Math 216
Differential Equations

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Theorem: t-derivative rule

Theorem. (t-derivative rule)
Suppose that $f(t)$ is continuous and has exponential order, and that $f'(t)$ is continuous.

Then, for sufficiently large s,

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0).$$

If $f'(t)$ has exponential order and $f''(t)$ is continuous, then for sufficiently large s,

$$\mathcal{L}\{f''(t)\}(s) = s^2F(s) - sf(0) - f'(0)$$

Example

Example. Compute $\mathcal{L}\{t\cos\beta t\}$ and $\mathcal{L}\{t\sin\beta t\}$.

Solution. By linearity of $\mathcal{L}$,

$$\begin{aligned}\mathcal{L}\{(t\cos\beta t)'\} &= \mathcal{L}\{\cos\beta t\} - \beta\mathcal{L}\{t\sin\beta t\}\\ \mathcal{L}\{(t\sin\beta t)'\} &= \mathcal{L}\{\sin\beta t\} + \beta\mathcal{L}\{t\cos\beta t\}\end{aligned}$$

By the t-derivative rule.

$$\begin{aligned}\mathcal{L}\{(t\cos\beta t)'\} &= s\mathcal{L}\{t\cos\beta t\} - 0\\ \mathcal{L}\{(t\sin\beta t)'\} &= s\mathcal{L}\{t\sin\beta t\} - 0\end{aligned}$$

Equating,

$$\begin{aligned}\mathcal{L}\{t\cos\beta t\} &= \frac{s}{s(s^2+\beta^2)} - \frac{\beta}{s}\mathcal{L}\{t\sin\beta t\}\\ \mathcal{L}\{t\sin\beta t\} &= \frac{\beta}{s(s^2+\beta^2)} + \frac{\beta}{s}\mathcal{L}\{t\cos\beta t\}\end{aligned}$$

Example continued

Example continued.

$$\begin{aligned}\mathcal{L}\{t\cos\beta t\} &= \frac{1}{s^2+\beta^2} - \frac{\beta}{s}\mathcal{L}\{t\sin\beta t\}\\ \mathcal{L}\{t\sin\beta t\} &= \frac{\beta}{s}\left(\frac{1}{s^2+\beta^2}\right) + \frac{\beta}{s}\mathcal{L}\{t\cos\beta t\}\end{aligned}$$

Substitute for $\mathcal{L}\{t\sin\beta t\}$ in $\mathcal{L}\{t\cos\beta t\}$

$$\begin{aligned}\left(1+\frac{\beta^2}{s^2}\right)\mathcal{L}\{t\cos\beta t\} &= \frac{1}{s^2+\beta^2} - \frac{\beta^2}{s^2(s^2+\beta^2)}\\ &= \frac{s^2-\beta^2}{s^2(s^2+\beta^2)}\end{aligned}$$

Solve:

$$\mathcal{L}\{t\cos\beta t\} = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

Example continued

Example continued.

$$\mathcal{L}\{t\cos\beta t\} = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

$$\mathcal{L}\{t\sin\beta t\} = \frac{\beta}{s}\left(\frac{1}{s^2+\beta^2}\right) + \frac{\beta}{s}\mathcal{L}\{t\cos\beta t\}$$

Substitute for $\mathcal{L}\{t\sin\beta t\}$ in $\mathcal{L}\{t\cos\beta t\}$

$$\begin{aligned}\mathcal{L}\{t\sin\beta t\} &= \frac{\beta}{s}\left(\frac{1}{s^2+\beta^2}\right) + \frac{\beta}{s}\left(\frac{s^2-\beta^2}{(s^2+\beta^2)^2}\right)\\ &= \frac{\beta}{s}\left(\frac{(s^2+\beta^2)+(s^2-\beta^2)}{(s^2+\beta^2)^2}\right)\\ &= \frac{2\beta s}{(s^2+\beta^2)^2}\end{aligned}$$

Rational functions

We often must compute the inverse transform of rational functions

$$R(s) = \frac{P(s)}{Q(s)}$$

where

- $P(s)$ and $Q(s)$ are polynomials,
- The degree of $P(s)$ is less than that of the degree of $Q(s)$,
- $Q(s)$ factors into the product of linear and quadratic terms.

