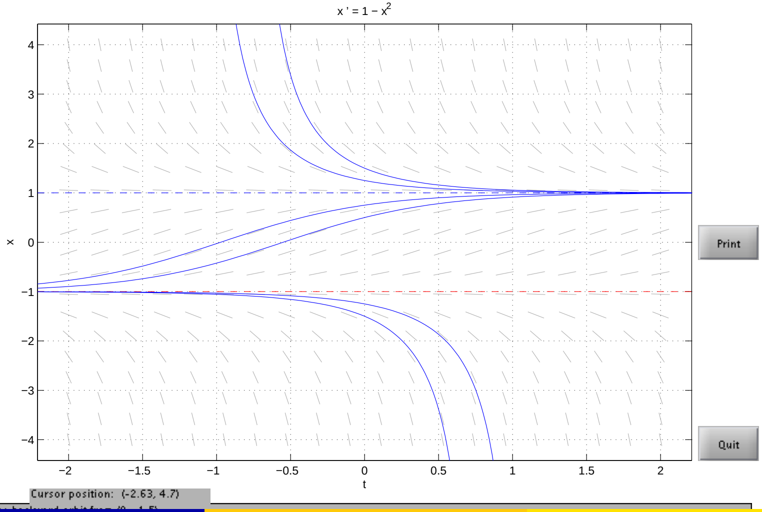
Bifurcation diagram

Bifurcation diagram for $x' = \mu - x^2$. Plotting critical points for each μ. The value $\mu = 0$ is called a bifurcation point – because there is a qualitative change in the behavior of the critical points for $\mu > 0$ and $\mu < 0$.



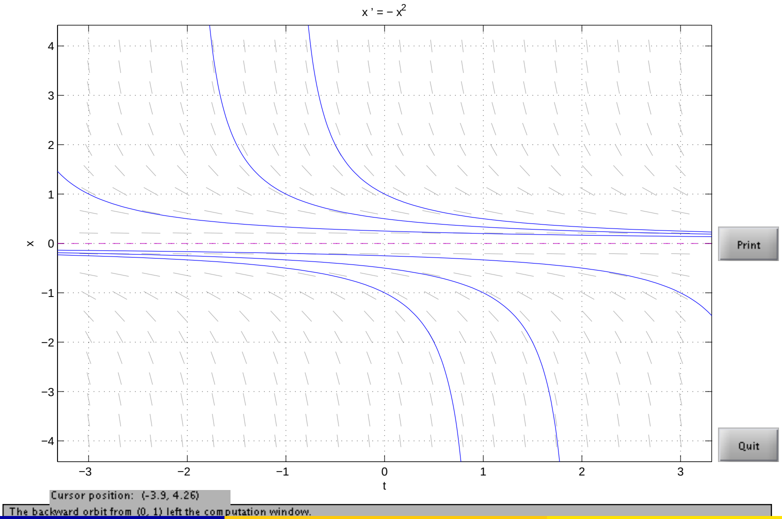
Direction field and solution plot

Solutions for $\frac{dx}{dy} = 1 - x^2$. The stable critical point is 1, the unstable critical point is 0.



