Math 216
Differential Equations

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

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Example

Example. A mass-spring system without damping starts at rest at equillibrium, and is acted on by an external driver which is terminated after 2π seconds.

$$x'' + 9x = f(t); \qquad x(0) = x'(0) = 0,$$

where the driver force is

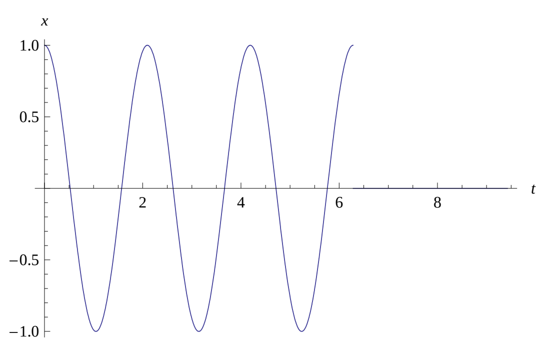
$$f(t) = \begin{cases} \cos 3t & \text{if } 0 \leqslant t \leqslant 2\pi \\ 0 & \text{otherwise .} \end{cases}$$

Compute the displacement function $x(t)$.

The external driver

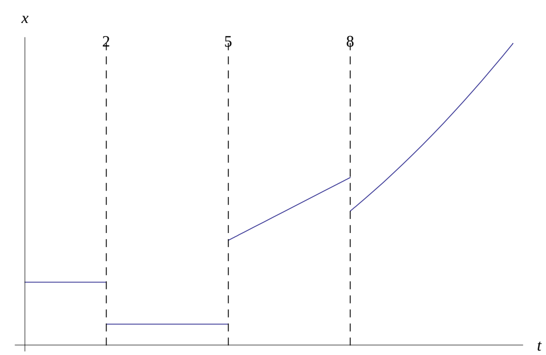
The driver is applied 2π seconds, then cut-off

$$f(t) = \quad = \begin{cases} \cos 3t & \text{if } 0 \leqslant t \leqslant 2\pi \\ 0 & \text{otherwise .} \end{cases}$$



A piecewise continuous functions

$$f(t) \quad = \begin{cases} 3 & \text{if } t < 2 \\ 1 & \text{if } 2 < t < 5 \\ t & \text{if } 5 < t < 8 \\ \frac{t}{10} & \text{if } 8 < t < 12 \end{cases}$$



Definition

Definition

A function $f(t)$ is piecewise continuous on the finite interval $[a, b]$ (where $a < b$) if f is continuous at all except the finitely many points $\{a, x_1, \ldots, x_k, b\}$ where each $a < x_i < b$ and

- Both one-sided limits $\lim_{\epsilon \to 0^+} f(x_i - \epsilon)$ and $\lim_{\epsilon \to 0^+} f(x_i + \epsilon)$ exists and are finite for each i (but not necessarily equal!),
- The one-sided limit $\lim_{\epsilon \to 0^+} f(a + \epsilon)$ exists and is finite,
- The one-sided limit $\lim_{\epsilon \to 0^+} f(b - \epsilon)$ exists and is finite.

A function $f(t)$ is piecewise continuous on the infinite interval $[0, \infty)$ if $f(t)$ is piecewise continuous on the finite intervals $[0, N]$ for each $N > 0$.

Examples

Example. The following function is piecewise continuous on $[0, \infty)$:

$$f(t) = \begin{cases} t & 0 \leqslant t \leqslant 4 \\ 5 & t > 4 \end{cases}$$

Example. The unit staircase function is piecewise continuous on $[0, \infty)$:

$$g(t) = n \qquad \text{if} \quad n - 1 \leqslant t < n, \quad n = 1, 2, 3, \ldots$$

Example. The following function is NOT piecewise continuous on $[0, \infty)$:

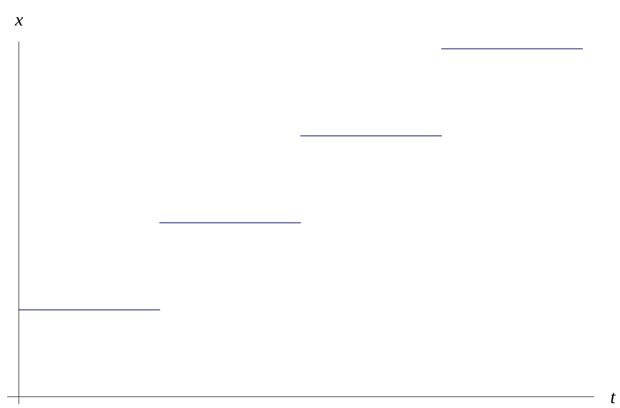
$$h(t) = t^{-\frac{1}{2}}$$

because it blows-up at 0.

