Math 216
Differential Equations

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Problem

Problem. Solve the initial value problem

$$x'' + x = g(t); \qquad x(0) = x'(0) = 0.$$

Solution. Take the Laplace transform of both sides

$$s^2 X(s) + X(s) = G(s),$$

where $\mathcal{L}\{g(t)\} = G(s)$. Solve for $X(s)$

$$X(s) = \left(\frac{1}{s^2+1}\right) G(s).$$

Problem continued

Problem. We need to compute the inverse transform of

$$X(s) = \left(\frac{1}{s^2+1}\right) G(s).$$

Notice that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1},$$

Question. Is there a simple formula for $x(t)$ in terms of $\sin t$ and $g(t)$?

Answer. Yes, using the convolution of $\sin t$ and $g(t)$.

Convolution

Definition

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$.

The convolution of $f(t)$ and $g(t)$, denoted by $f * g$, is the function of t defined by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Example

Compute $t^2 * t$.

$$\begin{aligned} t^2 * t &= \int_0^t \tau^2(t-\tau)\,d\tau \\ &= \int_0^t (t\tau^2 - \tau^3)\,d\tau \\ &= \left(\frac{t\tau^3}{3} - \frac{\tau^4}{4}\right)\Big|_0^t \\ &= \frac{t^4}{3} - \frac{t^4}{4} \\ &= \frac{t^4}{12} \end{aligned}$$

Note that

$$\mathcal{L}\{t^2\}\cdot\mathcal{L}\{t\} = \frac{2}{s^3}\cdot\frac{1}{s^2} = \mathcal{L}\{\frac{t^4}{12}\} = \mathcal{L}\{t^2 * t\}.$$

Example

Compute $t * t^2$.

$$\begin{aligned} t * t^2 &= \int_0^t \tau(t-\tau)^2\,d\tau \\ &= \int_0^t (\tau^3 - 2t\tau^2 + t^2\tau)\,d\tau \\ &= \left(\frac{\tau^4}{4} - 2\frac{t\tau^3}{3} + \frac{t^2\tau^2}{2}\right)\Big|_0^t \\ &= \frac{t^4}{4} - 2\frac{t^4}{3} + \frac{t^4}{2} \\ &= \frac{t^4}{12} \end{aligned}$$

Thus, $t * t^2 = t^2 * t$.

Basic properties of convolution

Theorem

Let $f(t), g(t), h(t)$ be piecewise continuous on $[0,\infty)$. Then

$$\begin{aligned} &(a) && f * g = g * f, \\ &(b) && f * (g + h) = (f * g) + (f * h), \\ &(c) && f * 0 = 0. \end{aligned}$$

Proof of (a)

Prove. $f * g = g * f$.

By definition

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Change variables with $w = t - \tau$,

$$(f * g)(t) = \int_t^0 f(t-w)g(w)\,(-dw) = \int_0^t f(t-w)g(w)\,dw = g * f.$$

Proof of (b)

Prove. $f * (g + h) = (f * g) + (f * h)$.
By definition

$$\begin{aligned}(f*(g+h))(t) &= \int_0^t f(\tau)(g(t-\tau)+h(t-\tau))\,d\tau \\ &= \int_0^t (f(\tau)g(t-\tau)+f(\tau)h(t-\tau))\,d\tau \\ &= \int_0^t f(\tau)g(t-\tau)\,d\tau + \int_0^t f(\tau)h(t-\tau)\,d\tau \\ &= (f*g)(t)+(f*h)(t)\end{aligned}$$

Theorem

Theorem

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α.

Then,

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}.$$

Equivalently,

$$f * g = \mathcal{L}^{-1}\{F(s)G(s)\},$$

where $\mathcal{L}\{f\}(s) = F(s)$ and $\mathcal{L}\{g\}(s) = G(s)$.

See section 7.4 of Edwards and Penney for a proof.

Example 1

Example 1. Use the convolution theorem to find $\mathcal{L}^{-1}\{\frac{1}{(s^2+1)^2}\}$.