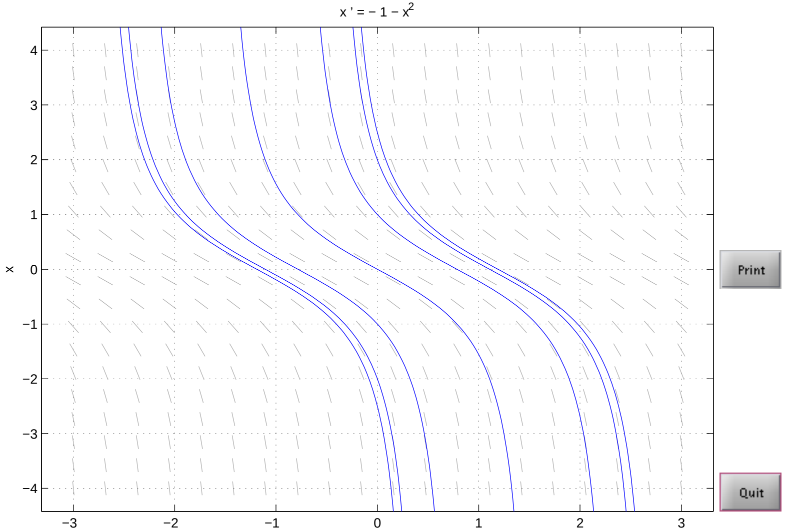
Direction field and solution plot

Solutions for $\frac{dx}{dy} = -x^2$. The semistable critical point is 0. There are no others.



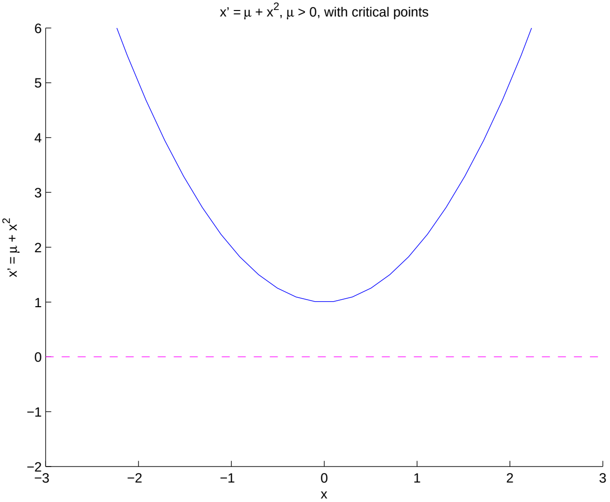
Direction field and solution plot

Solutions for $\frac{dx}{dy} = -1 - x^2$. There are no critical points.



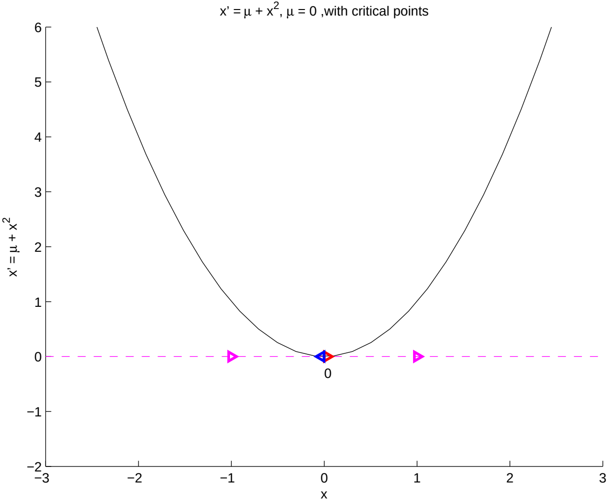
Phase plot for $\mu > 0$

Plot of $f(x) = \mu + x^2$ when $\mu > 0$.



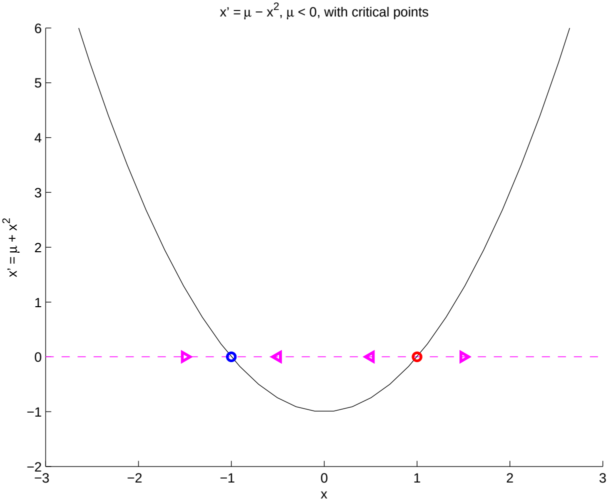
Phase plot for $\mu = 0$

Plot of $f(x) = \mu + x^2$ when $\mu = 0$.