Staircase function

The staircase function:

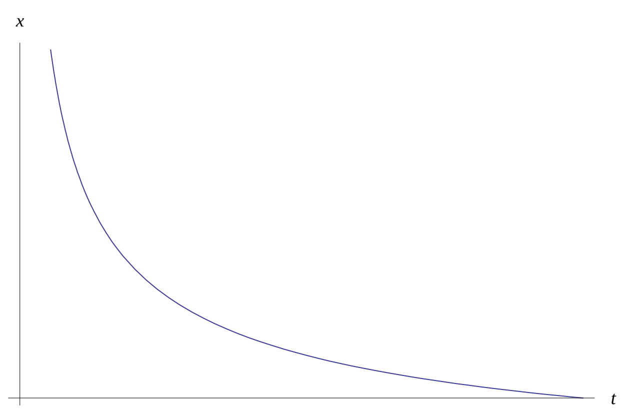
$$f(t) = n \qquad \text{if} \quad n - 1 \leqslant t < n, \quad n = 1, 2, 3, \ldots$$



A nonpiecewise continuous function

The following function is not piecewise continuous:

$$f(t) \quad = \quad t^{-\frac{1}{2}}.$$



Transform of a piecewise continuous function

Compute the Laplace transform of the following function:

$$g(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 2 \\ 2 & \text{if } 2 < t \leqslant 4 \\ 0 & \text{if } 4 < t \end{cases}$$

Apply the definition of Laplace transform

$$\begin{aligned} \mathcal{L}\{g(t)\}(s) &= \int_0^\infty e^{-st} g(t)\, dt \\ &= \int_2^4 2e^{-st}\, dt \\ &= \left[-\frac{2}{s} e^{-st} \right]_{t=2}^4 \\ &= \frac{2}{s}\left(e^{-2s} - e^{-4s}\right). \end{aligned}$$

Transform of a piecewise continuous function

Compute $\mathcal{L}\{g(t)\}$ where

$$g(t) = \begin{cases} t & 0 \leqslant t \leqslant 4 \\ 5 & t > 4 \end{cases}$$

Answer. Split the integral at the discontinuity at $t = 4$:

$$\begin{aligned} \mathcal{L}\{g(t)\}(s) &= \int_0^\infty e^{-st} g(t)\, dt \\ &= \int_0^4 e^{-st} t\, dt + \int_4^\infty 5e^{-st}\, dt \end{aligned}$$

Use integration by parts on the first integral

$$\begin{aligned} \mathcal{L}\{g(t)\}(s) &= \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^4 + \left[-\frac{5}{s} e^{-st} \right]_{t=4}^\infty \\ &= \left(-\frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + 0 + \frac{1}{s^2} \right) + \frac{5e^{-4s}}{s} \\ &= \frac{1}{s^2} + \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2}. \end{aligned}$$

Unit step function

Definition

The unit step function is defined by

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } 0 \leqslant t \end{cases}$$

Fact. Any piecewise continuous function can be expressed in terms of unit step functions.

Modifying the unit step function

We can move the jump of $u(t)$ to a new location:

$$u(t-a) = u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } a \leqslant t \end{cases}$$

We can modify the height of the jump to M

$$Mu(t) = \begin{cases} 0 & \text{if } t < 0 \\ M & \text{if } 0 \leqslant t \end{cases}$$

We can reverse the jump

$$1 - u(t-a) = \begin{cases} 1 & \text{if } t < a \\ 0 & \text{if } a \leqslant t \end{cases}$$

We can contain the jump between a and b (with $a < b$)

$$u(t-a) - u(t-b) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } a \leqslant t < b \\ 0 & \text{if } b \leqslant t \end{cases}$$

Example

Example. Write the following function in terms of the unit step function:

$$f(t) = \begin{cases} t^2 & \text{if } t < 4 \\ 6 & \text{if } 4 \leqslant t \end{cases}$$

- $t^2\big[1 - u(t-4)\big]$ is t if $t < 4$ and 0 otherwise,
- $6u(t-4)$ is 6 only when $4 \leqslant t$ and 0 otherwise.

