Math 216
Differential Equations

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Linearization

Definition. Let F, G be continuously differentiable, and (a, b) a critical point of the system:

$$x' = F(x, y), \qquad y' = G(x, y).$$

The linearization at (a, b) is the system of linear equations

$$\begin{aligned} u' &= F_x(a,b)u + F_y(a,b)v \\ v' &= G_x(a,b)u + G_y(a,b)v, \end{aligned}$$

In matrix notation, the linearization is

$$\mathbf{u}' = \mathbf{J}(a,b)\mathbf{u} \quad \text{where } \mathbf{J}(a,b) = \begin{bmatrix} F_x(a,b) & F_y(a,b) \\ G_x(a,b) & G_y(a,b) \end{bmatrix}$$

and has a critical point at the origin.

Key theorem for linearization

Theorem

Suppose F, G are continuously differentiable, and (a, b) is a critical point (so, $F(a, b) = G(a, b) = 0$) where $\det(\mathbf{J}(a, b)) \neq 0$.

If the eigenvalues of $\mathbf{J}(a, b)$ are distinct and not imaginary, then the trajectories to the nonlinear system near (a, b)

$$x' = F(x, y), \quad y' = G(x, y)$$

look like slightly distorted versions of the trajectories to the linearization near $(0, 0)$

$$\mathbf{u}' = \mathbf{J}(a, b)\mathbf{u}$$

That is, the critical point (a, b) of the nonlinear system has the same stability and type as the critical point $(0, 0)$ in the linearization.

Computing eigenvalues of 2×2 matrices

The characteristic polynomial for the real-valued matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A})$ where

$$\mathrm{tr}(\mathbf{A}) = a + d \quad \text{and} \quad \det(\mathbf{A}) = ad - bc.$$

The eigenvalues are

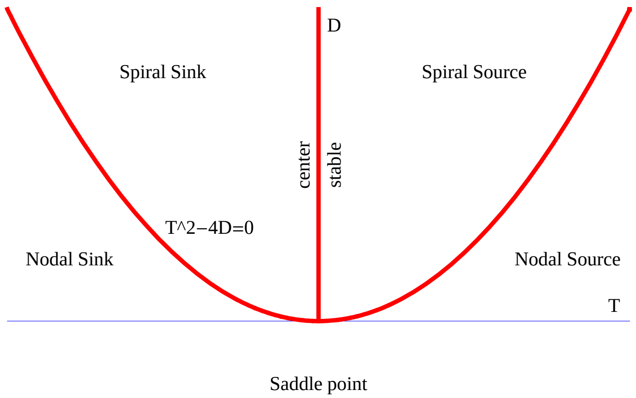
$$\frac{\mathrm{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}.$$

Summary

Distribution of critical points in the Trace-Determinant plane.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad T = \text{tr}(\mathbf{A}) = a + d, \quad D = \det(\mathbf{A}) = ad - bc.$$

Sensitive areas: Places where type of critical point sensitive to perturbations.



Logistic population model

Logistic growth. Recall the logistic population model for one species:

$$x' = \beta x - \delta x^2 \qquad \beta, \delta > 0,$$

where the birth rate is β and the death rate is δx.

Critical points. $0, \frac{\beta}{\delta}$ are the critical points,
0 is a source and $\frac{\beta}{\delta}$ is a sink.

Analysis. With no further interactions, the population will approach the stable population $\frac{\beta}{\delta}$. (See section 2.1).

General logistic population model with interaction

Consider two population $x(t), y(t)$ which interact.
Separate. Each population is modeled by the logistic equation:

$$\begin{aligned} \frac{dx}{dt} &= a_1 x - b_1 x^2 \\ \frac{dy}{dt} &= a_2 y - b_2 y^2 \end{aligned}$$

where $a_1, a_2, b_1, b_2 > 0$.

Interaction is proportional to the likelihood of a chance encounter, xy:

$$\begin{aligned} \frac{dx}{dt} &= a_1 x - b_1 x^2 - c_1 xy \\ \frac{dy}{dt} &= a_2 y - b_2 y^2 - c_2 xy \end{aligned}$$

where c_1, c_2 are nonzero real values.

Competition model with logistic growth

Competition Model. $c_1, c_2 > 0$ in the interaction model with logistic growth:

$$\begin{aligned} \frac{dx}{dt} &= a_1 x - b_1 x^2 - c_1 xy \\ \frac{dy}{dt} &= a_2 y - b_2 y^2 - c_2 xy \end{aligned}$$

Explanation. The two populations $x(t)$ and $y(t)$ are separately logistic populations (when no interaction occurs), but interaction hurts each population. They are in competition with each other.

