Math 216
Differential Equations

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

December 5, 2008

Translations in the t-domain

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

Compute.

$$\mathcal{L}\{4(t-1)^2u(t-1)\}(s) = \frac{8e^{-s}}{s^3}.$$

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

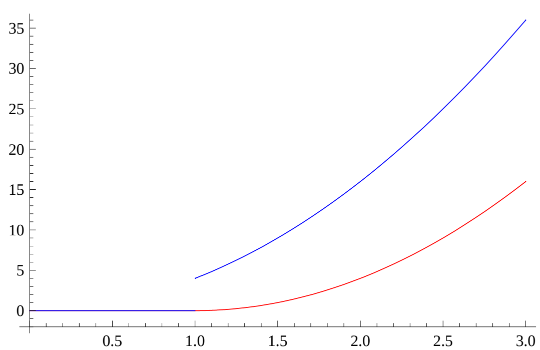
Compute.

$$\begin{aligned}\mathcal{L}\{4t^2u(t-1)\}(s) &= \mathcal{L}\{4((t+1)-1)^2u(t-1)\}(s)\\ &= e^{-s}\mathcal{L}\{4(t+1)^2\}(s)\\ &= e^{-s}\mathcal{L}\{4t^2+8t+4\}(s)\\ &= e^{-s}(\frac{8}{s^3}+\frac{8}{s^2}+\frac{4}{s})\end{aligned}$$

Translations in t-domain

A comparison of function:

$4(t-1)^2u(t-1)$ $\qquad$ $4t^2u(t-1)$



Example 1

Example 1. A mass-spring system without damping starts at rest at equillibrium, and is acted on by an external driver which is terminated after 2π seconds:

$$x'' + 9x = f(t); \qquad x(0) = x'(0) = 0,$$

where the driver force is

$$f(t) = \begin{cases} \cos 3t & \text{if } 0 \le t < 2\pi \\ 0 & \text{otherwise .} \end{cases}$$

What is the response of the system

Example 1 continued

Compute $\mathcal{L}\{f(t)\}$, where $f(t) = (\cos 3t)\big[1 - u(t - 2\pi)\big]$

$$\begin{aligned}
\mathcal{L}\{(\cos 3t)\big[1 - u(t-2\pi)\big]\} &= \mathcal{L}\{\cos 3t\} - \mathcal{L}\{(\cos 3t)u(t-2\pi)\} \\
&= \frac{s}{s^2+9} - e^{-2\pi s}\mathcal{L}\{\cos(3t+6\pi)\} \\
&= \frac{s}{s^2+9} - \frac{se^{-2\pi s}}{s^2+9} \\
&= \frac{s(1-e^{-2\pi s})}{s^2+9}.
\end{aligned}$$

Example 1 continued

Step 1. Compute the Laplace transform of

$$x'' + 9x = f(t); \qquad x(0) = x'(0) = 0,$$

Take the transform of each side:

$$s^2X(s) + 9X(s) = \frac{s(1-e^{-2\pi s})}{s^2+9}$$

Step 2. Solve for $X(s)$:

$$X(s) = \frac{s(1-e^{-2\pi s})}{(s^2+9)^2}.$$

Example 1 continued

Step 3. Compute the inverse transform of

$$X(s) = \frac{s(1-e^{-2\pi s})}{(s^2+9)^2}.$$

We need to determine the inverse transform of

$$G(s) = \frac{s}{(s^2+9)^2}$$

Look-up on a table:

$$\mathcal{L}^{-1}\{G(s)\}(t) = \frac{1}{6}t\sin 3t.$$

Example 1 continued

Step 3 continued. Compute the inverse transform of

$$X(s) = \frac{s(1-e^{-2\pi s})}{(s^2+9)^2} = G(s) - e^{-2\pi s}G(s).$$

The second term is a t-translation by 2π:

$$\begin{aligned}
\mathcal{L}^{-1}\{e^{-2\pi s}G(s)\}(t) &= u(t-2\pi)\big[\frac{1}{6}(t-2\pi)\sin(3t-6\pi)\big] \\
&= u(t-2\pi)\big[\frac{1}{6}(t-2\pi)\sin 3t\big].
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{L}^{-1}\{X(s)\}(t) &= \mathcal{L}^{-1}\{\frac{s(1-e^{-2\pi s})}{(s^2+9)^2}\}(t) \\
&= \frac{1}{6}t\sin 3t - u(t-2\pi)\frac{1}{6}(t-2\pi)\sin 3t.
\end{aligned}$$