We can rewrite $f(t)$ as

$$f(t) = t^2\big[1 - u(t-4)\big] + 6u(t-4).$$

Example

Example. Write the following function in terms of the unit step function:

$$f(t) = \begin{cases} t & \text{if } t < 1 \\ 2 - t & \text{if } 1 \leqslant t < 2 \\ 3 & \text{if } 2 \leqslant t \end{cases}$$

- $t\big[1 - u(t-1)\big]$ is t if $t < 1$ and 0 otherwise,
- $(2-t)\big[u(t-1) - u(t-2)\big]$ is $2 - t$ only when $1 \leqslant t < 2$ and 0 otherwise,
- $3u(t-2)$ is 1 only when $2 \leqslant t$ and 0 otherwise.

We can rewrite $f(t)$ as

$$f(t) = t\big[1 - u(t-1)\big] + (2-t)\big[u(t-1) - u(t-2)\big] + 3u(t-2).$$

Example

Example. Write the following function in terms of the unit step function:

$$f(t) = \begin{cases} \sin 2t & \text{if } 2\pi \leqslant t < 4\pi \\ 0 & \text{otherwise} \end{cases}$$

- $\sin 2t\big[u(t-2\pi) - u(t-4\pi)\big]$ is $\sin 2t$ only when $2\pi \leqslant t < 4\pi$ and 0 otherwise,

We can rewrite $f(t)$ as

$$f(t) = \sin 2t\big[u(t-2\pi) - u(t-4\pi)\big].$$

Intregration and piecewise discontinuous functions

Fact. Suppose $f(t)$ and $g(t)$ are piecewise continuous functions that agree everywhere on an interval $[a, b]$ except for the points $\{x_1, x_2, \ldots, x_k\}$. Then

$$\int_a^b f(t)\,dt = \int_a^b g(t)\,dt$$

If $f(t)$ and $g(t)$ agree everywhere except for their points of discontinuity then

$$\mathcal{L}\big\{f(t)\big\} = \mathcal{L}\big\{g(t)\big\}.$$

Moral. When computing the Laplace transform of a piecewise continuous function you do not need to bother what happens at points of discontinuity.

Transform of the unit step function

Compute the Laplace transform (where $a > 0$)

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } a \leqslant t \end{cases}$$

Answer.

$$\begin{aligned} \mathcal{L}\{u(t-a)\}(s) &= \int_0^\infty e^{-st} u(t-a)\, dt \\ &= \int_a^\infty e^{-st}\, dt + \int_0^a 0\, dt \\ &= \left[-\frac{e^{-st}}{s}\right]_{t=a}^\infty \\ &= \frac{e^{-as}}{s}. \end{aligned}$$

Transform of the unit step function

When $a > 0$ then

$$\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$$

Conversely, we will write

$$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a)$$

No continous function has the transform $\frac{e^{-as}}{s}$.

Translation in t

Theorem (Translation in the t domain)

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for all $s \geqslant \alpha \geqslant 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \qquad \text{for all } s \geqslant \alpha.$$

Conversely, an inverse transform is given by

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

Translations

***t*-domain.** Multiplying $f(t)$ by e^{at} translates in s-domain:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

***s*-domain.** Multiplying $F(s)$ by e^{-as} translates in t-domain:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

The presence of $u(t-a)$ “kills-off" $f(t)$ for $t < a$.

Proof of Theorem

Proof. By the definition of Laplace transform

$$\begin{aligned}\mathcal{L}\{f(t-a)u(t-a)\}(s) &= \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt \\ &= \int_a^\infty e^{-st} f(t-a)\,dt\end{aligned}$$

since $u(t-a) = 0$ when $t < a$ and 1 otherwise.
Let $v = t - a$, so $dv = dt$.

$$\begin{aligned}\mathcal{L}\{f(t-a)u(t-a)\}(s) &= \int_a^\infty e^{-st} f(t-a)\,dt \\ &= \int_0^\infty e^{-s(v+a)} f(v)\,dv \\ &= e^{-as}\int_0^\infty e^{-sv} f(v)\,dv \\ &= e^{-as}F(s).\end{aligned}$$

Computing transforms

Problem. In practice we usually need to compute the transform of a function expressed as $g(t)u(t-a)$ rather than as $f(t-a)u(t-a)$.