Logistic population model

Cooperation Model. $c_1, c_2 < 0$ in the interaction model with logistic growth:

$$\begin{aligned}\frac{dx}{dt} &= a_1 x - b_1 x^2 - c_1 xy \\ \frac{dy}{dt} &= a_2 y - b_2 y^2 - c_2 xy\end{aligned}$$

Explanation. The two populations $x(t)$ and $y(t)$ are separately logistic populations (when no interaction occurs), but interaction helps each population.

Logistic population model

Predator-Prey Model. $c_2 < 0 < c_1$ in the interaction model with logistic growth:

$$\begin{aligned}\frac{dx}{dt} &= a_1 x - b_1 x^2 - c_1 xy \\ \frac{dy}{dt} &= a_2 y - b_2 y^2 - c_2 xy\end{aligned}$$

Explanation. $c_2 < 0 < c_1$. The two populations $x(t)$ and $y(t)$ are separately logistic populations (when no interaction occurs), but the interaction is one of predation.

- $x(t)$ is hurt by the interaction, and is the prey population.
- $y(t)$ is helped by the interaction, and is the predator population.

Example of logistic cooperation model

Equation. Consider the cooperation model for species x, y:

$$\begin{aligned}\frac{dx}{dt} &= 30x - 3x^2 + xy = x(30 - 3x + y) \\ \frac{dy}{dt} &= 60y - 3y^2 + 4xy = y(60 - 3y + 4x)\end{aligned}$$

Critical points. $(0,0), (0,20), (10,0), (30,60)$.

Jacobian.

$$\mathbf{J}(x,y) = \begin{bmatrix} 30 - 6x + y & x \\ 4y & 60 - 6y + 4x \end{bmatrix}$$

Qualitative properties of example

Jacobian. Critical point: $(0,0)$. The stable solution $x \equiv 0, y \equiv 0$ is one where both populations are extinct.

$$\mathbf{J}(x,y) = \begin{bmatrix} 30 - 6x + y & x \\ 4y & 60 - 6y + 4x \end{bmatrix} \quad \mathbf{J}(0,0) = \begin{bmatrix} 30 & 0 \\ 0 & 60 \end{bmatrix}$$

Eigenvalues. At $(0,0)$: $\lambda = 30, 60$

Analysis. $(0,0)$ is a nodal source in the linearization; so it is a nodal source in the nonlinear system.

Qualitative properties of example

Jacobian. Critical point: $(0, 20), (10, 0)$. The stable solutions where one of the species is extinct.

$$\mathbf{J}(x, y) = \begin{bmatrix} 30 - 6x + y & x \\ 4y & 60 - 6y + 4x \end{bmatrix}$$

$$\mathbf{J}(0, 20) = \begin{bmatrix} 50 & 0 \\ 80 & -60 \end{bmatrix} \quad \mathbf{J}(10, 0) = \begin{bmatrix} -30 & 10 \\ 0 & 100 \end{bmatrix}$$

Eigenvalues. At $(0, 20)$: $\lambda = 50, -60$, At $(10, 0)$: $\lambda = -30, 100$

Analysis. Both $(0, 20)$ and $(10, 0)$ are saddlepoints (all trajectories are repulsed except those where one population is extinct) in the linearization; so it is a saddlepoint in the nonlinear system.

Qualitative properties of example

Jacobian. Critical point: $(30, 60)$. The only stable solution where the populations coexist: $x \equiv 30, y \equiv 60$.

$$\mathbf{J}(x, y) = \begin{bmatrix} 30 - 6x + y & x \\ 4y & 60 - 6y + 4x \end{bmatrix} \quad \mathbf{J}(30, 60) = \begin{bmatrix} -90 & 30 \\ 240 & -180 \end{bmatrix}$$

We can determine stability by computing the trace (T) and determinant (D):

$$T = -90 - 180 = -270 \quad D = (-90)(-180) - (30)(240) = 9000 \quad T^2 - 4D = 36900$$

Since discriminant is positive ($T^2 - 4D > 0$), $T < 0$ and $D > 0$, the eigenvalues are real and negative.

Analysis. $(30, 60)$ is a nodal sink in the linearization; so it is a nodal sink in the nonlinear system.

Analysis

Separately the populations, without interaction, would tend to logistic growth.