Example 1 completed

Example 1. The displacement of the mass-spring system without damping

$$x'' + 9x = f(t); \qquad x(0) = x'(0) = 0,$$

where the driver force is applied for 2π seconds

$$f(t) = \begin{cases} \cos 3t & \text{if } 0 \le t \le 2\pi \\ 0 & \text{otherwise .} \end{cases}$$

is given by

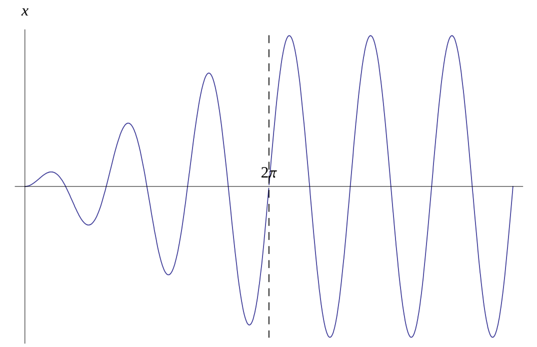
$$x(t) = \frac{1}{6} t \sin 3t - u(t - 2\pi)\frac{1}{6}(t - 2\pi)\sin 3t.$$

Since $u(t - 2\pi) = 0$ if $0 \le t < 2\pi$ and 1 otherwise:

$$x(t) = \begin{cases} \frac{1}{6} t \sin 3t & \text{if } 0 \le t < 2\pi \\ \frac{\pi}{3} \sin 3t & \text{otherwise} \end{cases}$$

Example 1 graph

$$x(t) = \begin{cases} \frac{1}{6} t \sin 3t & \text{if } 0 \le t < 2\pi \\ \frac{\pi}{3} \sin 3t & \text{otherwise} \end{cases}$$



Impulses

Problem. Suppose we apply a force $f(t)$ for a very short duration of time ϵ on a mass m, initially at rest, and which moves along the x-axis. What is the effect on the mass?

Solution. By Newton's second law

$$f(t) = mx''(t)$$

so, (let $t = 0$ be the first instant the force is applied)

$$\int_0^\epsilon f(t)\,dt = \int_0^\epsilon mx''(t)\,dt = mx'(\epsilon) - mx'(0) = mx'(\epsilon).$$

So, the value $\int_0^\epsilon f(t)\,dt$ is the amount of momentum imparted to m at the end of the time the force is applied.

Def. $\int_0^\epsilon f(t)\,dt$ is called the impulse of the force $f(t)$ in time ϵ.

Mass-spring problem

Problem. A mass of 1 kg is attached to a spring with elasticity 1. Initially, the mass is at rest at the equillibrium point. Every 2π seconds (starting at 0) a hammer blow delivers an impulse 1 to the mass. What is the response of the system after the second blow?

At the time of the first blow, the system is described by the equation

$$x'' + x = 0; \qquad x(0) = 0, x'(0) = 1.$$

Mass-spring problem

$$x'' + x = 0; \qquad x(0) = 0, x'(0) = 1.$$

Step 1. Compute the transform.

$$s^2 X(s) - x'(0) + X(s) = 0.$$

Step 2. Solve for $X(s)$.

$$X(s) = \frac{1}{s^2 + 1}$$

Step 3. Compute the inverse transform.

$$x(t) = \sin t.$$

Mass-spring problem – second blow

The second blow requires we recompute the new position and velocity at $t = 2\pi$.

$$\begin{aligned} x(t) &= \sin t \\ x'(t) &= \cos t \end{aligned}$$

The new displacement $y(t) = x(t + 2\pi)$ is given by

$$y'' + y = 0; \qquad y(0) = x(2\pi) = 0, y'(0) = 1 + x'(2\pi) = 2.$$

We impart an impulse of 1 at $t = 2\pi$ to the velocity $x'(2\pi) = 1$.