Partial fractions. Use the method of partial fractions to break-up a rational function into a sum of rational functions whose inverse can be readily determined.

Rule for linear factors

Rule 1. Suppose that we can write the rational function $R(s)$ as

$$R(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(s-a)^n Q'(s)}$$

where the factor $(s-a)$ does not occur in $Q'(s)$.

The partial fraction decomposition corresponding to the linear factor $(s-a)$ is

$$\frac{A_1}{s-a} + \frac{A_2}{(s-2)^2} + \ldots + \frac{A_n}{(s-a)^n}.$$

where $A_1, A_2, \ldots, A_n$ are constants.

Rule for quadratic factors

Rule 2. Suppose that we can write the rational function $R(s)$ as

$$R(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(as^2+bs+c)^n Q'(s)}$$

where the factor (as^2+bs+c) does not occur in $Q'(s)$.

The partial fraction decomposition corresponding to the quadratic factor (as^2+bs+c) is

$$\frac{A_1 s + B_1}{(as^2+bs+c)} + \frac{A_2 s + B_2}{(as^2+bs+c)^2} + \ldots + \frac{A_n s + B_n}{(as^2+bs+c)^n}.$$

where $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$ are constants.

Typical inverse transforms

Common inverse transforms with partial fraction expansion:

f(t)	F(s)	
$e^{\alpha t} t^n$	$\frac{n!}{(s-\alpha)^{n+1}}$	$(s > \alpha)$
$e^{\alpha t} \cos \beta t$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$	$(s > \alpha)$
$e^{\alpha t} \sin \beta t$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$	$(s > \alpha)$
$e^{\alpha t} \frac{1}{2\beta} t \sin \beta t$	$\frac{s-\alpha}{[(s-\alpha)^2+\beta^2]^2}$	$(s > \alpha)$
$e^{\alpha t} \frac{1}{2\beta^3} \left(\sin \beta t - \beta t \cos \beta t \right)$	$\frac{1}{[(s-\alpha)^2+\beta^2]^2}$	$(s > \alpha)$

Example 1: Distinct linear factors

Example 1. Determine $\mathcal{L}^{-1}\{F\}$, where

$$F(s) = \frac{7s-1}{(s+1)(s+2)(s-3)}.$$

Partial fraction. The expansion has the form

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}.$$

Eliminate the denominators by multiplying through:

$$7s - 1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$$

Example 1 continued

Method 1. This always works.
Equate coefficients on each side.

$$7s - 1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$$

Multiply out:

$$7s - 1 = (A+B+C)s^2 + (-A-2B+C)s + (-6A-3B+2C)$$

Equate coefficients:

$$\begin{aligned} A + B + C &= 0 \\ -A - 2B + 3C &= 7 \\ -6A - 3B + 2C &= -1 \end{aligned}$$

Solve:

$$A = 2, \quad B = -3, \quad C = 1.$$

Example 1 continued

Method 2. The Cover-up method works best for distinct linear factors.
Choose values of s to make system easy to solve.

$$7s - 1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$$

Equation is true for all s, so choose s to make life easy ☺.

Let $s = -1$ (eliminates B, C)

$$-8 = -4A \quad \text{or} \quad A = 2.$$

Let $s = -2$ (eliminates A, C)

$$-15 = 5B \quad \text{or} \quad B = -3.$$

Let $s = 3$ (eliminates A, B)

$$20 = 20C \quad \text{or} \quad C = 1.$$

Example 1 solution

We found $A = 2, B = -3, C = 1$, so

$$F(s) = \frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}.$$

Solution. Use the linearity of $\mathcal{L}^{-1}$ and table look-up.

$$\mathcal{L}^{-1}\{F(s)\} = 2e^{-t} - 3e^{-2t} + e^{3t}.$$

Example 2: Repeated linear factors

Example 2. Determine $\mathcal{L}^{-1}\{F\}$, where

$$F(s) = \frac{s^2 + 9s + 2}{(s-1)^2(s+3)}.$$

Partial fraction. The expansion has the form

$$\frac{s^2 + 9s + 2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}.$$

Eliminate the denominators by multiplying through:

$$s^2 + 9s + 2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2$$

Example 2 continued

Method 1. Equate coefficients on each side.