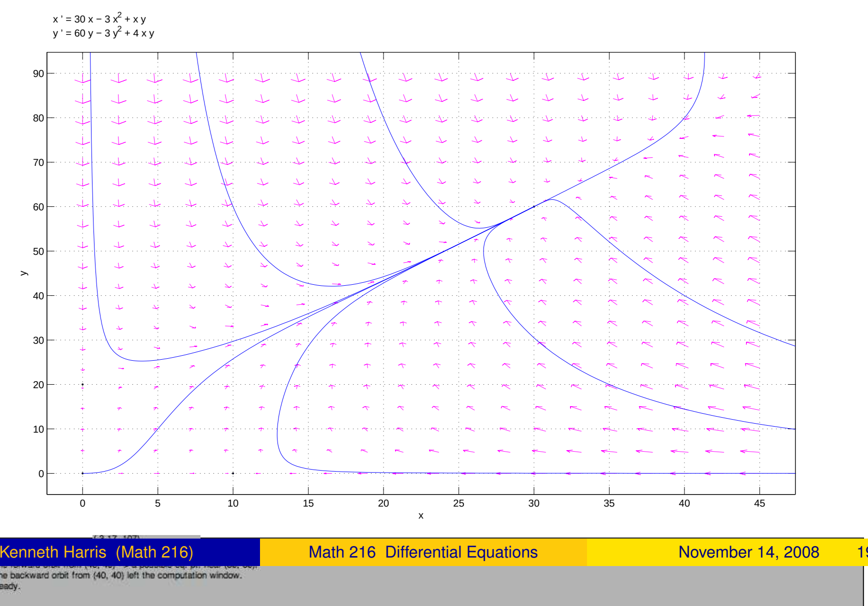
- $x(t) \to 10$ as $t \to \infty$
- $y(t) \to 20$ as $t \to \infty$

Interaction. Both populations are helped by the interaction

- $x(t) \to 30$ as $t \to \infty$
- $y(t) \to 60$ as $t \to \infty$

Direction field

Direction field. Trajectories are drawn to $(30, 60)$.



Example of logistic predator-prey model

Equation. Consider the logistic predator(y)-prey(x) model for two species:

$$\begin{aligned}\frac{dx}{dt} &= 30x - 2x^2 - xy = x(30 - 2x - y)\\ \frac{dy}{dt} &= 20y - 4y^2 + 2xy = y(20 - 4y + 2x)\end{aligned}$$

Critical points. $(0,0), (0,5), (15,0), (10,10)$.

Jacobian.

$$\mathbf{J}(x,y) = \begin{bmatrix} 30 - 4x - y & -x \\ 2y & 20 - 8y + 2x \end{bmatrix}$$

Qualitative properties of example

Jacobian. Critical point: $(0,0)$. The stable solution $x \equiv 0, y \equiv 0$ is one where both populations are extinct.

$$\mathbf{J}(x,y) = \begin{bmatrix} 30 - 4x - y & -x \\ 2y & 20 - 8y + 2x \end{bmatrix} \quad \mathbf{J}(0,0) = \begin{bmatrix} 30 & 0 \\ 20 & 0 \end{bmatrix}$$

Eigenvalues. At $(0,0)$: $\lambda = 30, 20$

Analysis. $(0,0)$ is a nodal source in the linearization; so it is a nodal source in the nonlinear system.

Qualitative properties of example

Jacobian. Critical point: $(0,5), (15,0)$. The stable solutions where one of the species is extinct.

$$\mathbf{J}(x,y) = \begin{bmatrix} 30 - 4x - y & -x \\ 2y & 20 - 8y + 2x \end{bmatrix}$$

$$\mathbf{J}(0,5) = \begin{bmatrix} 25 & 0 \\ 10 & -20 \end{bmatrix} \quad \mathbf{J}(15,0) = \begin{bmatrix} -30 & -15 \\ 0 & 110 \end{bmatrix}$$

Eigenvalues. At $(0,5)$: $\lambda = 25, -20$, At $(15,0)$: $\lambda = -30, 110$

Analysis. Both $(0,5)$ and $(15,0)$ are saddlepoints (all trajectories are repulsed except those where one population is extinct) in the linearization; so it is a saddlepoint in the nonlinear system.

Qualitative properties of example

Jacobian. Critical point: $(10,10)$. The only stable solution where the populations coexist: $x \equiv 10, y \equiv 10$.

$$\mathbf{J}(x,y) = \begin{bmatrix} 30 - 4x - y & -x \\ 2y & 20 - 8y + 2x \end{bmatrix} \quad \mathbf{J}(10,10) = \begin{bmatrix} -20 & -10 \\ 20 & -40 \end{bmatrix}$$

We can determine stability by computing the trace (T) and determinant (D):

$$T = -20 - 40 = -60 \quad D = (-20)(-40) - (-10)(20) = 1000 \quad T^2 - 4D = -400$$

Since discriminant is negative ($T^2 - 4D < 0$) and $T < 0$, the eigenvalues are complex with negative real component.

Analysis. $(10,10)$ is a spiral sink in the linearization; so it is a spiral sink in the nonlinear system.

Analysis

Separately the populations, without interaction, would tend to logistic growth.

