Math 216
Differential Equations

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Laplace transforms

The Laplace transform connects two worlds:

1. The t-domain (the real world problem)
   - t is real and positive.
   - Require solving ordinary differential equations.
   - The functions $f(t)$ in these equations can be complicated with nasty discontinuities. (Typical problems are where the functions $f(t)$ are signals.)
2. The s-domain (the world of the Laplace transform)
   - s is often taken to be complex in applications.
   - Require solving algebraic equations.
   - The functions $F(s)$ are often simple rational functions (i.e. poly/poly).

Strategy. Apply the Laplace transform $\mathcal{L}$ to an ODE in the t-domain, resulting in a simple algebraic equation in the s-domain, then return to reality using the “inverse Laplace transform" $\mathcal{L}^{-1}$.

Laplace transform

Definition The Laplace transform of a t-domain function $f(t)$ (defined on all positive reals) is given by the integral

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

The transform is like a hologram, in that each value $F(s)$, contains information about ALL values of $f(t)$.

A function $f(t)$ has a Laplace transform $F(s)$ if the integral exists for sufficiently large s.

Note. Note that the s-domain of the transform $F(s)$ is often taken to be the complex numbers with postive real components.

Existence of Laplace transform

Question. When do functions $f(t)$ have Laplace transforms?

Answer. $f(t)$ should be reasonably well behaved and must not grow too fast.

- Continuous functions are reasonably well-behaved. More generally, so are piecewise continuous functions.
- $f(t)$ should not grow faster than the exponential used in the Laplace integral. This means there is a real number α such that

  $$|f(t)| < e^{\alpha t} \quad \text{for all sufficiently large } t$$

  We then say $f(t)$ has exponential order α.
- If $f(t)$ is continuous and has exponential order α, then $\mathcal{L}\{f(t)\}(s) = F(s)$ exists for all $s > \alpha$ (see Theorem 2 of Section 7.1).

Note. All functions we have encountered so far in this class are reasonably well-behaved and have exponential order (for some α).

Computing the Laplace transform

We don't normally need to compute the Lapace transform by using the integral definition.

Instead, you look-up basic transforms in a table and apply general rules for extending these basic transforms. Here are two rules from last class:

1. (Linearity) $\mathcal{L}\{af(t) + bg(t)\}(s) = aF(s) + bG(s)$.
2. (s-shift) If a is any real number, then $\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$.

Some basic Laplace transforms

Some basic Laplace transforms:

f(t)	**F(s)**	**f(t)**	**F(s)**
$t^n \quad (n = 0, 1, 2 \ldots)$	$\frac{n!}{s^{n+1}}$	$t^n e^{at} \quad (n = 0, 1, 2 \ldots)$	$\frac{n!}{(s-a)^{n+1}}$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
$\sin bt$	$\frac{b}{s^2+b^2}$	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh bt$	$\frac{b}{s^2-b^2}$	$\cosh bt$	$\frac{s}{s^2-b^2}$
$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$	$t^{n-(1/2)} \ (n = 1, 2, \ldots)$	$\left(\frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2n-1}{2}\right) \frac{\sqrt{\pi}}{s^{n+(1/2)}}$

Theorem: t-derivative rule

Theorem (t-derivative rule)

Suppose that $f(t)$ is continuous and has exponential order, and that $f'(t)$ is continuous.

Then, for sufficiently large s,

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0).$$

Furthermore, if $f(t)$ has exponential order α, so that

$$|f(t)| < e^{\alpha t} \quad \text{for all sufficiently large } t$$

then $\mathcal{L}\{f'(t)\}(s)$ is defined for all $s > \alpha$.

Transforms of second-order derivatives

Example. Suppose $f(t)$ and $f'(t)$ are continuous and both have exponential order α, and that $f''(t)$ continuous. Then

$$\mathcal{L}\{f''(t)\}(s) = s^2F(s) - sf(0) - f'(0) \quad \text{where } s > \alpha$$

Proof. Apply the previous theorem twice

$$\begin{aligned} \mathcal{L}\{f''(t)\}(s) &= s\mathcal{L}\{f'(t)\}(s) - f'(0) \\ &= s(sF(s) - f(0)) - f'(0) \\ &= s^2F(s) - sf(0) - f'(0). \end{aligned}$$

Note. See the Corollary in Section 7.2 for a generalization to higher-order derivatives.

Example

Example 1. Apply the Laplace transform to the following IVP.

$$x'' + 16x = 0; \qquad x(0) = 5, x'(0) = 4.$$

Apply the transform; use linearity to distribute throughout:

$$\mathcal{L}\{x''\}(s) + 16X(s) = 0$$

Compute the transform of the second-order derivative

$$\mathcal{L}\{x''\}(s) = s^2X(s) - sx(0) - x'(0) = s^2X(s) - 5s - 2$$

Substitute

$$s^2X(s) - 5s - 4 + 16X(s) = 0$$

Simplify and solve for $X(s)$

$$\begin{aligned}(s^2+16)X(s) &= 5s+4\\ X(s) &= \frac{5s+4}{s^2+16}\end{aligned}$$

Recovering solutions

We started with a differential equation in the t-domain

$$x'' + 16x = 0; \qquad x(0) = 5, x'(0) = 4.$$

and found a solution (using only simple algebra) in the s-domain

$$X(s) = \frac{5s+4}{s^2+16}.$$

However, we need to be able to recover a solution $x(t)$ in the t-domain from $X(s)$ in the s-domain.