$$s^2 + 9s + 2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2$$

Multiply out:

$$s^2 + 9s + 2 = (A+C)s^2 + (2A + B - 2C)s + (-3A + 3B + C)$$

Equate coefficients:

$$\begin{aligned} A + C &= 1 \\ 2A + B - 2C &= 9 \\ -3A + 3B + C &= 2 \end{aligned}$$

Solve:

$$A = 2, \quad B = 3, \quad C = -1.$$

Example 2 continued

Method 2. Choose values of s to make life easy ☺.

$$s^2 + 9s + 2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2$$

Let $s = 1$ (eliminates A, C)

$$12 = 4B \quad \text{or} \quad B = 3.$$

Let $s = -3$ (eliminates A, B)

$$-16 = 16C \quad \text{or} \quad C = -1.$$

Let $s = 0$, $B = 3$, $C = -1$

$$2 = -3A + 9 - 1 \quad \text{or} \quad A = 2.$$

Example 2 solution

We found $A = 2, B = 3, C = -1$, so

$$F(s) = \frac{2}{s-1} + \frac{3}{(s-1)^2} - \frac{1}{s+3}.$$

Solution. Use the linearity of $\mathcal{L}^{-1}$ and table look-up.

$$\mathcal{L}^{-1}\{F(s)\} = 2e^t + 3te^t - e^{-3t}.$$

Example 3: Quadratic factors

Example 3. Determine $\mathcal{L}^{-1}\{F\}$, where

$$F(s) = \frac{2s^2 + 10s}{(s+1)(s^2 - 2s + 5)}.$$

Partial fraction. The expansion has the form

$$\frac{2s^2 + 10s}{(s+1)(s^2 - 2s + 5)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 - 2s + 5}.$$

Eliminate the denominators by multiplying through:

$$2s^2 + 10s = A(s^2 - 2s + 5) + (Bs + C)(s + 1)$$

Example 3 continued

Method 1. Equate coefficients on each side.

$$2s^2 + 10s = A(s^2 - 2s + 5) + (Bs + C)(s + 1)$$

Multiply out:

$$2s^2 + 10s = (A + B)s^2 + (-2A + B + C)s + (5A + C)$$

Equate coefficients:

$$\begin{aligned} A + B &= 2 \\ -2A + B + C &= 10 \\ 5A + C &= 0 \end{aligned}$$

Solve:

$$A = -1, \quad B = 3, \quad C = 5.$$

Example 3 continued

Method 2. Choose values of s to make life easy ☺.

$$2s^2 + 10s = A(s^2 - 2s + 5) + (Bs + C)(s + 1)$$

Let $s = -1$ (eliminates B, C)

$$-8 = 8A \quad \text{or} \quad A = -1$$

Let $s = 0$, $A = -1$ (eliminates A, B)

$$0 = -5 + C \quad \text{or} \quad C = 5.$$

Let $s = 1$, $A = -1$, $C = 5$

$$12 = -4 + 2B + 10 \quad \text{or} \quad B = 3.$$

Example 3 solution

We found $A = -1, B = 3, C = 5$, so

$$F(s) = \frac{-1}{s+1} + \frac{3s+5}{s^2 - 2s + 5}.$$

We need the second denominator in the form: $[(s-\alpha)^2 + \beta^2]$.
Complete the square, and rewrite:

$$\begin{aligned} F(s) &= \frac{-1}{s+1} + \frac{3(s-1)+8}{[(s-1)^2 + 2^2]} \\ &= \frac{-1}{s+1} + 3\frac{s-1}{[(s-1)^2 + 2^2]} + 4\frac{2}{[(s-1)^2 + 2^2]} \end{aligned}$$

Solution. Use the linearity of $\mathcal{L}^{-1}$ and table look-up.

$$\mathcal{L}^{-1}\{F(s)\} = -e^{-t} + 3e^t \cos 2t + 4e^t \sin 2t.$$

Example 1

Example 1. Solve using the Laplace transform.

$$x'' - 6x' + 9x = 6t^2 e^{3t}; \quad x(0) = x'(0) = 0.$$

Step 1. Apply the Laplace transform to both equations.

$$s^2 X - 6sX + 9X = 6\left(\frac{2}{(s-3)^3}\right)$$

Step 2. Solve for X.

$$X(s) = \frac{12}{(s-3)^5}$$

Step 3. Compute the inverse transform.

$$x(t) = \frac{1}{2} t^4 e^{3t}$$

since $\mathcal{L}\{t^4\} = \frac{4!}{s^5}$.