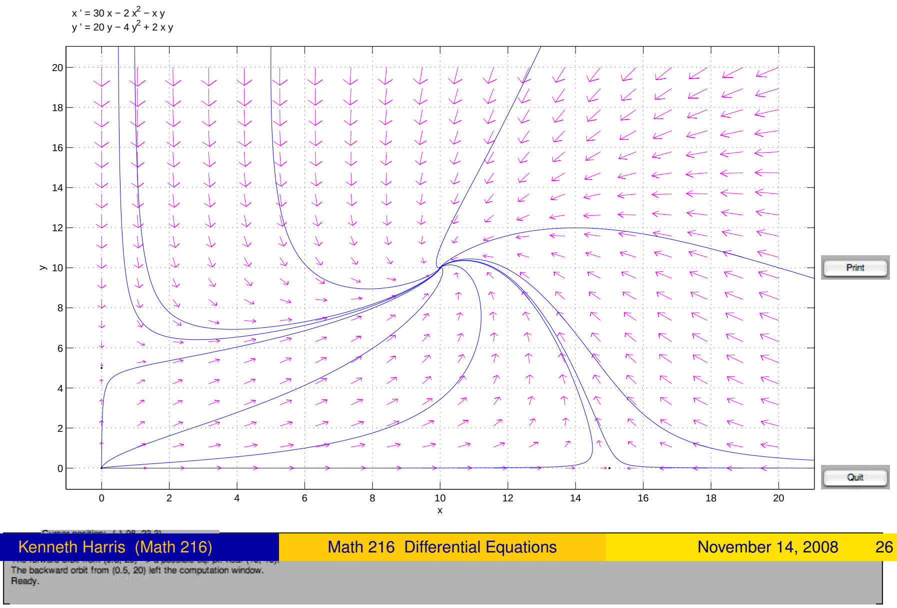
- $x(t) \to 15$ as $t \to \infty$
- $y(t) \to 5$ as $t \to \infty$

Interaction. Predators (y) are helped and prey (x) are hurt by the interaction

- $x(t) \to 10$ as $t \to \infty$
- $y(t) \to 10$ as $t \to \infty$

Direction field

Direction field. Trajectories are drawn to $(10, 10)$.



Method: Sketching trajectories

Sketching (in a qualitative way) solution curves for autonomous systems:

$$x' = F(x, y), \quad y' = G(x, y).$$

Step 1. Find all critical points (a, b) where $F(a, b) = G(a, b) = 0$.

Step 2. For each critical point (a, b), compute the linearization matrix

$$\begin{bmatrix} F_x(a,b) & F_y(a,b) \\ G_x(a,b) & G_y(a,b) \end{bmatrix}$$

and verify it is invertible.

Step 3. Determine the type and sign of the eigenvalues.

- If real: are they distinct? what are the signs?
- If complex: what is the sign of the real component?

Note: it is not necessary to determine the actual values of the eigenvalues.

Method: Sketching trajectories

- If the eigenvalues are distinct and non-imaginary then you can continue to **Step 4**. Otherwise, this is a borderline case, so the subsequent steps do not apply.

Step 4. Determine the stability and type of the critical point based on the type and sign of the eigenvalues in **Step 3**. We have either a spiral point, saddlepoint, or improper node in the linearization, and so in the nonautonomous system.

Step 5. In the xy-plane, mark the critical points. Around each, sketch the trajectories of the linearization, including the direction of motion.

Step 6. Sketch in some other trajectories to fill out the picture, making them compatible with the behavior of trajectories around critical points.

Borderline cases

The borderline cases occur when the linearization has imaginary eigenvalues, or one eigenvalue.

1. Imaginary eigenvalue: linearization has a center at origin; autonomous system could have a spiral sink or a spiral source.
2. One eigenvalue: linearization has a proper node (a star point) or an improper node. The autonomous system has the same stability properties, but could be a saddlepoint, sink node or source node.

Analysis. In a borderline case you generally must result to numerical computation. Sometimes you can provide an explicit or implicit solution to the first-order equation:

$$\frac{dy}{dx} = \frac{F(x,y)}{G(x,y)}.$$

Simple predator-prey model

General Model. We can look at alternative models by loosening the restriction that a_1, a_2, b_1, b_2 are postive in

$$\begin{aligned}\frac{dx}{dt} &= a_1 x - b_1 x^2 - c_1 xy \\ \frac{dy}{dt} &= a_2 y - b_2 y^2 - c_2 xy.\end{aligned}$$

For example, when $b_1 = b_2 = 0$, we have a natural growth model for the population.

- $x(t) = e^{a_1 t}$ without interaction,
- $y(t) = e^{a_2 t}$ without interaction.

Simple predator-prey model

Lotka-Volterra equations. This predator-prey model with natural growth was first investigated in the mid-twenties.

$$\begin{aligned}\frac{dx}{dt} &= a_1 x - c_1 xy = x(a_1 - c_1 y) \quad a_1, c_1 \geq 0 \\ \frac{dy}{dt} &= -a_2 y + c_2 xy = -y(a_2 - c_2 x) \quad a_2, c_2 \geq 0\end{aligned}$$

Assumptions.

