Math 216
Differential Equations

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Improper integral

Recall the definition of the improper integral (taken over an infinite interval)

$$\int_a^\infty g(t)\,dt = \lim_{b\to\infty} \int_a^b g(t)\,dt.$$

Notation. We will also write the following abbreviation for the limit:

$$\Big[g(t)\Big]_a^\infty = \lim_{b\to\infty} \Big(g(b) - g(a)\Big)$$

Laplace transform

Definition

Let $f(t)$ be a function defined for all $t \geq 0$. The Laplace transform of f is the function F defined by the integral:

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

The domain of $F(s)$ is all values of s for which the integral exists.

The value s is a parameter, and you treat it like a constant in the right-side integral.

Note. It is possible to take the domain of s to be complex numbers, and sometimes this is convenient – see Examples 2 and 5 of Section 7.1.

Laplace transform

Note. If the Laplace transform of $f(t)$ is defined at a:

$$\mathcal{L}\{f(t)\}(a) = \int_0^\infty e^{-at} f(t)\,dt < \infty,$$

then $\mathcal{L}\{f(t)\}(s)$ is defined for all $s \geq a$.

It is a good idea to state the domain on which the Laplace transform is defined:

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-at} f(t)\,dt \qquad \text{where } s > a.$$

Laplace transform

Notation. The Laplace transform of f is written using a capital F:

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-at} f(t)\, dt \qquad \text{where } s > a.$$

The book often drops reference to "s": $\mathcal{L}\{f(t)\}$. This is still a function of s, not a function of t.

Examples. Let $x(t)$ and $g(t)$ be defined for $t \geq 0$.

- We write $X(s)$ for $\mathcal{L}\{x(t)\}(s)$.
- We write $G(s)$ for $\mathcal{L}\{g(t)\}(s)$.

Advantage Laplace transform

Advantage. The Laplace transform replaces linear constant coefficient differential equations in the t-domain by simpler algebraic equations in the s-domain.

Derivatives. If $\mathcal{L}\{x(t)\} = X(s)$, then $\mathcal{L}\{x'(t)\} = sX(s) - x(0)$.

Equations. We will transform a differential equation in the t-domain

$$x'(t) + 27x(t) = g(t) \qquad x(0) = x_0,$$

to an algebraic equation in the s-domain using the Laplace transform

$$sX(s) - x_0 + 27X(s) = G(s).$$

Solve for $X(s)$ and convert back to the t-domain to get a solution $x(t)$.

Example: e^{at}

Evaluate $\mathcal{L}\{e^{at}\}$ where $a \geq 0$.
Solution

$$\begin{aligned}
\mathcal{L}\{e^{at}\}(s) &= \int_0^\infty e^{-st} e^{at}\, dt \\
&= \int_0^\infty e^{at-st}\, dt \\
&= \int_0^\infty e^{(a-s)t}\, dt \\
&= \left[\frac{1}{a-s} e^{(a-s)t}\right]_0^\infty \\
&= \lim_{b\to\infty} \left(\frac{1}{a-s} e^{(a-s)b} - \frac{1}{a-s}\right)
\end{aligned}$$

If $s > a$, then $a - s < 0$ and $e^{(a-s)b} \to 0$ as $b \to \infty$.

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a} \qquad \text{where } s > a.$$

Integration by parts

Evaluate the following using integration by parts:

$$\int_0^\infty e^{-st} f(t)\, dt.$$

Answer.

$$\int_0^\infty e^{-st} f(t)\, dt = \left[-\frac{1}{s} e^{-st} f(t)\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f'(t)\, dt,$$

using the substitutions

- $u = f(t), \qquad du = f'(t)dt$
- $dv = e^{-st}dt, \qquad v = -\frac{1}{s}e^{-st}$.

