Math 216
Differential Equations

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Old mass-spring systems

Old model. We have been assuming the the response of the spring was constant regardless of its elongation or compression:

$$mx'' + cx' + kx = F(t) \qquad \text{where } m, c, k > 0.$$

- $x(t)$ is the displacement from the equillibrium position,
- m is the mass on the spring,
- c is damping,
- k is the spring elasticity, and obeys Hooke's law $F_{spring} = -kx$
- F is the external force applied to the system.

Nonlinear elasticity

New model. In actual systems, there is a progressive stiffening or weakening of the spring as it is elongated or compressed. This is a nonlinear response.

We assume the spring reacts symmetrically, the simplest modification:

$$F_{spring} = -kx + \beta x^3$$

so, the nonlinear mass-spring system is

$$mx'' + cx' + kx - \beta x^3 = F(t) \qquad \text{where } m, c, k > 0.$$

- $\beta = 0$: original linear equation.
- $\beta < 0$: spring becomes increasingly stiffer as it is elongated or depressed (hard spring),
- $\beta > 0$: spring becomes increasingly less resilient as it is elongated or depressed (soft spring).

Nonlinear elasticity

Nonlinear model. We will consider nonlinear models without external forces:

$$mx'' + cx' + kx - \beta x^3 = 0$$

Normal Form. We convert this to a first-order system by introducing velocity, $y = y(t)$:

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\frac{c}{m}y - \frac{k}{m}x + \frac{\beta}{m}x^3 \end{aligned}$$

Linearization

Nonlinear model. We start without damping:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\frac{k}{m}x + \frac{\beta}{m}x^3 = \frac{x}{m}(\beta x^2 - k)$$

Critical points. $(0, 0)$ is a critical point.

- $\beta < 0$ (hard spring): $(0, 0)$ is the only critical point.
- $\beta > 0$ (soft spring): $(\pm\sqrt{\frac{k}{\beta}}, 0)$ are also critical points.

Linearization. The Jacobian is

$$\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} + \frac{3\beta}{m}x^2 & 0 \end{bmatrix}$$

Critical point at origin

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{k}{m}x + \frac{\beta}{m}x^3$$

Linearization at $(0, 0)$.

$$\mathbf{J}(x, y) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} + \frac{3\beta}{m}x^2 & 0 \end{bmatrix} \quad \mathbf{J}(0, 0) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}$$

The characteristic equation is $\lambda^2 + \omega^2$, where $\omega^2 = \frac{k}{m}$ (natural frequency). The eigenvalues are $\lambda = \pm\omega i$.

Analysis. The linearization has a stable center at the origin. The critical point of the nonlinear system at $(0, 0)$ could be either a stable center or spiral point (stable or unstable). We need to analyze solutions to obtain further information.

Solutions at origin

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{k}{m}x + \frac{\beta}{m}x^3$$

Solutions. We use the chain rule to rewrite as a first order system (with independent variable x and dependent variable y)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{k}{m}x + \frac{\beta}{m}x^3}{y}.$$

Separating variables

$$my\,dy = -kx + \beta x^3\,dx.$$

Integrating

$$\frac{1}{2}my^2 + \frac{1}{2}kx^2 - \frac{1}{4}\beta x^4 = E$$

where E is a an arbitrary constant of integration.

Conservation of energy

Position-velocity solutions to

$$mx'' + kx - \beta x^3 = 0$$

satisfy the conservation of energy

$$\frac{1}{2}my^2 + \frac{1}{2}kx^2 - \frac{1}{4}\beta x^4 = E$$

where E is the total energy,

- Kinetic energy (due to inertia) is $\frac{1}{2}my^2$,
- Potential energy (due to spring potential) is $\frac{1}{2}kx^2 - \frac{1}{4}\beta x^4$

Analysis. Position-velocity trajectories are along paths of constant energy.

Hard spring

Hard springs get stiffer as they are stretched or compressed:

$$F_{spring} = -kx - |\beta|x^3 \qquad \text{where } \beta < 0.$$

The nonlinear undamped mass-spring system is

$$mx'' + kx + |\beta|x^3 = 0 \qquad \text{where } m, c, k > 0.$$

As a system of equations:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{k}{m}x + \frac{|\beta|}{m}x^3$$

Critical point. The only critical point is at $(0,0)$, which is a center in the linearization.