Mass-spring problem

$$y'' + y = 0; \qquad y(0) = 0, y'(0) = 2.$$

Step 1. Compute the transform.

$$s^2 Y(s) - y'(0) + Y(s) = 0.$$

Step 2. Solve for $Y(s)$.

$$Y(s) = \frac{2}{s^2 + 1}$$

Step 3. Compute the inverse transform.

$$y(t) = 2 \sin t.$$

However, we want $x(t) = y(t - 2\pi)$ (where $t \geq 2\pi$)

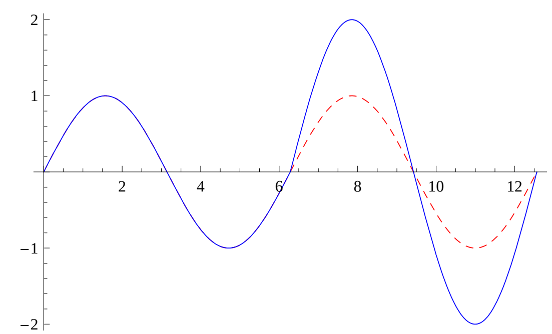
$$x(t) = 2\sin(t - 2\pi) = 2 \sin t.$$

Mass-spring problem

Displacement after two blows

$$x(t) = \begin{cases} \sin t & 0 < t < 2\pi \\ 2 \sin t & 2\pi \leq t \end{cases}$$

sin t is dashed.



Mass-spring problem

After two blows, the displacement of the mass is given by

$$x(t) = \begin{cases} \sin t & 0 < t < 2\pi \\ 2\sin t & 2\pi \le t \end{cases}$$

Question. What is the displacement after 3 blows? 4 blows? n blows?

There is a better way to solve this problem.

A useful fiction

Invention. Let $\delta(t)$ be a "function" which represents a force imparting an instantaneous impulse of 1. What properties should $\delta(t)$ have?

Property 1. Since $\delta(t)$ is instantaneous:

$$\delta(t) = 0 \qquad \text{for all } t \neq 0,$$

What about $t = 0$? Lets leave it undefined for the moment.

Property 2. Since $\delta(t)$ represents an impulse of 1 applied at $t = 0$:

$$\int_b^a \delta(t)\, dt = 1 \qquad \text{whenver } a < 0 < b.$$

Note. $\delta(0)$ cannot have any intelligible value for both properties to be true!!

The key property of δ

Question. How do we capture $\delta(t) \neq 0$?
Use the mean value theorem for integrals.

Let $f(t)$ be any continuous function, and $a < 0 < b$. By the MVT, there is a T with $a < T < b$ with

$$\int_a^b f(t)\delta(t)\, dt = f(T)\int_a^b \delta(t)\, dt = f(T) \cdot 1 = f(T).$$

(using **Property 2**).

By shrinking (a, b) containing 0, we force $T \to 0$, so $f(T) \to f(0)$.
Since $\delta(t) = 0$ whenever $t \neq 0$ (**Property 1**),

$$\int_a^b f(t)\delta(t)\, dt = \int_c^d f(t)\delta(t)\, dt.$$

whenever $c < 0 < d$. So,

$$\int_a^b f(t)\delta(t)\, dt = f(0).$$

The Dirac delta function

Definition

The Dirac delta function $\delta(t)$ is characterized by the following three properties (where a is any real value):

Property 1. $\delta(t-a) = 0$ whenever $t \neq a$.

Property 2. For any $\epsilon > 0$,

$$\int_{a-\epsilon}^{a+\epsilon} \delta(t-a)\, dt = 1.$$

Property 3. For any $\epsilon > 0$ and $f(t)$ continuous around 0,

$$\int_{a-\epsilon}^{a+\epsilon} f(t)\delta(t-a)\, dt = f(a).$$

The Dirac delta "function" models a sudden force applied to a system at $t = a$, which we treat as instantaneous.

The Laplace transform

We can compute the Laplace transform $\mathcal{L}\{\delta(t-a)\}$ using properties 1 and 3:

$$\begin{aligned}\mathcal{L}\{\delta(t-a)\} &= \int_0^\infty e^{-st}\delta(t-a)\,dt \\ &= \int_{-\infty}^\infty e^{-st}\delta(t-a)\,dt \\ &= e^{-as}.\end{aligned}$$

So, for any real number a,

$$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as} \qquad \text{for all } s > 0$$

Example 1

Example 1. Find a solution $x(t)$ to

$$x''(t) + x(t) = \delta(t) \qquad x(0) = x'(0) = 0.$$

Step 1. Apply the Laplace transform

$$s^2X(s) + X(s) = e^{-0s} = 1.$$

Step 2. Solve for $X(s)$.