1. Prey $(x(t))$ has an unlimited food supply and would grow at the natural growth rate $x' = a_1 x$ unless subject to predation.
2. Predator $(y(t))$ has no other food source than $x(t)$, so would starve at the natural growth rate $y' = -a_2 y$ unless prey present.
3. Rate of predation upon the prey is proportional to the rate at which the predators and the prey meet (xy). The interaction leads to
   - Decline in prey population $-c_1 xy$,
   - Increase in predator population $c_2 xy$.

Example of Lotka-Volterra equation

Equation.

$$\begin{aligned}\frac{dx}{dt} &= 4x - xy = x(4 - y) \\ \frac{dy}{dt} &= -16y + 2xy = -y(16 - 2x)\end{aligned}$$

Critical points. $(0,0)$, $(8,4)$.

Qualitative properties of example

$$\begin{aligned}\frac{dx}{dt} &= 4x - xy = x(4-y)\\ \frac{dy}{dt} &= -16y + 2xy = -y(16-2x)\end{aligned}$$

Jacobian. Critical point: $(0,0)$. The stable solution $x \equiv 0, y \equiv 0$ is one where both populations go extinct.

$$\mathbf{J}(x,y) = \begin{bmatrix} 4-y & -x \\ 2y & -16+2x \end{bmatrix} \quad \mathbf{J}(-1,1) = \begin{bmatrix} 4 & 0 \\ 0 & -16 \end{bmatrix}$$

Eigenvalues. At (0,0): $\lambda = 4, -16$

Analysis. $(0,0)$ is a unstable saddlepoint in the linearization; so it is an unstable saddlepoint in the autonomous system.

Qualitative properties of example

$$\begin{aligned}\frac{dx}{dt} &= 4x - xy = x(4-y)\\ \frac{dy}{dt} &= -16y + 2xy = -y(16-2x)\end{aligned}$$

Jacobian. Critical point: $(8,4)$. The stable solution $x \equiv 8, y \equiv 4$ is one where both populations coexist permanently.

$$\mathbf{J}(x,y) = \begin{bmatrix} 4-y & -x \\ 2y & -16+2x \end{bmatrix} \quad \mathbf{J}(8,4) = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$$

Eigenvalues. At (8,4): $\lambda = \pm 8i$

Analysis. $(8,4)$ is a stable center in the linearization; we can draw **no conclusions about stability** in the autonomous system:
it could be a stable center, stable spiral sink or unstable spiral source.

Finding implicit solutions

Since $(8,4)$ is a center, we cannot determine the stability of

$$\begin{aligned}\frac{dx}{dt} &= 4x - xy = x(4-y)\\ \frac{dy}{dt} &= -16y + 2xy = -y(16-2x)\end{aligned}$$

without looking at solutions.
By the Chain rule

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-y(16-2x)}{x(4-y)}$$

We want solutions x, y to this ordinary first order equation.

Finding implicit solutions

Solve.

$$\frac{dy}{dx} = \frac{-y(16-2x)}{x(4-y)}$$

Answer. Use separation of variables:

$$\frac{y-4}{y}dy = \frac{16-2x}{x}dx$$

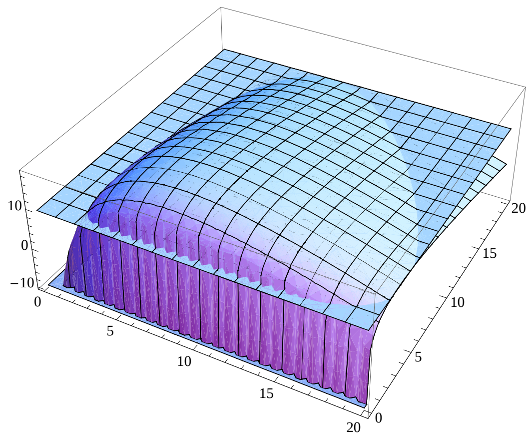
So, an implicit solution for x, y is given (for each constant C) by

$$y - 4\ln y + 2x - 16\ln x = C.$$

We can determine C from an initial value $x(0), y(0)$ (population at $t=0$).

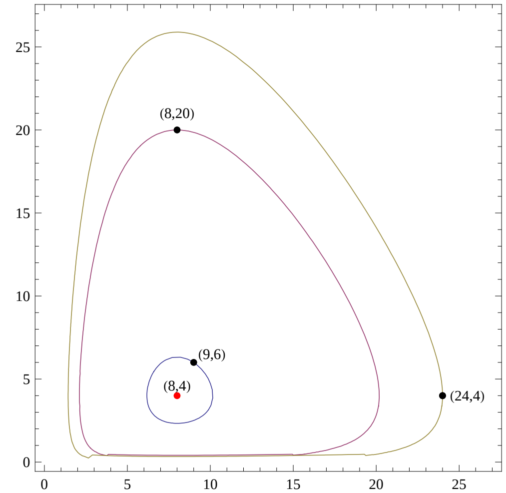
Three dimensional plot

Implicit plot of $y - 4\ln y + 2x - 16\ln x$. Solutions are planes $z = C$.
Here: $z = 9.139$ corresponding to $x(0) = 8, y(0) = 20$.