Solution. Let $g(t+a) = f(t)$, so that $f(t-a) = g(t)$. Then

$$\begin{aligned}\mathcal{L}\{g(t)u(t-a)\}(s) &= \mathcal{L}\{f(t-a)u(t-a)\}(s) \\ &= e^{-as}\mathcal{L}\{f(t)\} \\ &= e^{-as}\mathcal{L}\{g(t+a)\}(s)\end{aligned}$$

So,

$$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s).$$

Example 1

Example 1. Compute the Laplace transform of $t^2u(t-1)$.

Answer. Let $g(t) = t^2$ and $a = 1$. Then

$$g(t+a) = g(t+1) = (t+1)^2 = t^2 + 2t + 1.$$

Compute

$$\mathcal{L}\{g(t+1)\}(s) = \mathcal{L}\{t^2+2t+1\}(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}.$$

So, the Laplace transform is

$$\begin{aligned}\mathcal{L}\{t^2u(t-1)\}(s) &= e^{-s}\mathcal{L}\{t^2+2t+1\}(s) \\ &= e^{-s}\Big(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\Big).\end{aligned}$$

Example 2

Example 2. Compute the Laplace transform of $(\cos t)u(t-\pi)$.

Answer. Let $g(t) = \cos t$ and $a = \pi$. Then

$$g(t+\pi) = \cos(t+\pi) = -\cos t.$$

Compute

$$\mathcal{L}\{g(t+\pi)\}(s) = \mathcal{L}\{-\cos t\}(s) = -\frac{s}{s^2+1}.$$

So, the Laplace transform is

$$\begin{aligned}\mathcal{L}\{(\cos t)u(t-\pi)\}(s) &= e^{-\pi s}\mathcal{L}\{g(t+\pi)\}(s) \\ &= -e^{-\pi s}\frac{s}{s^2+1}.\end{aligned}$$

Example 3: Inverse transform

Example 3. Compute the inverse transform of $\mathcal{L}^{-1}\{\frac{e^{-2s}}{s^2}\}$.

Answer. Let $F(s) = \frac{1}{s^2}$. Then

$$f(t) = \mathcal{L}^{-1}\{\frac{1}{s^2}\}(t) = t.$$

The shift is $a = 2$, so

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{e^{-2s}}{s^2}\} &= f(t-2)u(t-2) \\ &= (t-2)u(t-2).\end{aligned}$$

Example 4: Inverse transform

Example 4. Compute the inverse transform of $\dfrac{s(e^{-\pi s} + e^{-2\pi s})}{s^2+4}$.

Answer. Let $F(s) = \frac{s}{s^2+4}$. Then

$$f(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+4}\}(t) = \cos 2t.$$

The shifts are $a = \pi$ and $a = 2\pi$, so

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{s(e^{-\pi s} + e^{-2\pi s})}{s^2+4}\} &= \mathcal{L}^{-1}\{\frac{se^{-\pi s}}{s^2+4}\} + \mathcal{L}^{-1}\{\frac{se^{-2\pi s}}{s^2+4}\} \\ &= f(t-\pi)u(t-\pi) + f(t-2\pi)u(t-2\pi) \\ &= \cos(2t-2\pi)u(t-\pi) - \cos(2t-4\pi)u(t-\pi) \\ &= \cos 2t\big[u(t-\pi) - u(t-2\pi)\big]\end{aligned}$$

Example 1

Example 1. A mass-spring system without damping starts at rest at equillibrium, and is acted on by an external driver which is terminated after 2π seconds.

$$x'' + 9x = f(t); \qquad x(0) = x'(0) = 0,$$

where the driver force is

$$f(t) = \begin{cases}\cos t & \text{if } 0 \leqslant t \leqslant 2\pi \\ 0 & \text{otherwise}.\end{cases}$$

Compute the displacement function $x(t)$.

Example 1 continued

Compute $\mathcal{L}\{f(t)\}$.