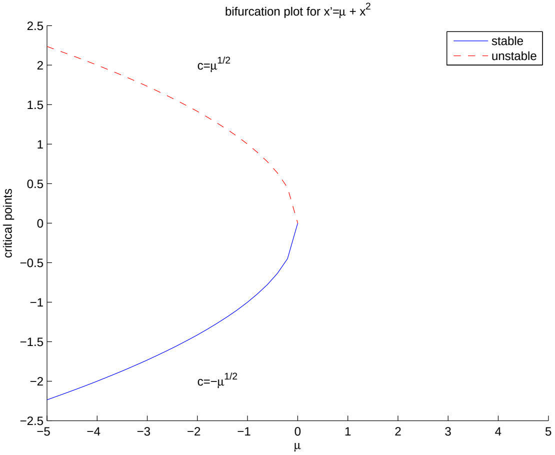
Phase plot for $\mu < 0$

Plot of $f(x) = \mu + x^2$ when $\mu < 0$.



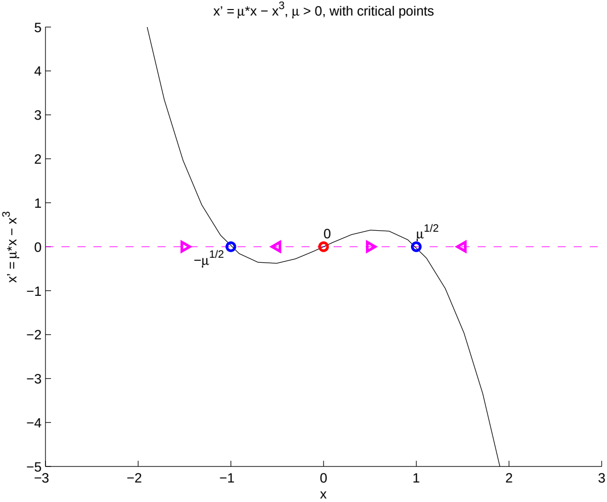
Bifurcation diagram

Bifurcation diagram for $x' = \mu + x^2$. Here, $\mu = 0$ is a bifurcation point.



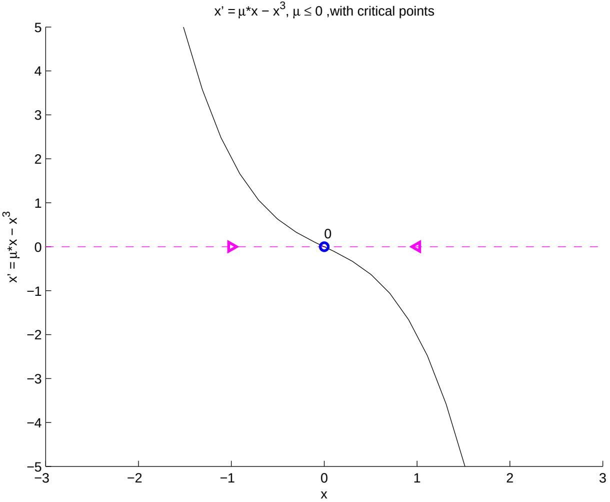
Phase plot for $\mu < 0$

Plot of $f(x) = \mu x - x^3$ when $\mu < 0$.



Phase plot for $\mu \geq 0$

Plot of $f(x) = \mu x - x^3$ when $\mu \geq 0$.



Bifurcation diagram

Bifurcation diagram for $x' = \mu x - x^3$. Here, $\mu = 0$ is called a (transcritical) bifurcation point – 0 changes its stability property to $\sqrt{\mu}$ and $-\sqrt{\mu}$. This is called a pitchfork diagram.