Write

$$\frac{1}{(s^2+1)^2} = \left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+1}\right)$$

Since $\mathcal{L}\{\sin t\}(s) = \frac{1}{s^2+1}$,

$$\mathcal{L}^{-1}\{\frac{1}{(s^2+1)^2}\} = \sin t * \sin t$$

Example 1 continued

Compute:

$$\begin{aligned}\sin t * \sin t &= \int_0^t \sin\tau \sin(t-\tau)\,dt \\ &= \frac{1}{2}\int_0^t (\cos(2\tau - t) - \cos t)\,d\tau \\ &= \frac{1}{2}\left(\frac{\sin(2\tau-t)}{2}\right)\Big|_0^t - \frac{1}{2}\tau\cos t\Big|_0^t \\ &= \frac{1}{2}\left(\frac{\sin t}{2} - \frac{\sin(-t)}{2}\right) - \frac{1}{2}t\cos t \\ &= \frac{\sin t - t\cos t}{2}\end{aligned}$$

Note: $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\beta-\alpha) - \cos(\beta+\alpha))$.

Trigonometry

Use Euler's formula

$$e^{\alpha i} = \cos\alpha + i \sin\alpha$$

to verify the identities

$$\begin{aligned} \sin\alpha \sin\beta &= \frac{1}{2}\big(\cos(\beta-\alpha) - \cos(\beta+\alpha)\big) \\ \cos\alpha \cos\beta &= \frac{1}{2}\big(\cos(\beta-\alpha) + \cos(\beta+\alpha)\big) \\ \cos\alpha \sin\beta &= \frac{1}{2}\big(\sin(\beta-\alpha) + \sin(\beta+\alpha)\big) \end{aligned}$$

Example 2

Example 2. Use the convolution theorem to find $\mathcal{L}^{-1}\{\frac{1}{(s-a)(s-b)}\}$ when $a \neq b$.

Write

$$\frac{1}{(s-a)(s-b)} = \Big(\frac{1}{s-a}\Big)\Big(\frac{1}{s-b}\Big).$$

Since $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, by the convolution theorem

$$\mathcal{L}^{-1}\Big\{\frac{1}{(s-a)(s-b)}\Big\}(t) = e^{at} * e^{bt}.$$

Example 2 continued

Compute:

$$\begin{aligned} e^{at} * e^{bt} &= \int_0^t e^{a\tau} e^{b(t-\tau)}\, d\tau \\ &= \int_0^t e^{(a-b)\tau} e^{bt}\, d\tau \\ &= \frac{e^{bt}}{a-b}\Big(e^{(a-b)\tau}\Big)\Big|_0^t \\ &= \frac{e^{bt}}{a-b}\Big(e^{(a-b)t} - 1\Big) \\ &= \frac{1}{a-b}\big(e^{at} - e^{bt}\big). \end{aligned}$$

Problem

Problem. Let $f(t)$ have Laplace transform $F(s)$, and $g(t) = 1$ (the function which is constantly 1). What is $f * g$?

Answer.

$$\begin{aligned} f * g &= \int_0^t f(\tau) g(t-\tau)\, d\tau \\ &= \int_0^t f(\tau)\, d\tau \end{aligned}$$

but,

$$\begin{aligned} \mathcal{L}\{f * 1\} &= \mathcal{L}\{f(t)\}(s) \cdot \mathcal{L}\{1\}(s) \\ &= F(s) \cdot \frac{1}{s}. \end{aligned}$$

So,

$$\mathcal{L}\Big\{\int_0^t f(\tau)\, d\tau\Big\} = \frac{F(s)}{s}.$$

Equivalently,

Example

Compute the inverse transform of $\frac{1}{s(s+1)(s-2)}$.

Answer. Use the previous result with $F(s) = \frac{1}{(s+1)(s-2)}$.
By partial fractions (or Example 2 previously)

$$\frac{1}{(s+1)(s-2)} = \frac{1}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right);$$

So,

$$\mathcal{L}^{-1}\{\frac{1}{(s+1)(s-2)}\} = \frac{1}{3}(e^{2t} - e^{-t})$$

Example continued

Apply the previous

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{1}{s(s+1)(s-2)}\}(t) &= \int_0^t \frac{1}{3}(e^{2\tau} - e^{-\tau})\, d\tau \\ &= \frac{1}{3}\left(\frac{1}{2}e^{2\tau} + e^{-\tau}\right)\Big|_0^t \\ &= \frac{1}{3}\left(\frac{1}{2}e^{2t} + e^{-t} - \frac{3}{2}\right).\end{aligned}$$