This requires an inverse Laplace transformation:

$$\mathcal{L}^{-1}\{X(s)\}(t) = x(t).$$

Inverse Laplace transforms

We cannot expect to recover a solution $x(t)$ from its transform $X(s)$ exactly, unless it is continuous.

(Reason: $X(s)$ is an integral, so altering some of the values of $x(t)$ won't change $X(s)$).

Theorem. If $F(s)$ is a Laplace transform of a continuous function $f(t)$, then $f(t)$ is unique:

if $F(s) = \mathcal{L}\{g(t)\}(s)$ and g is continuous then $f = g$.

Definition. If $F(s) = \mathcal{L}\{f(t)\}(s)$ and f is continuous, then we write

$$\mathcal{L}^{-1}\{F(s)\}(t) = f(t).$$

Example

Example. Compute $\mathcal{L}^{-1}\{\frac{s}{s^2+4^2}\}$.

Answer. Look-up in your table:

$$\mathcal{L}^{-1}\{\frac{s}{s^2+4^2}\}(t) = \cos 4t$$

Example. Compute $\mathcal{L}^{-1}\{\frac{4}{s^2+4^2}\}$.

Answer. Look-up in your table:

$$\mathcal{L}^{-1}\{\frac{4}{s^2+4^2}\}(t) = \sin 4t$$

Finding inverse transforms

Finding the inverse transform is an art (just like computing integrals).

- Ultimately, the only way to find the inverse of a Laplace transform $F(s)$ is to look-up a known inverse in a transform table (such as the back cover of the text).
- We often have to massage $F(s)$ using general techniques such as linearity which justifies carving $F(s)$ into parts.

Theorem. (Linearity of Inverse transform)
For any constants a, b,

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}.$$

Proof. Since $\mathcal{L}$ is linear, this relation is forced.

Example

Example 1 continued. Find the inverse transform of $X(s)$ where

$$X(s) = \frac{5s+4}{s^2+16}$$

We need to be able to recognize known transforms. Simplify:

$$X(s) = 5\frac{s}{s^2+4^2} + \frac{4}{s^2+4^2}$$

Look-up in a table

$$\mathcal{L}^{-1}\{\frac{s}{s^2+4^2}\}(t) = \cos 4t \qquad \mathcal{L}^{-1}\{\frac{4}{s^2+4^2}\}(t) = \sin 4t$$

Use the linearity of the inverse transform:

$$\mathcal{L}^{-1}\{X(s)\}(t) = 5\cos 4t + \sin 4t.$$

Example 1 continued

Example 1 continued. Solve

$$x'' + 16x = 0; \qquad x(0) = 5, x'(0) = 4.$$

Step 1. Convert to an algebraic equation in the s-domain

$$s^2X(s) - 5s - 4 + 16X(s) = 0$$

Step 2. Solve for the unknown transform $X(s)$

$$X(s) = \frac{5s+4}{s^2+16}$$

Step 3. Apply inverse transform to recover a solution in the t-domain

$$x(t) = 5\cos 4t + \sin 4t.$$

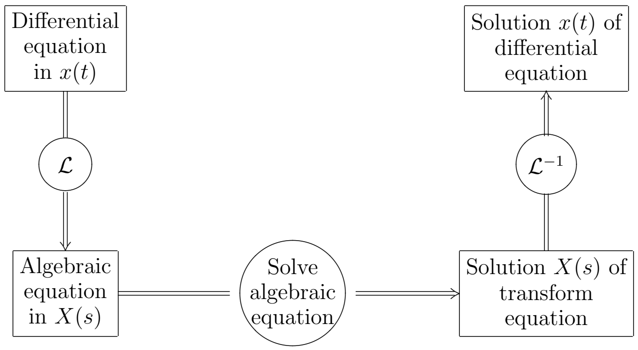
Method

To solve an initial value problem:

1. **Apply** the Laplace transform to both sides. Use the initial conditions to rewrite equation to an algebraic equation in an unknown transform.
2. **Solve** the algebraic equation for the unknown transform.
3. **Find** the inverse transform of the solution to obtain the solution to the differential equation.

   This may require massaging the transform into a linear combination of recognizable forms.