Example 2

Example 2. Solve using the Laplace transform.

$$x'' - 4x' + 4x = 4e^{2t}; \quad x(0) = -1, x'(0) = -4.$$

Step 1. Apply the Laplace transform:

$$(s^2 X + s + 4) - 4(sX + 1) + 4X = \frac{4}{s-2}.$$

Simplify

$$s^2 X - 4sX + 4X = \frac{4}{s-2} - s.$$

Step 2. Solve for $X(s)$.

$$X(s) = \frac{4}{(s-2)^3} - \frac{s}{(s-2)^2}.$$

Example 2 continued

Step 3. Find the inverse transformation.

$$X(s) = \frac{4}{(s-2)^3} - \frac{s}{(s-2)^2}.$$

For the first term $\mathcal{L}\{t^2\} = \frac{2!}{s^3}$; the second term requires partial fractions.

$$\frac{s}{(s-2)^2} = \frac{A}{s-2} + \frac{B}{(s-2)^2}$$

Cross multiply.

$$-s = A(s-2) + B$$

So, $A = -1$ and $B = -2$. Thus,

$$x(t) = e^{2t}\left(2t^2 - 2t - 1\right).$$

Mass-spring system

Example 1. Consider the following mass-spring system without damping, but with forced oscillations:

$$x'' + 9x = 8\cos t, \quad x(0) = x'(0) = 0.$$

Natural frequency. Note that the natural frequency is $\omega_0 = 3$ (the frequency of oscillation when there is no driver); and the driver frequency is $\omega = 1$. So, there will be no resonance.
Step 1. Apply the Laplace transform:

$$s^2 X(s) + 9X(s) = \frac{8s}{s^2+1}.$$

Step 2. Solve for $X(s)$:

$$X(s) = \frac{8s}{(s^2+9)(s^2+1)}.$$

Example 1 continued

Step 3. Find the inverse transform of

$$X(s) = \frac{8s}{(s^2+9)(s^2+1)}.$$

Use partial fractions to rewrite

$$\frac{8s}{(s^2+9)(s^2+1)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+1}.$$

Cross multiply

$$8s = (As+B)(s^2+1) + (Cs+D)(s^2+9).$$

Equate coefficients. This leads to four equations:

$$0 = A + C \qquad 0 = B + D$$
$$8 = A + 9C \qquad 0 = 9B + D$$

So, $A = -1$, $C = 1$, $B = D = 0$. Thus,

$$X(s) = -\frac{s}{s^2+9} + \frac{s}{s^2+1}.$$

Example completed

Step 3 continued. Find the inverse transform of

$$X(s) = -\frac{s}{s^2+9} + \frac{s}{s^2+1}$$

Each summand can be found on a table. The inverse transform is

$$x(t) = \cos t - \cos 3t,$$

which is the solution to

$$x'' + 9x = 8\cos t, \quad x(0) = x'(0) = 0.$$

Mass-spring system

Example 2. Consider the following mass-spring system without damping, but with forced oscillations:

$$x'' + 9x = 6\cos 3t, \quad x(0) = x'(0) = 0.$$

Natural frequency. Note that the natural frequency is $\omega_0 = 3$ (the frequency of oscillation when there is no driver); and the driver frequency is $\omega = 3$. So, there is resonance.
Step 1. Apply the Laplace transform:

$$s^2 X(s) + 9X(s) = \frac{6s}{s^2+9}.$$

Step 2. Solve for $X(s)$:

$$X(s) = \frac{6s}{(s^2+9)^2}.$$

Example 2 continued

Step 3. Find the inverse transform for

$$X(s) = \frac{6s}{(s^2+9)^2}.$$

Look-up on a table.

$$\mathcal{L}\{t \sin \beta t\} = \frac{2\beta s}{(s^2+\beta^2)^2}.$$

The inverse transform is

$$x(t) = t \sin 3t.$$