Trajectories

Solutions obey the conservation of energy (where x is position and y is velocity)

$$\frac{1}{2}my^2 + \frac{1}{2}kx^2 - \frac{1}{4}\beta x^4 = E$$

Like the linear system, these solutions are periodic. (Recall, the linear system is when $\beta = 0$.)

Trajectories in the Position-Velocity plane are similar to the linear system with two exceptions

1. Trajectories are flatter quartic ovals (not quadratic ellipses),
2. The period and amplitude of the nonlinear system depends on the initial position x_0 and initial velocity y_0. (In the linear system, the period is $T = 2\pi\sqrt{\frac{k}{m}}$, so independent of the initial conditions.)

Example

Example. Hard spring: $\beta = -8$, $m, k = 2$:

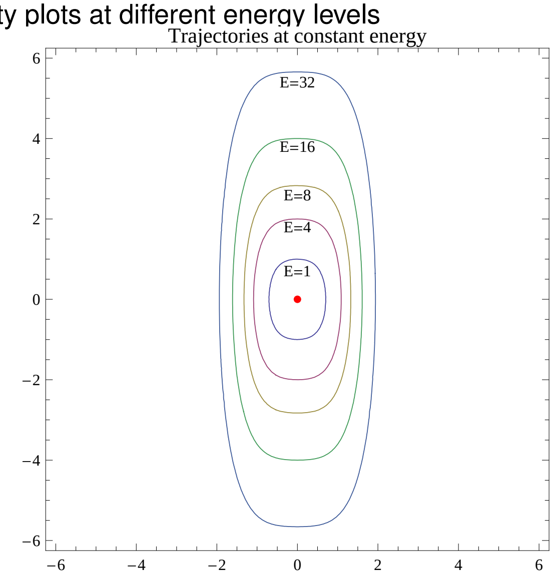
$$2x'' + 2x + 8x^3 = 0.$$

Solution. Position x and velocity y solutions satisfy constant energy E:

$$y^2 + x^2 + 2x^4 = E$$

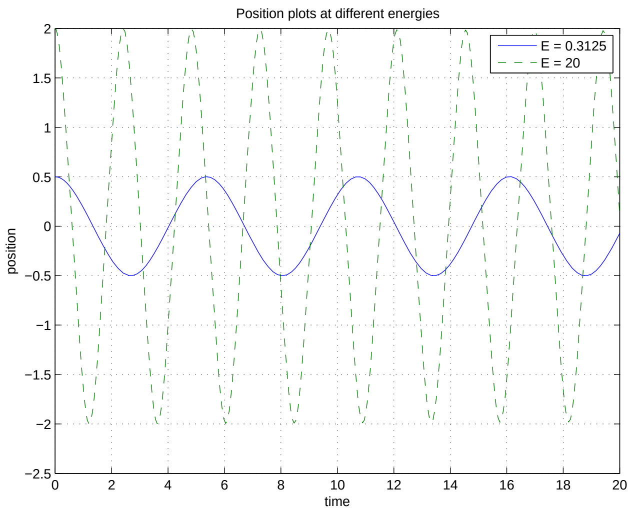
Position-Velocity plots

Position-Velocity plots at different energy levels



Trajectories

Plot of position at different energies: 0.3125 and 20.
Period decreases, amplitude increases at higher energy.



Soft spring

Soft springs weaken as they are stretched or compressed:

$$F_{spring} = -kx + \beta x^3 \quad \text{where } \beta > 0,$$

so, the nonlinear undamped mass-spring system is

$$mx'' + kx - \beta x^3 = 0 \quad \text{where } m, c, k, \beta > 0.$$

The behavior of the spring depends on displacement x.

- When $kx > \beta x^3$ spring's force is directed toward equillibrium position.
- When $kx = \beta x^3$ spring exerts no force.
- When $kx < \beta x^3$ spring's force repulses from equillibrium position.

Soft spring

Soft spring model

$$mx'' + kx - \beta x^3 = 0 \quad \text{where } m, c, k, \beta > 0.$$

As a system of equations:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{k}{m}x + \frac{|\beta|}{m}x^3 = -\frac{x}{m}(k - \beta x^2)$$

Critical point. There are critical points at $(0, 0)$ (a center in the linearization) and $(\pm\sqrt{\frac{k}{\beta}}, 0)$.