Let $g(t) = \cos t$, so

$$g(t+2\pi) = \cos(t+2\pi) = \cos t.$$

So,

$$\begin{aligned}\mathcal{L}\{(\cos t)\big[1 - u(t-2\pi)\big]\} &= \mathcal{L}\{\cos t\} - \mathcal{L}\{(\cos t)u(t-2\pi)\} \\ &= \frac{s}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\cos(t+2\pi)\} \\ &= \frac{s}{s^2+1} - \frac{se^{-2\pi s}}{s^2+1} \\ &= \frac{s(1-e^{-2\pi s})}{s^2+1}.\end{aligned}$$

Example 1 continued

Step 1. Compute the Laplace transform of

$$x'' + 9x = f(t); \qquad x(0) = x'(0) = 0,$$

Take the transform of each side:

$$s^2 X(s) + 9X(s) = \frac{s(1 - e^{-2\pi s})}{s^2 + 1}$$

Step 2. Solve for $X(s)$:

$$X(s) = \frac{s(1 - e^{-2\pi s})}{(s^2 + 9)(s^2 + 1)}.$$

Example 1 continued

Step 3. Compute the inverse transform of

$$X(s) = \frac{s(1 - e^{-2\pi s})}{(s^2 + 9)(s^2 + 1)}.$$

We need to determine the inverse transform of

$$G(s) = \frac{s}{(s^2 + 9)(s^2 + 1)}$$

Use partial fractions or the convolution:

$$\mathcal{L}^{-1}\{G(s)\}(t) = \cos(3t) * \cos t = \frac{1}{8}(\cos t - \cos 3t).$$

Example 1 continued

Step 3 continued. Compute the inverse transform of

$$X(s) = \frac{s(1 - e^{-2\pi s})}{(s^2 + 9)(s^2 + 1)} = G(s) - e^{-2\pi s} G(s).$$

The second term is a t-translation by 2π:

$$\begin{aligned}
\mathcal{L}^{-1}\{se^{-2\pi s} G(s)\}(t) &= u(t - 2\pi)\Big[\frac{1}{8}\big(\cos(t - 2\pi) - \cos(3t - 6\pi)\big)\Big] \\
&= u(t - 2\pi)\Big[\frac{1}{8}(\cos t - \cos 3t)\Big].
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{L}^{-1}\{X(s)\}(t) &= \mathcal{L}^{-1}\Big\{\frac{s(1 - e^{-2\pi s})}{(s^2 + 9)(s^2 + 1)}\Big\}(t) \\
&= \big[1 - u(t - 2\pi)\big]\Big[\frac{1}{8}(\cos t - \cos 3t)\Big].
\end{aligned}$$

Example 1 completed

Example 1. The displacement of the mass-spring system without damping

$$x'' + 9x = f(t); \qquad x(0) = x'(0) = 0,$$

where the driver force is applied for 2π seconds

$$f(t) = \begin{cases} \cos t & \text{if } 0 \leqslant t \leqslant 2\pi \\ 0 & \text{otherwise} \end{cases}.$$

is given by

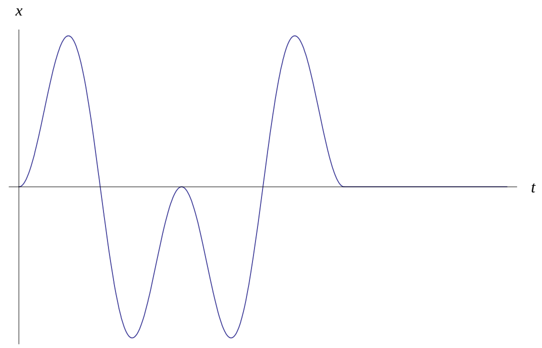
$$x(t) = \big[1 - u(t - 2\pi)\big]\Big[\frac{1}{8}(\cos t - \cos 3t)\Big].$$

Since $1 - u(t - 2\pi) = 1$ if $0 \leqslant t < 2\pi$ and 0 otherwise:

$$x(t) = \begin{cases} \frac{1}{8}(\cos t - \cos 3t) & \text{if } 0 \leqslant t < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Example 1 graph

$$x(t) = \begin{cases} \frac{1}{8}(\cos t - \cos 3t) & \text{if } 0 \leqslant t < 2\pi \\ 0 & \text{otherwise} \end{cases}$$



Example 2

Example 1. A mass-spring system without damping starts at rest at equillibrium, and is acted on by an external driver which is terminated after 2π seconds.

$$x'' + 9x = f(t); \qquad x(0) = x'(0) = 0,$$

where the driver force is

$$f(t) = \begin{cases} \cos 3t & \text{if } 0 \leqslant t \leqslant 2\pi \\ 0 & \text{otherwise .} \end{cases}$$

Compute the displacement function $x(t)$.