$$X(s) = \frac{1}{s^2+1}$$

Step 3. Apply the inverse transform.

$$x(t) = \sin t.$$

Example 1 revisited

We have two ways of thinking about an instantaneous force applied to our mass-spring system:
Method 1. As a change in the initial velocity

$$x''(t) + x(t) = 0 \qquad x(0) = 0, x'(0) = 1.$$

Method 1. As an impulse using δ

$$x''(t) + x(t) = \delta(t) \qquad x(0) = x'(0) = 0.$$

The solution in either case is

$$x(t) = \sin t$$

Example 2

There is nothing special about Example 1. Suppose we apply an impulse to a system which changes the velocity at $t = 0$ to v_0.

Apply the Laplace transform to the system

$$ax'' + bx' + cx = f(t) \qquad x(0) = 0, x'(0) = v_0$$

Solution. Apply the transform.

$$as^2X(s) - av_0 + bsX(s) + cX(s) = F(s).$$

Solve for $X(s)$.

$$X(s) = \frac{F(s) + av_0}{as^2 + bs + c}.$$

Example 2

Redo the problem using the impulse av_0 –
the momentum in the mass-spring system.

Apply the Laplace transform to the system

$$ax'' + bx' + cx = f(t) + (av_0)\delta(t) \qquad x(0) = x'(0) = 0.$$

Solution. Apply the transform.

$$as^2X(s) + bsX(s) + cX(s) = F(s) + av_0e^{-s0} = F(s) + av_0.$$

Solve for $X(s)$.

$$X(s) = \frac{F(s) + av_0}{as^2 + bs + c}.$$

So, we can think of an instantaneous force as a sudden change of velocity at $t = 0$, or as an impulse changing momentum and instantaneously applied at $t = 0$ using the Dirac delta function.

Mass-spring problem

Problem. A mass of 1 kg is attached to a spring with elasticity 1. Initially, the mass is at rest at the equillibrium point. Every 2π seconds (starting at 0) a hammer blow delivers an impulse 1 to the mass. What is the response of the system after two blows?

System. We can re-write the problem as one system:

$$x'' + x = \delta(t) + \delta(t - 2\pi); \qquad x(0) = 0, x'(0) = 0.$$

Mass-spring problem

$$x'' + x = \delta(t) + \delta(t - 2\pi); \qquad x(0) = 0, x'(0) = 0.$$

Step 1. Apply the Laplace transform.

$$s^2X(s) + X(s) = 1 + e^{-2\pi s}$$

Step 2. Solve for $X(s)$.

$$X(s) = \frac{1 + e^{-2\pi s}}{s^2 + 1}.$$

Step 3. Apply the Laplace inverse transform.

$$x(t) = \sin t + u(t - 2\pi)\sin(t - 2\pi) = \sin t + u(t - 2\pi)\sin t$$

This agrees with our first computation.

Mass-spring problem

The third blow comes at $t = 4\pi$, the system is described by

$$x'' + x = \delta(t) + \delta(t - 2\pi) + \delta(t - 4\pi); \qquad x(0) = 0, x'(0) = 0.$$

After applying the Laplace transform and solving for $X(s)$:

$$X(s) = \frac{1 + e^{-2\pi s} + e^{-4\pi s}}{s^2 + 1}.$$

The solution is given by

$$\begin{aligned} x(t) &= \sin t + u(t - 2\pi)\sin t + u(t - 4\pi)\sin(t - 4\pi) \\ &= \sin t + u(t - 2\pi)\sin t + u(t - 4\pi)\sin t \end{aligned}$$

Equivalently,

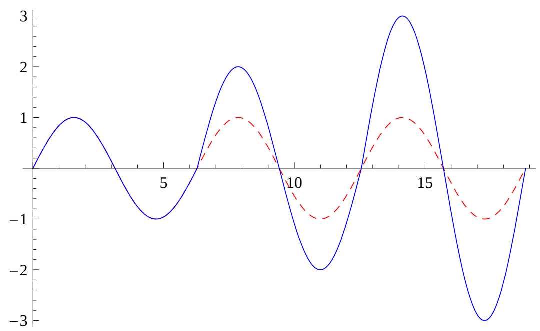
$$x(t) = \begin{cases} \sin t & 0 \le t < 2\pi \\ 2\sin t & 2\pi \le t < 4\pi \\ 3\sin t & 4\pi \le t < 6\pi \end{cases}$$

Mass-spring problem

Displacement after three blows

$$x(t) = \begin{cases} \sin t & 0 \le t < 2\pi \\ 2\sin t & 2\pi \le t < 4\pi \\ 3\sin t & 4\pi \le t < 6\pi \end{cases}$$

sin t is dashed.