Trajectories obey the conservation of energy

$$\frac{1}{2}my^2 - \frac{1}{2}kx^2 + \frac{1}{4}\beta x^4 = E$$

Analysis of other critical points

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{k}{m}x + \frac{\beta}{m}x^3$$

Linearization at $(\pm\sqrt{\frac{k}{\beta}}, 0)$.

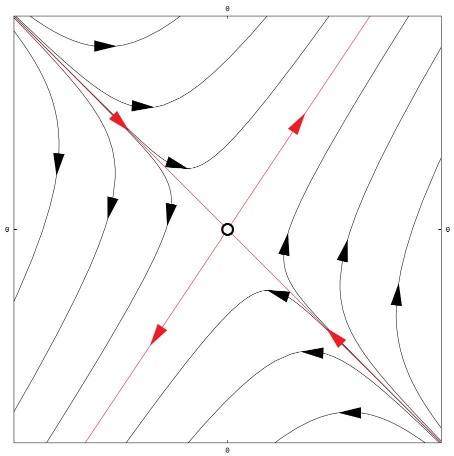
$$\mathbf{J}(x, y) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} + \frac{3\beta}{m}x^2 & 0 \end{bmatrix} \quad \mathbf{J}(\pm\sqrt{\frac{k}{\beta}}, 0) = \begin{bmatrix} 0 & 1 \\ \frac{2k}{m} & 0 \end{bmatrix}$$

The characteristic equation is $\lambda^2 - \frac{2k}{m}$:
there are two real eigenvalues of opposite signs.

Analysis. The nonlinear system has saddlepoints at $(\pm\sqrt{\frac{k}{\beta}}, 0)$.

Saddlepoints

Linearization. Eigenvectors correspond to the lines which separate different behavior of the system. These lines separate different behavior in trajectories.



Separatices

Separatrices are the nonlinear deformations of the eigenvectors for the saddlepoint in the linearization.

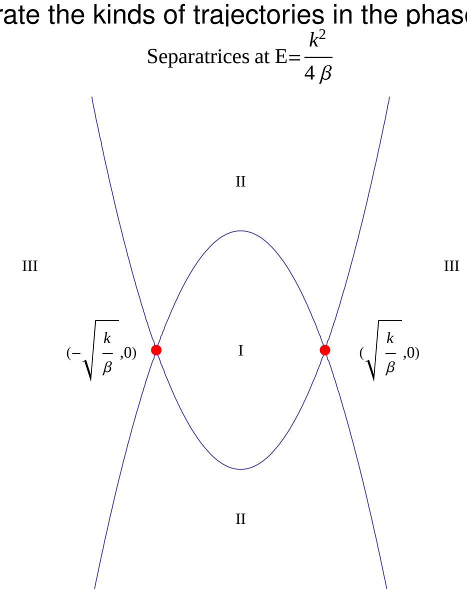
They occur at the same energy level as the critical points $(\pm\sqrt{\frac{k}{\beta}}, 0)$.

$$E = \frac{1}{2}\frac{k^2}{\beta} - \frac{1}{4}\frac{k^2}{\beta} = \frac{1}{4}\frac{k^2}{\beta}$$

The separatrices separate different kinds of behavior of trajectories in the nonlinear system.

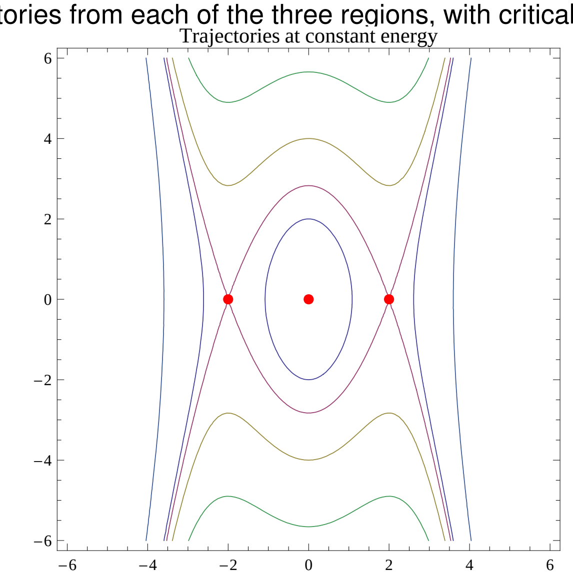
Separatrices

Separatrices separate the kinds of trajectories in the phase plane .