Example: $\sin kt$, $\cos kt$

Evaluate. $\mathcal{L}\{\sin kt\}$ and $\mathcal{L}\{\cos kt\}$.
Answer. Use integration by parts.

$$\begin{aligned}
\mathcal{L}\{\sin kt\}(s) &= \left[-\frac{1}{s}e^{-st}\sin kt\right]_0^\infty + \frac{k}{s}\int_0^\infty e^{-st}\cos kt\,dt \\
&= \frac{k}{s}\mathcal{L}\{\cos kt\}(s) \\
\mathcal{L}\{\cos kt\}(s) &= \left[-\frac{1}{s}e^{-st}\cos kt\right]_0^\infty - \frac{k}{s}\int_0^\infty e^{-st}\sin kt\,dt \\
&= \frac{1}{s} - \frac{k}{s}\mathcal{L}\{\sin kt\}(s)
\end{aligned}$$

Solve for $\mathcal{L}\{\sin kt\}$ and $\mathcal{L}\{\cos kt\}$:

$$\begin{aligned}
\mathcal{L}\{\sin kt\}(s) &= \frac{k}{s^2+k^2} \qquad \text{where } s > 0 \\
\mathcal{L}\{\cos kt\}(s) &= \frac{s}{s^2+k^2} \qquad \text{where } s > 0
\end{aligned}$$

Example: 1, t

Evaluate. $\mathcal{L}\{1\}$, where 1 is the constant function.
Answer. $\mathcal{L}\{1\}(s) = \mathcal{L}\{e^{0t}\}(s) = \frac{1}{s-0} = \frac{1}{s}$ where $s > 0$.

Evaluate. $\mathcal{L}\{t\}$.
Answer. Use integration by parts

$$\begin{aligned}
\mathcal{L}\{t\}(s) &= \left[-\frac{1}{s}e^{-st}t\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\
&= \frac{1}{s}\mathcal{L}\{1\}(s) \\
&= \frac{1}{s^2} \qquad \text{where } s > 0.
\end{aligned}$$

Note that

$$\begin{aligned}
\left[-\frac{1}{s}e^{-st}t\right]_0^\infty &= \lim_{b\to\infty}\left(-\frac{1}{s}e^{-sb}b + 0\right) \\
&= \lim_{b\to\infty} -\frac{b}{se^{sb}} = 0.
\end{aligned}$$

Example: t^n

Evaluate. $\mathcal{L}\{t^n\}$, where $n > 0$ is an integer.
Answer. Let $n > 0$. Use integration by parts.

$$\begin{aligned}
\mathcal{L}\{t^n\}(s) &= \left[-\frac{1}{s}e^{-st}t^n\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt \\
&= \frac{n}{s}\mathcal{L}\{t^{n-1}\}(s)
\end{aligned}$$

since

$$\begin{aligned}
\left[-\frac{1}{s}e^{-st}t^n\right]_0^\infty &= \lim_{b\to\infty}\left(-\frac{1}{s}e^{-sb}b^n + 0\right) \\
&= \lim_{b\to\infty} -\frac{b^n}{se^{sb}} = 0.
\end{aligned}$$

So, when $n > 0$, we have

$$\mathcal{L}\{t^n\}(s) = \frac{n}{s}\mathcal{L}\{t^{n-1}\}(s)$$

Example: t^n

Evaluate. $\mathcal{L}\{t^n\}$, where $n > 0$ is an integer.
Continued. Let $n > 0$.

$$\begin{aligned}
\mathcal{L}\{t^n\}(s) &= \frac{n}{s}\mathcal{L}\{t^{n-1}\}(s) \\
&= \frac{n}{s}\left(\frac{n-1}{s}\right)\mathcal{L}\{t^{n-2}\}(s)\Big) \\
&= \frac{n\cdot(n-1)}{s^2}\mathcal{L}\{t^{n-2}\}(s) \\
&= \ldots \text{ continue for } n-2 \text{ more steps} \\
&= \frac{n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1}{s^n}\mathcal{L}\{1\}(s) \\
&= \frac{n!}{s^n}\left(\frac{1}{s}\right) \\
&= \frac{n!}{s^{n+1}} \qquad \text{where } s > 0.
\end{aligned}$$

Translation in s

Theorem (Translation in s)

Suppose $\mathcal{L}\{f(t)\}(s)$ exists for all $s > a$. Then

$$\mathcal{L}\{e^{\alpha t}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-\alpha) \qquad \text{where } s > \max\{a, \alpha\}$$

Proof.

$$\begin{aligned} \mathcal{L}\{e^{\alpha t}f(t)\}(s) &= \int_0^\infty e^{-st}e^{\alpha t}f(t)\,dt \\ &= \int_0^\infty e^{-(s-\alpha)t}f(t)\,dt \\ &= \mathcal{L}\{f(t)\}(s-\alpha) \end{aligned}$$

where $s > \max\{a, \alpha\}$.