Laplace transform for solving IVPs



Example 2

Example 2. Solve

$$y'' - y = -t; \qquad y(0) = 0, y'(0) = 1.$$

Step 1. Apply the Laplace transform using linearity

$$\mathcal{L}\{y''\}(s) - Y(s) = -\mathcal{L}\{t\}(s) = -\frac{1}{s^2}.$$

Apply the t-derivative rule

$$\mathcal{L}\{y''\}(s) = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s) - 1$$

Substitute into $\mathcal{L}\{y''\}(s)$

$$s^2 Y(s) - 1 - Y(s) = (s^2 - 1)Y(s) - 1 = -\frac{1}{s^2}.$$

Step 2 Solve for the unknown transform $Y(s)$:

$$Y(s) = \frac{1}{s^2 - 1}\left(1 - \frac{1}{s^2}\right) = \frac{1}{s^2 - 1}\left(\frac{s^2 - 1}{s^2}\right) = \frac{1}{s^2}.$$

Example 2 continued

Step 3. Find the inverse transform of

$$Y(s) = \frac{1}{s^2}.$$

Look-up the inverse transform in a table:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = t$$

So,

$$y(t) = t$$

is the solution to the IVP

$$y'' - y = -t; \qquad y(0) = 0, y'(0) = 1.$$

Example 3

Example 3. Solve

$$x'' + 2x' + x = 3te^{-t}; \qquad x(0) = 4, x'(0) = 2.$$

Step 1. Apply the Laplace transform.

$$\mathcal{L}\{x''\} + 2\mathcal{L}\{x'\} + X(s) = \mathcal{L}\{3te^{-t}\} = \frac{3}{(s+1)^2}$$

Apply the t-derivative rule to compute transform of derivatives:

$$\begin{aligned} \mathcal{L}\{x''\} &= s^2 X(s) - 4s - 2 \\ \mathcal{L}\{x'\} &= sX(s) - 4 \end{aligned}$$

Substitute and gather $X(s)$ terms

$$s^2 X(s) + 2sX(s) + X(s) - 4s - 10 = \frac{3}{(s+1)^2}$$

Example 3 continued

Step 2. Solve for the transform $X(s)$.

$$X(s) = \frac{3}{(s+1)^4} + \frac{4s+10}{(s+1)^2}$$

Step 3. We need to do some massaging to find the inverse of $X(s)$:

$$X(s) = \frac{3}{(s+1)^4} + \frac{4(s+1)+6}{(s+1)^2}$$

or equivalently

$$X(s) = \frac{4}{s+1} + \frac{6}{(s+1)^2} + \frac{3}{(s+1)^4}$$

Example 3 continued

Continued. Our s-solution consists of the sum of:

$$\frac{4}{s+1} \quad \frac{6}{(s+1)^2} \quad \frac{3}{(s+1)^4}$$

We can use table look-up and s-translations (for appropriate a):

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Here, we want $n = 0, 1, 3$ and $a = -1$.

$$\mathcal{L}\{e^{-t}\} = \frac{1}{(s+1)} \quad \mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2} \quad \mathcal{L}\{t^3e^{-t}\} = \frac{6}{(s+1)^4}$$

Our t-solution is

$$x(t) = (4 + 6t + \frac{1}{2}t^3)e^{-t}$$

Example 4

Example 4. Solve

$$x' + 3x = e^{-t}; \qquad x(0) = 5.$$

Step 1. Apply the Laplace transform

$$sX(s) - 5 + 3X(s) = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

using the t-derivative rule and table look-up of $\mathcal{L}\{e^{at}\}$ with $a = -1$.

Step 2. Solve for $X(s)$.

$$\begin{aligned}(s+3)X(s) &= 5 + \frac{1}{s+1} \\ X(s) &= \frac{5}{s+3} + \frac{1}{(s+1)(s+3)}.\end{aligned}$$

Example 4 continued

Step 3. We need to massage the right-side into a linear combination of recognizable forms

$$X(s) = \frac{5}{s+3} + \frac{1}{(s+1)(s+3)}.$$

We will break-up the second term using partial fractions.

Partial Fractions. We want A and B such that

$$\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

The general method of cross multiplying and setting like terms equal:

$$1 = A(s+3) + B(s+1)$$

So, $0 = A + B$ and $1 = 3A + B$.

Cover-up method

Alternative method. Use the cover-up method to solve for A and B

$$1 = A(s+3) + B(s+1)$$

Set $s+1=0$ or $s=-1$

$$1 = A(-1+3)+0 \qquad A = \frac{1}{2}$$

Covers up occurrences of $(s+1)$ and eliminate unwanted coefficients.

Set $s+3=0$, so that $s=-3$:

$$1 = 0 + B(-3+1) \qquad B = -\frac{1}{2}$$

Note. The method will not always work.

Example 4 continued

Re-write the s-solution

$$\begin{aligned} X(s) &= \frac{5}{s+3} + \frac{1}{(s+1)(s+3)} \\ X(s) &= \frac{5}{s+3} + \frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}}{s+3} \\ X(s) &= \frac{\frac{9}{2}}{s+3} + \frac{\frac{1}{2}}{s+1}. \end{aligned}$$

Get the inverse transforms using linearity, table look-up, and s-translation

$$x(t) = \frac{9}{2}e^{-3t} + \frac{1}{2}e^{-t}.$$

$x(t)$ is the t-solution to

$$x' + 3x = e^{-t}; \qquad x(0) = 5.$$