Trajectories

Several trajectories from each of the three regions, with critical points.



Separatrices: Three regions

There are three regions of behavior determined by the separatrices.

I Periodic. Trajectories where $0 \leq E < 8$. Mass oscillates around central equillibrium.

II Unbounded. Trajectories where $E > 8$. Velocity is too great and carries mass beyond the spring threshold. (In physical applications: spring breaks!!)

III Unbounded. Trajectories where $E < 0$. Physically impossible in mass-spring. Mass too far away from central equillibrium point and spring acts repulsively.

Note. The critical points $(\pm\sqrt{\frac{k}{\beta}}, 0)$ occur where the mass is sufficiently far that the spring exerts no force on the mass. Beyond this distance spring repulses the mass (no physical analogy with mass-spring).

Example

Example. Soft spring: $\beta = 2$, $m = 2, k = 8$:

$$2x'' + 8x - 2x^3 = 0.$$

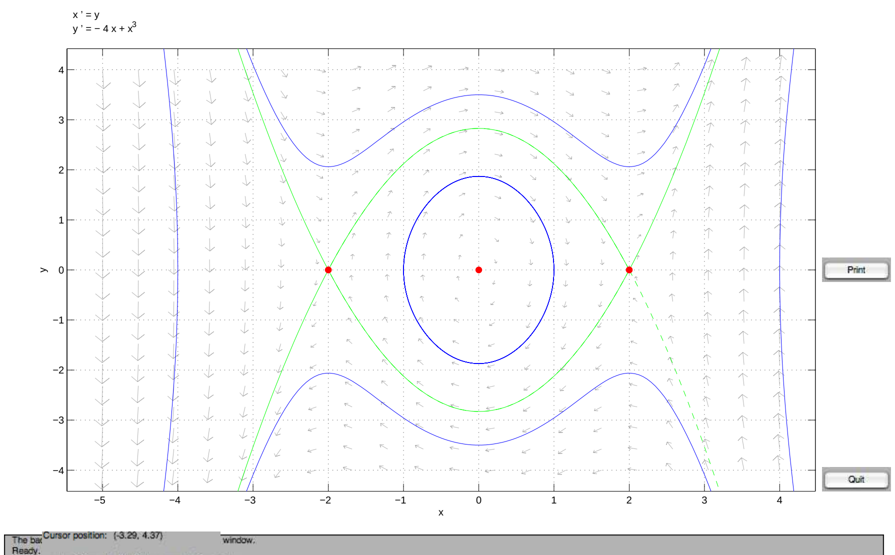
Critical points. $(0, 0)$, $(2, 0)$, $(-2, 0)$

Solution. Position x and velocity y at constant energy E:

$$y^2 + 4x^2 - \frac{1}{2}x^4 = E$$

Differential fields with separatices

Separatrices plotted on with critical points



Amplitude and period

Periodic behavior. At sufficiently low energy, trajectories are periodic. Like the hard spring model, both amplitude and period depend upon the initial position and velocity.

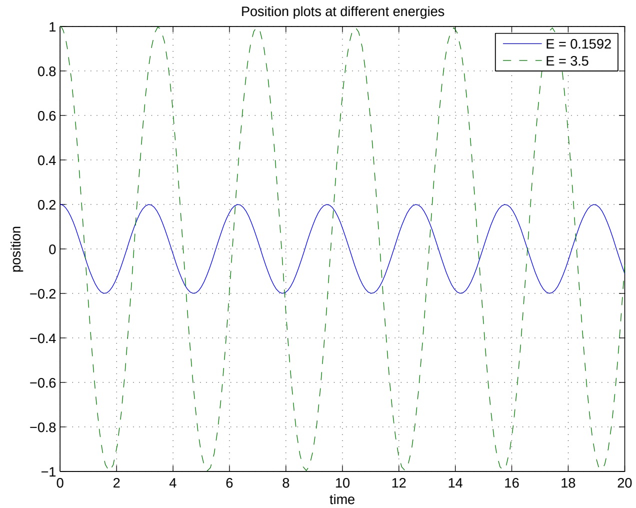
However,

- Amplitude increases at higher energy.
- Period decreases at higher energy.