Examples

Example. Compute $\mathcal{L}\{e^{\alpha t}\cos kt\}$ for $\alpha > 0$.
Answer. Recall, $\mathcal{L}\{\cos kt\}(s) = \frac{s}{s^2+k^2}$ where $s > 0$.

$$\begin{aligned} \mathcal{L}\{e^{\alpha t}\cos kt\}(s) &= \mathcal{L}\{\cos kt\}(s-\alpha) \\ &= \frac{s-\alpha}{(s-\alpha)^2+k^2} \quad \text{where } s > \alpha. \end{aligned}$$

Example. Compute $\mathcal{L}\{e^{\alpha t}\sin kt\}$ for $\alpha > 0$.
Answer. Recall, $\mathcal{L}\{\sin kt\}(s) = \frac{k}{s^2+k^2}$ where $s > 0$.

$$\begin{aligned} \mathcal{L}\{e^{\alpha t}\sin kt\}(s) &= \mathcal{L}\{\sin kt\}(s-\alpha) \\ &= \frac{k}{(s-\alpha)^2+k^2} \quad \text{where } s > \alpha. \end{aligned}$$

Linearity

Theorem (Linearity)

Suppose $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exists for all $s > a$. Let c, d be constants. Then

$$\mathcal{L}\{bf(t) + cg(t)\} = b\mathcal{L}\{f(t)\} + c\mathcal{L}\{g(t)\} \qquad \text{where } s > a$$

Proof. Use linearity of integration.

$$\begin{aligned} \mathcal{L}\{bf(t) + cg(t)\} &= \int_0^\infty e^{-st}(bf(t) + cg(t))\,dt \\ &= b\int_0^\infty e^{-st}f(t)\,dt + c\int_0^\infty e^{-st}g(t)\,dt \\ &= b\mathcal{L}\{f(t)\} + c\mathcal{L}\{g(t)\} \end{aligned}$$

where $s > a$.

Example

Example. Compute $\mathcal{L}\{11 + 5e^{4t} - 6\sin 2t\}$.
Answer. Use Linearity

$$\mathcal{L}\{11 + 5e^{4t} - 6\sin 2t\} = 11\mathcal{L}\{1\} + 5\mathcal{L}\{e^{4t}\} - 6\mathcal{L}\{\sin 2t\}.$$

Recall, for $s > 4$ (required for e^{4t}) we have

$$\mathcal{L}\{1\}(s) = \frac{1}{s} \quad \mathcal{L}\{e^{4t}\}(s) = \frac{1}{s-4} \quad \mathcal{L}\{\sin 2t\}(s) = \frac{2}{s^2+2^2}.$$

Substitute

$$\begin{aligned} \mathcal{L}\{11 + 5e^{4t} - 6\sin 2t\}(s) &= 11\left(\frac{1}{s}\right) + 5\left(\frac{1}{s-4}\right) - 6\left(\frac{2}{s^2+2^2}\right) \\ &= \frac{11}{s} + \frac{5}{s-4} - \frac{12}{s^2+2^2} \end{aligned}$$

where $s > 4$.

Definition

Definition

A function $f(t)$ is piecewise continuous on the finite interval $[a, b]$ (where $a < b$) if f is continuous at all except the finitely many points $\{a, x_1, \ldots, x_k, b\}$ where each $a < x_i < b$ and

- Both one-sided limits $\lim_{\epsilon \to 0^+} f(x_i - \epsilon)$ and $\lim_{\epsilon \to 0^+} f(x_i + \epsilon)$ exists and are finite for each i (but not necessarily equal!),
- The one-sided limit $\lim_{\epsilon \to 0^+} f(a + \epsilon)$ exists and is finite,
- The one-sided limit $\lim_{\epsilon \to 0^+} f(b - \epsilon)$ exists and is finite.