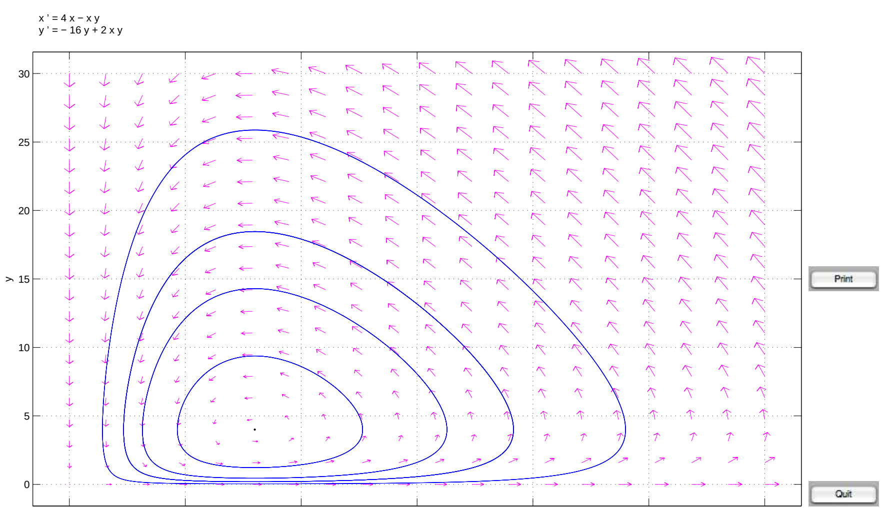
Three trajectories

Three trajectories with initial values: $(9,6), (8,20), (24,4)$.



Direction field

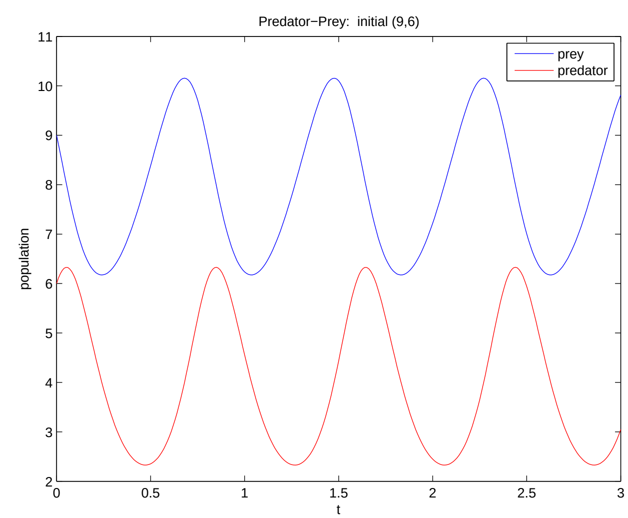
Direction field. I had to change the solver for `pplane` to Runge-Kutta and step size to 0.005 to get accurate renderings of trajectories.



Solution through $(9,6)$

Solution trajectories for $x(t), y(t)$ for initial populations: $x(0) = 9$, $y(0) = 6$.

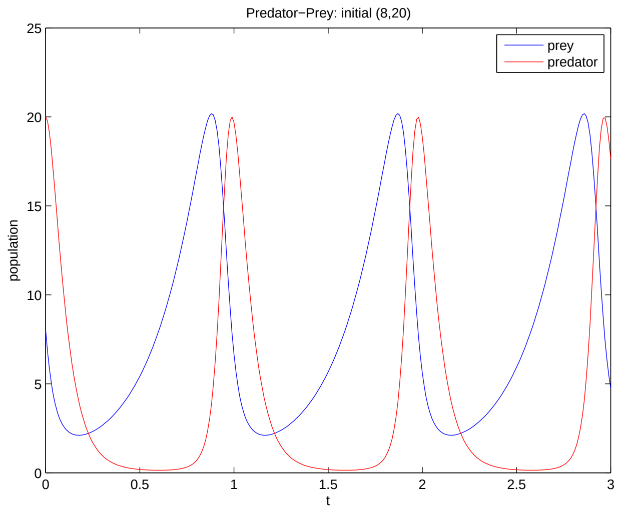
Generated using Runge-Kutta approximation, `rk2.m`.



Solution through $(8, 20)$

Solution trajectories for $x(t), y(t)$ for initial populations: $x(0) = 8$, $y(0) = 20$.