This makes sense: when the mass is farther from the equillibrium the spring's force is weaker.

Trajectories

Plot of position at different energies: 0.1592 and 3.5.
Period increases, amplitude increases at higher energy.



Example

Example.

$$\begin{aligned} x' &= 4xy \\ y' &= x^2 - y^2 \end{aligned}$$

Critical points. The only critical point is $(0, 0)$.

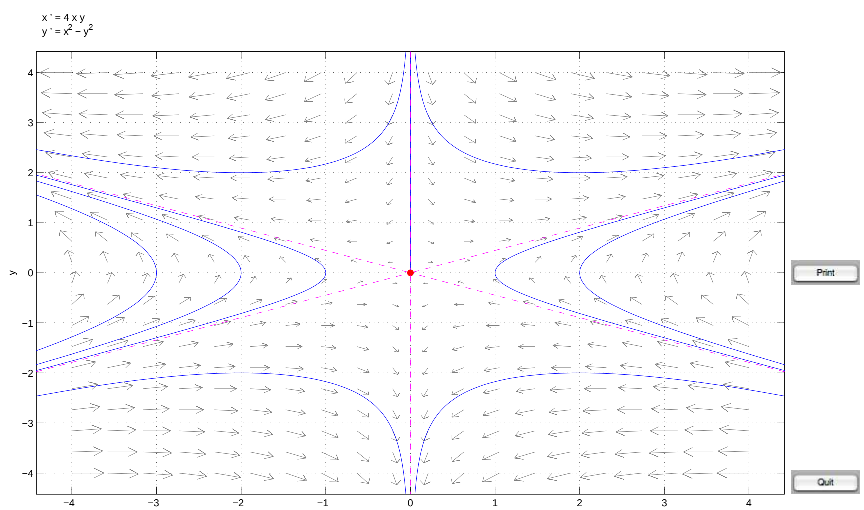
Jacobian.

$$\mathbf{J}(x, y) = \begin{bmatrix} 4y & 4x \\ 2x & -2y \end{bmatrix} \qquad \mathbf{J}(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We cannot use the linearization to get any information about the critical point $(0, 0)$, since $\det(\mathbf{J}(0, 0)) = 0$ and so $\mathbf{J}(0, 0)$ is not invertible.

Phase plot

Phase plot shows six solutions lying on rays. (The mauve-colored dashed lines.) These are separatrices. Linear systems have at most four.



Ray solutions

Lets look at the ray solutions of

$$x' = 4xy, \quad y' = x^2 - y^2$$

Case 1. If $x = 0$ then $x' = 0$ (so, x never changes!) and $y' = -y^2$.
We can solve $y' = -y^2$ using separation of variables:

$$y(t) = \frac{1}{t - c}$$

where c is an arbitrary constant.

Analysis. One kind of trajectory lives on the y-axis:

$$x(t) = 0, \quad y(t) = \frac{1}{t - c}.$$

Ray solutions

Lets look at the ray solutions of

$$x' = 4xy, \quad y' = x^2 - y^2$$

Case 2. Suppose $y = mx$ and lets find m. So, $\frac{dy}{dx} = m$ and

$$\frac{dx}{dt} = 4mx^2, \quad \frac{dy}{dt} = (1-m)x^2$$

By the chain rule

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(1-m)x^2}{4mx^2} = \frac{1-m}{4m}$$

Thus, $5m^2 - 1 = 0$, and $m = \pm\frac{1}{\sqrt{5}}$.

Analysis. A second kind of trajectory lives on the lines with slope $m = \pm\frac{1}{\sqrt{5}}$.

Lorenz system

Example. In the late fifties the meteorologist Edward Lorenz came upon the following nonlinear autonomous three-dimensional system when modeling atmospheric turbulence beneath a thunderhead (where air is cooled from above and warmed from below).

$$\begin{aligned} x' &= -ax + ay \\ y' &= rx - y - xz \\ z' &= -bz + xy \end{aligned}$$

where a, b, r are positive constants.