So,

$$\mathcal{L}^{-1}\{\frac{1}{s(s+1)(s-2)}\}(t) = \frac{1}{6}\left(e^{2t} + 2e^{-t} - 3\right).$$

Theorem

Theorem

Let $f(t)$ be piecewise continuous for $t \geq 0$ and of exponential order α. Then

$$\mathcal{L}\{t\, f(t)\}(s) = -F'(s) \qquad \text{for all } s > \alpha.$$

Equivalently,

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}(t)$$

By repeated application,

$$\mathcal{L}\{t^n\, f(t)\}(s) = (-1)^n F^{(n)}(s) \qquad \text{for all } s > \alpha.$$

Problem revisited

Problem revisited. Solve the initial value problem

$$x'' + x = g(t); \qquad x(0) = x'(0) = 0.$$

Solution. Take the Laplace transform of both sides

$$s^2X(s) + X(s) = G(s),$$

where $\mathcal{L}\{g(t)\} = G(s)$. Solve for $X(s)$

$$X(s) = \left(\frac{1}{s^2+1}\right)G(s).$$

Compute the inverse transform using the previous theorem,

$$x(t) = \sin t * g(t).$$

Example 1

Compute $\mathcal{L}\{t\cos kt\}$ given that $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$.

Answer. Use the previous theorem:

$$\begin{aligned}
\mathcal{L}\{t\cos kt\} &= -\Big(\mathcal{L}\{\cos kt\}\Big)' \\
&= -\Big(\frac{s}{s^2+k^2}\Big)' \\
&= \frac{-1}{s^2+k^2} + \frac{2s^2}{(s^2+k^2)^2} \\
&= \frac{-s^2-k^2}{(s^2+k^2)^2} + \frac{2s^2}{(s^2+k^2)^2} \\
&= \frac{s^2-k^2}{(s^2+k^2)^2}.
\end{aligned}$$

Example 2

Compute $\mathcal{L}\{t^2\cos kt\}$.

Answer. Use the previous theorem:

$$\begin{aligned}
\mathcal{L}\{t(t\cos kt)\} &= -\Big(\mathcal{L}\{t\cos kt\}\Big)' \\
&= \Big(\frac{k^2-s^2}{(s^2+k^2)^2}\Big)' \\
&= \frac{-2s}{(s^2+k^2)^2} - \frac{4s(k^2-s^2)}{(s^2+k^2)^3} \\
&= \frac{-2s(s^2+k^2)+4s(s^2-k^2)}{(s^2+k^2)^3} \\
&= \frac{2s^3-6k^2s}{(s^2+k^2)^3}
\end{aligned}$$

Example 1

Example 1. Solve

$$tx'' + (t-1)x' + x = 0; \qquad x(0) = 0$$

Step 1. Apply the Laplace transform.

$$\mathcal{L}\{tx''\} + \mathcal{L}\{tx'\} - \mathcal{L}\{x'\} + X(s) = 0.$$

Example 1 continued

Compute.

$$\begin{aligned}
\mathcal{L}\{x'\} &= sX(s) - x(0) = sX(s) \\
\mathcal{L}\{tx'\} &= -\Big(\mathcal{L}\{x'\}\Big)' \\
&= -\Big(sX(s)\Big)' \\
&= -X(s) - sX'(s) \\
\mathcal{L}\{tx''\} &= -\Big(\mathcal{L}\{x''\}\Big)' \\
&= -\Big(s^2X(s) - sx(0) - x'(0)\Big)' \\
&= -2sX(s) - s^2X'(s).
\end{aligned}$$

Example 1 continued

Substitute.

$$\begin{aligned}\mathcal{L}\{x'\} &= sX(s) - x(0) = sX(s)\\ \mathcal{L}\{tx'\} &= -X(s) - sX'(s)\\ \mathcal{L}\{tx''\} &= -2sX(s) - s^2X'(s).\end{aligned}$$

So,

$$\begin{aligned}0 &= \mathcal{L}\{tx''\} + \mathcal{L}\{tx'\} - \mathcal{L}\{x'\} + X(s)\\ &= \big(-2sX(s) - s^2X'(s)\big) + \big(-X(s) - sX'(s)\big) - sX(s) + X(s)\\ &= -s(s+1)X'(s) - 3sX(s)\end{aligned}$$

Example 1 continued

Step 2. Solve for $X(s)$ in the linear first-order equation

$$-s(s+1)X'(s) - 3sX(s) = 0.$$

Equivalently,

$$X'(s) + \frac{3}{s+1}X(s) = 0.$$

So,

$$X(s) = \frac{C}{(s+1)^3}$$

where C is the constant of integration.