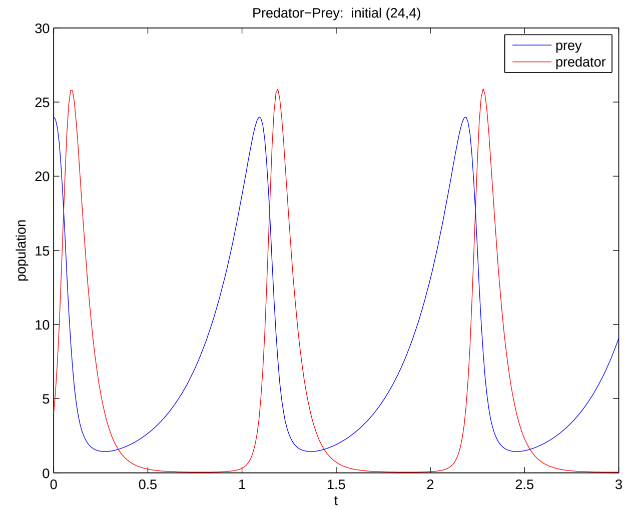
Generated using Runge-Kutta approximation, `rk2.m`.



Solution through $(24, 4)$

Solution trajectories for $x(t), y(t)$ for initial populations: $x(0) = 20$, $y(0) = 4$.

Generated using Runge-Kutta approximation, `rk2.m`.



Example of logistic predator-prey model

Equation. Consider the logistic predator(y)-prey(x) model for two species:

$$\frac{dx}{dt} = 30x - 2x^2 - xy = x(30 - 2x - y)$$
$$\frac{dy}{dt} = 80y - 4y^2 + 2xy = y(80 - 4y + 2x)$$

Critical points. $(0, 0), (0, 20), (15, 0), (4, 22)$.

Jacobian.

$$\mathbf{J}(x, y) = \begin{bmatrix} 30 - 4x - y & -x \\ 2y & 80 - 8y + 2x \end{bmatrix}$$

Qualitative properties of example

Jacobian. Critical point: $(0, 0)$. The stable solution $x \equiv 0, y \equiv 0$ is one where both populations are extinct.

$$\mathbf{J}(x, y) = \begin{bmatrix} 30 - 4x - y & -x \\ 2y & 80 - 8y + 2x \end{bmatrix} \quad \mathbf{J}(0, 0) = \begin{bmatrix} 30 & 0 \\ 80 & 0 \end{bmatrix}$$

Eigenvalues. At $(0, 0)$: $\lambda = 30, 80$

Analysis. $(0, 0)$ is a nodal source in the linearization; so it is a nodal source in the nonlinear system.

Qualitative properties of example

Jacobian. Critical point: $(0,20),(15,0)$. The stable solutions where one of the species is extinct.

$$\mathbf{J}(x,y)=\begin{bmatrix}30-4x-y & -x\\ 2y & 80-8y+2x\end{bmatrix}$$

$$\mathbf{J}(0,20)=\begin{bmatrix}10 & 0\\ 40 & -80\end{bmatrix}\quad \mathbf{J}(15,0)=\begin{bmatrix}-30 & -15\\ 0 & 110\end{bmatrix}$$

Eigenvalues. At $(0,20)$: $\lambda=10,-80$, At $(15,0)$: $\lambda=-30,110$

Analysis. Both $(0,20)$ and $(15,0)$ are saddlepoints (all trajectories are repulsed except those where one population is extinct) in the linearization; so it is a saddlepoint in the nonlinear system.

Qualitative properties of example

Jacobian. Critical point: $(4,22)$. The only stable solution where the populations coexist: $x\equiv 4, y\equiv 22$.

$$\mathbf{J}(x,y)=\begin{bmatrix}30-4x-y & -x\\ 2y & 80-8y+2x\end{bmatrix}\quad \mathbf{J}(4,22)=\begin{bmatrix}-8 & -4\\ 44 & -88\end{bmatrix}$$

We can determine stability by computing the trace (T) and determinant (D):

$$T=-8-88=-96\quad D=(-8)(-88)-(-4)(44)=880\quad T^2-4D=5696$$

Since discriminant is positive ($T^2-4D>0$), $T<0$ and $D>0$, the eigenvalues are real and negative.

Analysis. $(4,22)$ is a nodal sink in the linearization; so it is a nodal sink in the nonlinear system.

Analysis

Separately the populations, without interaction, would tend to logistic growth.