A function $f(t)$ is piecewise continuous on the infinite interval $[0, \infty)$ if $f(t)$ is piecewise continuous on the finite intervals $[0, N]$ for each $N > 0$.

Examples

Example. The following function is piecewise continuous on $[0, \infty)$:

$$f(t) = \begin{cases} t & 0 \leq t \leq 4 \\ 5 & t > 4 \end{cases}$$

Example. The unit staircase function is piecewise continuous on $[0, \infty)$:

$$g(t) = n \qquad \text{if} \quad n - 1 \leq t < n, \quad n = 1, 2, 3, \ldots$$

Example. The following function is NOT piecewise continuous on $[0, \infty)$:

$$h(t) = t^{-\frac{1}{2}}$$

because it blows-up at 0.

Example on piecewise continuous function

Compute $\mathcal{L}\{g(t)\}$ where

$$g(t) = \begin{cases} t & 0 \leq t \leq 4 \\ 5 & t > 4 \end{cases}$$

Answer. Split the integral at the discontinuity at $t = 4$:

$$\begin{aligned} \mathcal{L}\{g(t)\}(s) &= \int_0^\infty e^{-st} g(t)\, dt \\ &= \int_0^4 e^{-st} t\, dt + \int_4^\infty 5e^{-st}\, dt \end{aligned}$$

Use integration by parts on the first integral

$$\begin{aligned} \mathcal{L}\{g(t)\}(s) &= \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^4 + \left[-\frac{5}{s} e^{-st} \right]_4^\infty \\ &= \left(-\frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + 0 + \frac{1}{s^2} \right) + \frac{5e^{-4s}}{s} \\ &= \frac{1}{s^2} + \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2}. \end{aligned}$$

Integrals of piecewise continuous functions

Fact. If $f(t)$ is piecewise continuous on the finite interval $[a, b]$, then

$$\int_a^b e^{-st} f(t)\, dt$$

exists and is finite.

However, even if $f(t)$ is piecewise continuous on $[0, \infty)$,

$$\int_0^\infty e^{-st} f(t)\, dt$$

may not exists for any s.

We must be sure that $f(t)$ does not grow too fast.

Definition of exponential order

Definition

A function $f(t)$ is said to be of exponential order α (where $\alpha > 0$) if there exist positive constants T and M such that

$$|f(t)| < Me^{\alpha t} \qquad \text{for all } t > T.$$

We say $f(t)$ is of exponential order (without specifying α) if there is some $\alpha > 0$ such that $f(t)$ is of exponential order α.

Note. If $f(t)$ is of exponential order α, it will eventually settle down and grow no faster than $e^{\alpha t}$ (up to a constant factor).

Examples of exponential order

Example. $e^{\alpha t}$ (for $\alpha > 0$) is of exponential order α. Let $M = 2$, $T = 0$:

$$|e^{\alpha t}| < 2e^{\alpha t} \qquad \text{for all } t > 0.$$

Example. $e^{5t} \cos 2t$ is of exponential order 5. Let $M = 2$, $T = 0$:

$$|e^{\alpha t} \cos 2t| < 2e^{5t} \qquad \text{for all } t > 0.$$

Example. If $f(t) < M$ for all $t > 0$, then $e^{\alpha t} f(t)$ is of exponential order α.

$$|e^{\alpha t} f(t)| < Me^{\alpha t} \qquad \text{for all } t > 0.$$

Determining exponential order

The following is useful for determining when functions have exponential order:

Theorem

A function $f(t)$ is of exponential order if and only if there is a constant $\alpha > 0$ such that

$$\lim_{t\to\infty} \frac{|f(t)|}{e^{\alpha t}} < \infty$$

(The limit must exist and be finite.)

If the above condition is met, $f(t)$ has exponential order α.