See Section 1.5 of E+P for solving linear first-order equations.

Example 1 continued

Step 3. Compute $\mathcal{L}^{-1}\{\frac{C}{(s+1)^3}\}$.

$$\mathcal{L}^{-1}\{\frac{C}{(s+1)^3}\} = \frac{C}{2}t^2e^{-t}.$$

So, a general solution for $x(0) = 0$ is $x(t) = Ct^2e^{-t}$.

- We never used $x'(0)$.
- For every choice of C, $x'(0) = 0$. So, $x(0) = 0$ and $x'(0) = 0$ do not uniquely determine $x(t)$.
- There are no solutions for $x(0) = 0$ and $x'(0) = p$ where $p \neq 0$.

Example 1 completed

Step 4. Find the solution where $x(1) = \frac{1}{e}$.

The general solution is $x(t) = Ct^2e^{-t}$; so,

$$\begin{aligned}x(1) &= Ce^{-1}\\ x(1) &= \frac{1}{e},\end{aligned}$$

so, $C = 1$. Thus,

$$x(t) = t^2e^{-t}$$

is the unique solution to the initial value problem

$$tx'' + (t-1)x' + x = 0; \qquad x(0) = 0, x(1) = \frac{1}{e}.$$

Example 2

Example 1. Solve

$$tx'' + (3t-1)x' + 3x = 0; \qquad x(0) = 0$$

Step 1. Apply the Laplace transform.

$$\mathcal{L}\{tx''\} + 3\mathcal{L}\{tx'\} - \mathcal{L}\{x'\} + X(s) = 0.$$

Example 2 continued

Compute.

$$\begin{aligned}
\mathcal{L}\{x'\} &= sX(s) - x(0) = sX(s) \\
\mathcal{L}\{tx'\} &= -\Big(\mathcal{L}\{x'\}\Big)' \\
&= -\Big(sX(s)\Big)' \\
&= -X(s) - sX'(s) \\
\mathcal{L}\{tx''\} &= -\Big(\mathcal{L}\{x''\}\Big)' \\
&= -\Big(s^2X(s) - sx(0) - x'(0)\Big)' \\
&= -2sX(s) - s^2X'(s).
\end{aligned}$$

Example 2 continued

Substitute.

$$\begin{aligned}
\mathcal{L}\{x'\} &= sX(s) - x(0) = sX(s) \\
\mathcal{L}\{tx'\} &= -X(s) - sX'(s) \\
\mathcal{L}\{tx''\} &= -2sX(s) - s^2X'(s).
\end{aligned}$$

So,

$$\begin{aligned}
0 &= \mathcal{L}\{tx''\} + 3\mathcal{L}\{tx'\} - \mathcal{L}\{x'\} + 3X(s) \\
&= \big(-2sX(s) - s^2X'(s)\big) + \big(-3X(s) - 3sX'(s)\big) - sX(s) + 3X(s) \\
&= -s(s+3)X'(s) - 3sX(s)
\end{aligned}$$

Example 2 continued

Step 2. Solve for $X(s)$ in the linear first-order equation

$$-s(s+3)X'(s) - 3sX(s) = 0.$$

Equivalently,

$$X'(s) + \frac{3}{s+3}X(s) = 0.$$

So,

$$X(s) = \frac{C}{(s+3)^3}$$

where C is the constant of integration.

See Section 1.5 of E+P for solving linear first-order equations.

Example 1 continued

Step 3. Compute $\mathcal{L}^{-1}\{\frac{C}{(s+3)^3}\}$.

$$\mathcal{L}^{-1}\{\frac{C}{(s+3)^3}\} = \frac{C}{2}t^2e^{-3t}.$$

A general solution to the equation

$$tx'' + (3t-1)x' + 3x = 0; \qquad x(0) = 0$$

is

$$x(t) = Ct^2e^{-3t}$$