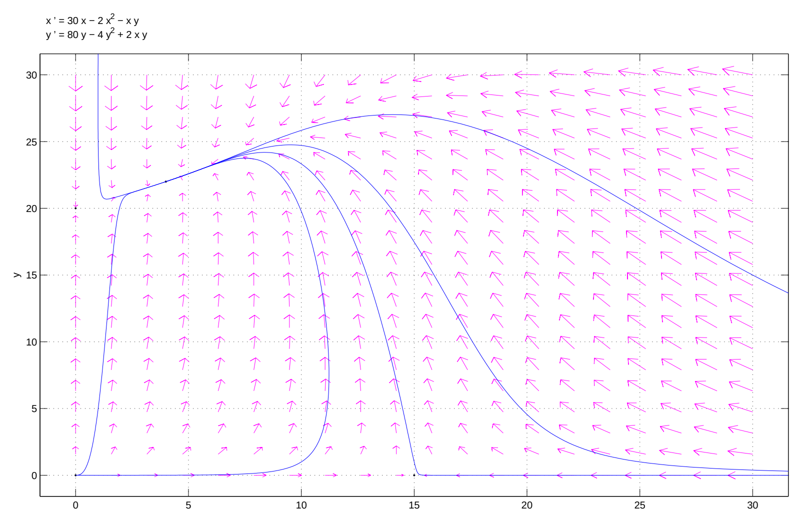
- $x(t)\to 15$ as $t\to\infty$
- $y(t)\to 20$ as $t\to\infty$

Interaction. Predators (y) are helped and prey (x) are hurt by the interaction

- $x(t)\to 4$ as $t\to\infty$
- $y(t)\to 22$ as $t\to\infty$

Direction field

Direction field. Trajectories are drawn to $(4,22)$.



Example of logistic competition model

Equation. Consider the competition model for species x, y:

$$\frac{dx}{dt} = 60x - 3x^2 - 4xy = x(60 - 3x - 4y)$$
$$\frac{dy}{dt} = 42y - 3y^2 - 2xy = y(42 - 3y - 2x)$$

Critical points. $(0,0), (0,14), (20,0), (12,6)$.

Jacobian.
$$\mathbf{J}(x,y) = \begin{bmatrix} 60 - 6x - 4y & -4x \\ -2y & 42 - 6y - 2x \end{bmatrix}$$

Qualitative properties of example

Jacobian. Critical point: $(0,0)$. The stable solution $x \equiv 0, y \equiv 0$ is one where both populations are extinct.

$$\mathbf{J}(x,y) = \begin{bmatrix} 60 - 6x - 4y & -4x \\ -2y & 42 - 6y - 2x \end{bmatrix} \quad \mathbf{J}(0,0) = \begin{bmatrix} 60 & 0 \\ 42 & 0 \end{bmatrix}$$

Eigenvalues. At $(0,0)$: $\lambda = 60, 42$

Analysis. $(0,0)$ is a nodal source in the linearization; so it is a nodal source in the nonlinear system.

Qualitative properties of example

Jacobian. Critical point: $(0,14), (20,0)$. The stable solutions where one of the species is extinct.

$$\mathbf{J}(x,y) = \begin{bmatrix} 60 - 6x - 4y & -4x \\ -2y & 42 - 6y - 2x \end{bmatrix}$$
$$\mathbf{J}(0,14) = \begin{bmatrix} 4 & 0 \\ -28 & 0 - 42 \end{bmatrix} \quad \mathbf{J}(20,0) = \begin{bmatrix} -20 & -80 \\ 0 & 2 \end{bmatrix}$$

Eigenvalues. At $(0,14)$: $\lambda = 4, -42$, At $(20,0)$: $\lambda = -20, 2$

Analysis. Both $(0,14)$ and $(20,0)$ are saddlepoints (all trajectories are repulsed except those where one population is extinct) in the linearization; so it is a saddlepoint in the nonlinear system.

Qualitative properties of example

Jacobian. Critical point: $(12,6)$. The only stable solution where the populations coexist: $x \equiv 12, y \equiv 6$.

$$\mathbf{J}(x,y) = \begin{bmatrix} 60 - 6x - 4y & -4x \\ -2y & 42 - 6y - 2x \end{bmatrix} \quad \mathbf{J}(12,6) = \begin{bmatrix} -36 & -48 \\ -12 & -18 \end{bmatrix}$$

We can determine stability by computing the trace (T) and determinant (D):

$$T = -36 - 18 = -54 \quad D = (-36)(-18) - (-12)(-48) = 72 \quad T^2 - 4D = 2628$$

Since discriminant is positive ($T^2 - 4D > 0$), $T < 0$ and $D > 0$, the eigenvalues are real and negative.

Analysis. $(12,6)$ is a nodal sink in the linearization; so it is a nodal sink in the nonlinear system.

Analysis

Separately the populations, without interaction, would tend to logistic growth.

- $x(t) \to 20$ as $t \to \infty$
- $y(t) \to 14$ as $t \to \infty$

Interaction. Both populations are hurt by the interaction

- $x(t) \to 12$ as $t \to \infty$
- $y(t) \to 6$ as $t \to \infty$

Direction field

Direction field. Trajectories are drawn to $(12, 6)$.