Mass-spring problem

The nth blow comes at $t = 2(n-1)\pi$, the system is described by

$$x'' + x = \delta(t) + \delta(t-2\pi) + \delta(t-4\pi) \ldots + \delta(t-2n\pi); \quad x(0) = 0, x'(0) = 0.$$

After applying the Laplace transform and solving for $X(s)$:

$$X(s) = \frac{1 + e^{-2\pi s} + e^{-4\pi s} + \ldots + e^{-2n\pi}}{s^2+1}.$$

The solution is given by

$$x(t) \quad = \quad \sin t + u(t-2\pi)\sin t + u(t-4\pi)\sin t + \ldots + u(t-2n\pi)\sin t$$

Equivalently,

$$x(t) = \begin{cases} \sin t & 0 \le t < 2\pi \\ 2\sin t & 2\pi \le t < 4\pi \\ 3\sin t & 4\pi \le t < 6\pi \\ \ldots & \\ n\sin t & 1(n-1)\pi \le t < 2n\pi \end{cases}$$

Generalized functions

- The delta function was first introduced by P.A.M Dirac in the 1930's.
- The "delta function" is not really a function (whose domain and range are the real numbers), but it can be made rigorous as a generalized function, or distribution.
- Intuitively, the delta function is an impulse, a mathematical abstraction. Think of a baseball bat transferring force to a baseball in an instantaneous packet, an impulse. You can then calculate the flight of the ball without worrying about how the bat transferred energy to the ball during the brief moment of contact.
- See the *Notes on the Dirac Delta Function* by Kurt Bryan under Lecture 37 for Friday December 3.

Approximating δ

- We can approximate the the Dirac delta function, $\delta(t)$ by narrowing our time frame (like instant replay), and imagine instead a constant force applied at a very short time whose impulse is 1.
- Let

  $$d_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & \text{if } 0 \le t < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

  so that

  $$\int_{-\infty}^{\infty} d_\epsilon(t)\,dt = 1$$

- We can think of $\delta(t)$ as the limit of these shorter and shorter snapshots of the energy transfer:

  $$\delta(t) = \lim_{t\to\infty} d_\epsilon(t)$$

  This is still an abstraction, since $\delta(t) = \infty$!!

Property 3 of δ

We can use this approximation of $\delta(t)$ to prove the key **Property 3**: and any interval $c < a < d$:

$$\int_d^c f(t)\delta(t-a)\,dt = f(a).$$

which we used in justifying the Laplace transform

$$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$$

Property 3 of δ

Let $f(t)$ be a continuous function and $c < a < d$.

$$\begin{aligned}\int_d^c \delta(t)f(t)\,dt &= \lim_{\epsilon\to 0}\int_d^c \delta_\epsilon(t)f(t)\,dt\\ &= \lim_{\epsilon\to 0}\int_{a-\epsilon}^{a+\epsilon} \delta_\epsilon(t)f(t)\,dt\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)\,dt\end{aligned}$$

Remember that $\delta_\epsilon(t-a) = \frac{1}{\epsilon}$ for $a \le t < a+\epsilon$ and 0 otherwise.

Let $F(t)$ be the antiderivative for $f(t)$. We continue:

$$\begin{aligned}\int_d^c \delta(t)f(t)\,dt &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)\,dt\\ &= \lim_{\epsilon\to 0}\frac{F(a+\epsilon)-F(a)}{\epsilon}\\ &= F'(a) = f(a).\end{aligned}$$

Justification of δ

- Instead of working with $\delta(t)$, you could work with $\delta_\epsilon(t)$ instead, since
$$\delta_\epsilon(t) = \frac{1}{\epsilon}\big(1 - u(t-\epsilon)\big);$$
then take limits as $\epsilon \to 0$.
- Since $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$, which is a nice function, it is easier to continue the fiction of using $\delta(t)$ and transfer to the s-domain using the Laplace transform.
- Still, you might sleep better knowing that δ can be put on a more rigorous standing.