L'ôpital's Theorem

L'ôpital's Theorem. Let $f(t)$ and $g(t)$ be differentiable functions such that $\lim f(t)$ and $\lim g(t)$ are either both zero or both $\pm\infty$. (The bounds on the limits $t \to b$ must be the same, where b can be any real value or $\pm\infty$.)

If

$$\lim \frac{f'(t)}{g'(t)} \quad \text{exists}$$

(it could be finite or infinite), then

$$\lim \frac{f(t)}{g(t)} = \lim \frac{f'(t)}{g'(t)}.$$

Example

Example. For any integer n, t^n is of exponential order.

Let $\alpha = 1$ (any $\alpha > 0$ will do). Then

$$\lim_{t\to\infty} \frac{t^n}{e^{\alpha t}} = 0.$$

To see this we can apply L'Hôpital's theorem n times:

$$\begin{aligned}\lim_{t\to\infty} \frac{t^n}{e^{\alpha t}} &= \lim_{t\to\infty} \frac{nt^{n-1}}{\alpha e^{\alpha t}} \\ &= \lim_{t\to\infty} \frac{n(n-1)t^{n-2}}{\alpha^2 e^{\alpha t}} \\ &= \dots \text{ apply L'Hôpital } n-2 \text{ more times} \\ &= \lim_{t\to\infty} \frac{n!}{\alpha^n e^{\alpha t}} = 0.\end{aligned}$$

Example

Example. e^{t^2} is NOT of exponential order.

Let $\alpha > 0$. Then

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{\alpha t}} = \lim_{t\to\infty} e^{t^2-\alpha t} = \infty$$

since $(t^2 - \alpha t) \to \infty$ as $t \to \infty$.

Existence theorem

Theorem

If $f(t)$ satisfies the following

(i) *$f(t)$ is piecewise continuous on $[0,\infty)$, and*

(ii) *$f(t)$ is of exponential order α,*

then $\mathcal{L}\{f(t)\}(s)$ exists for all $\alpha > s$.

Furthermore, if conditions (i) and (ii) are met, then

$$\lim_{s\to\infty} \mathcal{L}\{f(t)\}(s) = 0.$$

In fact, under these conditions,

$$\lim_{s\to\infty} s \cdot \mathcal{L}\{f(t)\}(s) < \infty.$$

The proof is given in Section 7.1 of the text, where this is stated as Theorem 2 and its Corollary.

Conditions not necessary

The two conditions of the Existence Theorem

(i) $f(t)$ is piecewise continuous on $[0,\infty)$, and

(ii) $f(t)$ is of exponential order α,

are not necessary for $\mathcal{L}\{f(t)\}$ to exist for sufficiently large s.

Example (i). $\mathcal{L}\{t^{-\frac{1}{2}}\}(s) = \sqrt{\frac{\pi}{s}}$, but the function $t^{-\frac{1}{2}}$ is not piecewise continuous on $[0,\infty)$. Note that

$$\lim_{s\to\infty} s \cdot \mathcal{L}\{t^{-\frac{1}{2}}\}(s) = \lim_{s\to\infty} \sqrt{s\pi} = \infty.$$

Example (ii). Let $f(t) = e^{t^2} \cos e^{t^2}$. We will see on Friday that $\mathcal{L}\{f(t)\}$ exists. You can verify that it is not of exponential order.

Show $\mathcal{L}\{t^{-\frac{1}{2}}\}(s) = \sqrt{\frac{\pi}{s}}$ for $s > 0$.
Answer. By definition

$$\mathcal{L}\{t^{-\frac{1}{2}}\}(s) = \int_0^\infty e^{-st} t^{-\frac{1}{2}}\, dt.$$

Use change of variables with $st = x^2$, so

$$\mathcal{L}\{t^{-\frac{1}{2}}\}(s) = 2s^{-\frac{1}{2}} \int_0^\infty e^{-x^2}\, dx$$

However, from Calculus III, the Gaussian integral can be evaluated

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

So,

$$\begin{aligned} \mathcal{L}\{t^{-\frac{1}{2}}\}(s) &= 2s^{-\frac{1}{2}} \cdot \frac{\sqrt{\pi}}{2} \\ &= \sqrt{\frac{\pi}{s}}. \end{aligned}$$