Motivation. What drove Lorenz was the search for an equation which would model some of the unpredictable behavior which we normally associate with the weather.

Basic properties of the Lorenz system

Example. Fix $a, b > 0$ in the system

$$\begin{aligned} x' &= -ax + ay \\ y' &= rx - y - xz \\ z' &= -bz + xy \end{aligned}$$

Critical points. When $r > 1$, there are three critical points:

$$(0,0,0), (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1), (-\sqrt{b(r-1)}), -\sqrt{b(r-1)}, r-1)$$

The critical points are unstable for most values of r.

Trajectories stay close to the origin, regardless of where the initial point is chosen. They wrap crazily around the second two unstable critical points.

Basic properties of the Lorenz system

Example. $a = 10, b = \frac{8}{3}, r = 28$:

$$\begin{aligned} x' &= -10x + 10y \\ y' &= 28x - y - xz \\ z' &= -\frac{8}{3}z + xy \end{aligned}$$

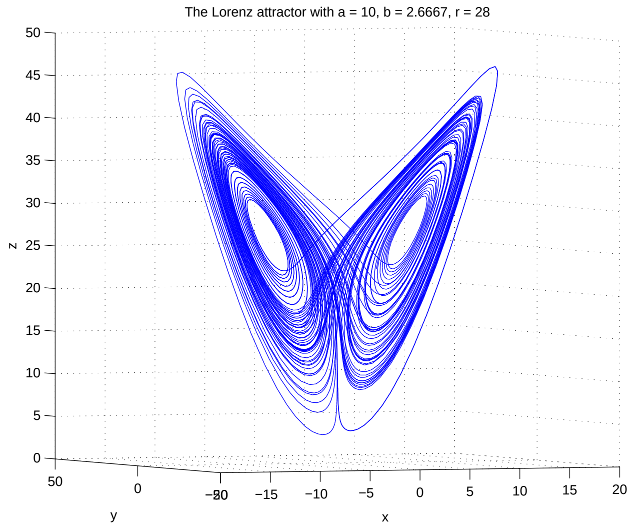
Trajectories. All trajectories eventually crazily orbit the unstable critical points:

$$(6\sqrt{2}, 6\sqrt{2}, 27), \quad (-6\sqrt{2}, -6\sqrt{2}, 27)$$

producing a butterfly-like pattern.

Phase plot

3-D Phase plot shows the butterfly-like trajectory.



Basic properties of the Lorenz system

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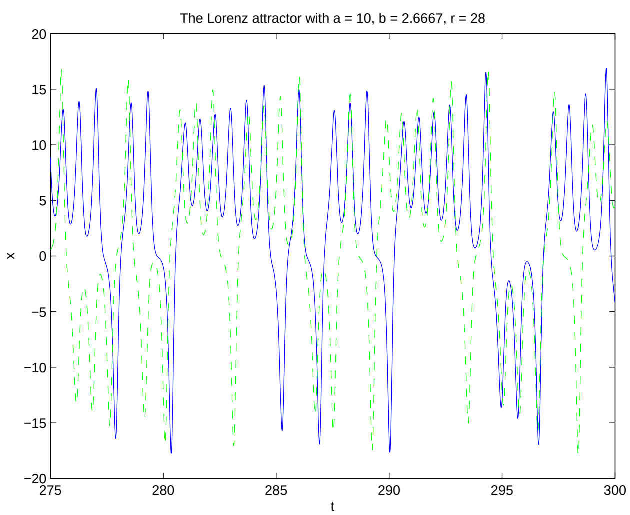
Chaos. Even very small perturbations of the starting position lead to dramatically different orbits. In fact, the disturbance is compounded exponentially, leading to the butterfly effect:

- A butterfly flapping its wings in Brazil can produce a tornado in Texas. (due to Lorenz)

This strong dependence of outcomes on very slightly differing initial conditions is a hallmark of the mathematical behavior known as chaos.

Time plot of x-coordinate

The initial value of the Dashed green trajectory differed by 0.0001 in the x-coordinate only. Note that the x-value is oscillating around $\pm 6\sqrt{2}$.



Time plot of z-coordinate

The initial value of the Dashed green trajectory differed by 0.0001 in the x-coordinate only. Note that the z-value is oscillating around 27.