Example 2 continued

Compute $\mathcal{L}\{f(t)\}$.

Let $g(t) = \cos 3t$, so

$$g(t + 2\pi) = \cos(3t + 6\pi) = \cos 3t.$$

So,

$$\begin{aligned} \mathcal{L}\{(\cos 3t)[1 - u(t - 2\pi)]\} &= \mathcal{L}\{\cos 3t\} - \mathcal{L}\{(\cos 3t)u(t - 2\pi)\} \\ &= \frac{s}{s^2 + 9} - e^{-2\pi s}\mathcal{L}\{\cos(3t + 6\pi)\} \\ &= \frac{s}{s^2 + 9} - \frac{se^{-2\pi s}}{s^2 + 9} \\ &= \frac{s(1 - e^{-2\pi s})}{s^2 + 9}. \end{aligned}$$

Example 2 continued

Step 1. Compute the Laplace transform of

$$x'' + 9x = f(t); \qquad x(0) = x'(0) = 0,$$

Take the transform of each side:

$$s^2 X(s) + 9X(s) = \frac{s(1 - e^{-2\pi s})}{s^2 + 9}$$

Step 2. Solve for $X(s)$:

$$X(s) = \frac{s(1 - e^{-2\pi s})}{(s^2 + 9)^2}.$$

Example 2 continued

Step 3. Compute the inverse transform of

$$X(s) = \frac{s(1-e^{-2\pi s})}{(s^2+9)^2}.$$

We need to determine the inverse transform of

$$G(s) = \frac{s}{(s^2+9)^2}$$

Look-up on a table:

$$\mathcal{L}^{-1}\{G(s)\}(t) = \frac{1}{6}t\sin 3t.$$

Example 2 continued

Step 3 continued. Compute the inverse transform of

$$X(s) = \frac{s(1-e^{-2\pi s})}{(s^2+9)^2} = G(s) - e^{-2\pi s}G(s).$$

The second term is a t-translation by 2π:

$$\begin{aligned}\mathcal{L}^{-1}\{e^{-2\pi s}G(s)\}(t) &= u(t-2\pi)\big[\frac{1}{6}(t-2\pi)\cos(3t-6\pi)\big] \\ &= u(t-2\pi)\big[\frac{1}{6}(t-2\pi)\cos 3t\big].\end{aligned}$$

So,

$$\begin{aligned}\mathcal{L}^{-1}\{X(s)\}(t) &= \mathcal{L}^{-1}\{\frac{s(1-e^{-2\pi s})}{(s^2+9)^2}\}(t) \\ &= \frac{1}{6}t\cos 3t - u(t-2\pi)\frac{1}{6}(t-2\pi)\cos 3t.\end{aligned}$$

Example 2 completed

Example 2. The displacement of the mass-spring system without damping

$$x'' + 9x = f(t); \qquad x(0) = x'(0) = 0,$$

where the driver force is applied for 2π seconds

$$f(t) = \begin{cases}\cos 3t & \text{if } 0 \leqslant t \leqslant 2\pi \\ 0 & \text{otherwise .}\end{cases}$$

is given by

$$x(t) = \frac{1}{6}t\cos 3t - u(t-2\pi)\frac{1}{6}(t-2\pi)\cos 3t.$$

Since $u(t-2\pi) = 0$ if $0 \leqslant t < 2\pi$ and 1 otherwise:

$$x(t) = \begin{cases}\frac{1}{6}t\cos 3t & \text{if } 0 \leqslant t < 2\pi \\ \frac{\pi}{3}\cos 3t & \text{otherwise}\end{cases}$$

Example 2 graph

$$x(t) = \begin{cases}\frac{1}{6}t\cos 3t & \text{if } 0 \leqslant t < 2\pi \\ \frac{\pi}{3}\cos 3t & \text{otherwise}\end{cases}$$