Math 216
Differential Equations

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

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Population models

Consider a population $P(t)$ changing over time only due to births and deaths (no immigration or emigration). We track the growth of the population due to births and deaths by the (possibly changing) birth rate and death rate:

- $\beta(t)$ is the number of births per unit population per unit of time at time t.
- $\delta(t)$ is the number of deaths per unit population per unit of time at time t.

Assuming β, δ, P are all continuous, the number of births and deaths which occur in the time interval $[t, \Delta t]$ is approximately

$$\text{births: } \beta(t) \cdot P(t) \cdot \Delta t \qquad \text{deaths: } \delta(t) \cdot P(t) \cdot \Delta t$$

ConcepTest

Problem. A mutual fund account is allowed to accumulate interest (no funds deposited or withdrawn). The interest rate, $r(t)$, changes over time, reflecting how well, or poorly, the fund is performing. Interest is compounded continuously (meaning interest is continuously being added to the account).

Question. Assume the amount in the account at any time, $A(t)$, and the interest rate $r(t)$ are continuous functions. What is the approximate amount the account has changed in the time interval $[t, \Delta t]$.

Answer. $r(t) \cdot A(t) \cdot \Delta t$.

Natural growth equation

The amount in the account at time $t + \Delta t$ is

$$P(t + \Delta t) \approx P(t) + \left[\beta(t) \cdot P(t) \cdot \Delta t - \delta(t) \cdot P(t) \cdot \Delta t\right].$$

The change in the population, ΔP, in the interval $[t, \Delta t]$ is

$$\Delta P \approx \left[\beta(t) - \delta(t)\right] P(t) \cdot \Delta t.$$

The rate of change of the population is approximately

$$\frac{\Delta P}{\Delta t} \approx \left[\beta(t) - \delta(t)\right] P(t).$$

If the population function $P(t)$ is differentiable, the error in this approximation approaches zero as $\Delta t \to 0$:

$$\frac{dP}{dt} = \lim_{t \to 0} \frac{\Delta P}{\Delta t} = \left[\beta(t) - \delta(t)\right] P(t).$$

Natural growth equation

The natural growth equation is differential equation

$$\frac{dP}{dt} = (\beta - \delta)P.$$

(where $\beta(t) = \beta$ and $\delta(t) = \delta$). It is first-order, linear and separable.

If β and δ are constant, then the equation reduces to

$$\frac{dP}{dt} = kP \qquad \text{where } k = \beta - \delta.$$

If we know the initial population $P(0) = P_0$, then the unique solution is

$$P(t) = P_0 e^{kt}.$$

ConcepTest

Question. How do we know that there is a unique solution, over the entire real line, for the IVP

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0?$$

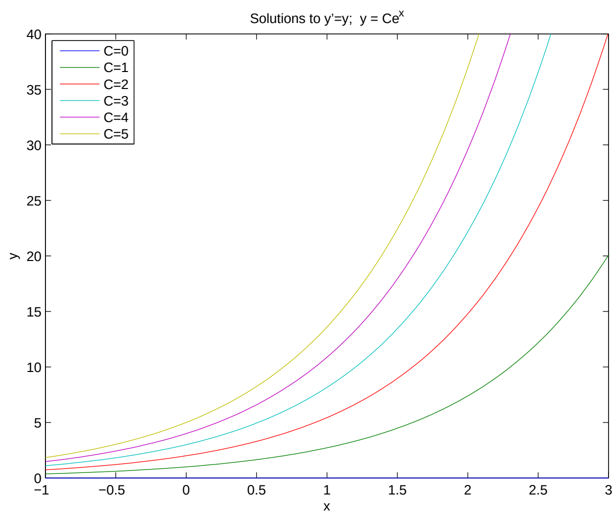
Answer. The equation is a first-order linear equation:

$$\frac{dP}{dt} - kP = 0.$$

The coefficient functions are both constant functions, so continuous over the entire real line. Theorem 1 of Section 1.5 then guarantees a unique solution over the entire real line through the point $(0, A_0)$.

Graphical solution various parameter values

Solutions to the ODE $y' = y$. Note that the solutions are parallel, except the singular solution (see `fig5.m`).



ConcepTest

Problem. Let $A(t)$ be the amount in a bank account at time t. Suppose the account pays at the annual fixed interest rate r compounded continuously. Assume that interest is allowed to accumulate, and a year deposit q is made and that this is added continuously over time.

Question. Determine the differential equation which models the bank account. Suppose the initial balance is A_0. What is the unique solution to the equation given this initial data?

Answer. The equation modeling the bank account is given by the IVP

$$\frac{dA}{dt} = rA + q, \qquad A(0) = A_0.$$

The general solution is

$$A(t) = Ce^{rt} - \frac{q}{r}$$

From the given initial value, $C = A_0 + \frac{q}{r}$.

The most general version of model

The most general form of the population model (with unlimited growth) is of the form

$$\frac{dy}{dt} = r(t)y + q(t)$$

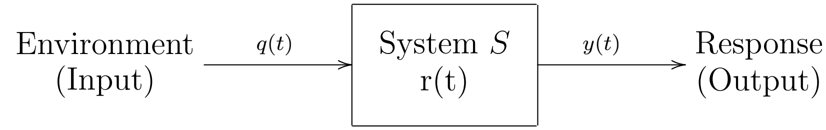
where r, q are functions over time expressible as

- $r(t)$: rate per unit of population per time at time t, (growth by the system
- $q(t)$: rate per time at time t

The ODE is first-order and linear, so is guaranteed to have a unique through any point (y, t) – provided r and q are continuous.

This is a model of a system S where $r(t)$ is the growth rate of the system and $q(t)$ is the contribution to S from outside the system. If r (or q) also depends upon the size of the populations, $r = r(y, t)$ then the model will no longer be linear.

Model of ideal system



Example applications

- Populations with immigration/emigration: $r(t) = \beta(t) - \delta(t)$ where β is the birth rate and δ is the death rate.
- Interest rate in a bank account with deposts/withdraws: $\delta(t) = 0$ in the population model.
- Radioactive decay: $\beta = 0$ and $q(x) = 0$ in the population model.

Example: Rabbits

Problem. Consider a prolific breed of rabbits whose birth and death rates are proportional to the size of the rabbit population. Suppose the initial population is P_0. Find the unique solution to the IVP modeling this rabbit population.

- Let $P(t)$ be the rabbit population at time t.
- $\beta(t) = bP(t)$, b is the constant of proportionality.
- $\delta(t) = dP(t)$, d is the constant of proportionality.

The initial value problem is given by

$$\frac{dP}{dt} = (bP - dP)\,P = kP^2$$

where $k = b - d$.

This equation is not linear, but it is separable.

Analysis of Rabbit Problem

$$\frac{dP}{dt} = kP^2, \qquad P(0) = P_0.$$

Apply the method of separation of variables:

$$\int P^{-2}\,dP = \int k\,dt + C.$$

Since $\int P^{-2}\,dP = -\frac{1}{P}$,

$$\begin{aligned} -\frac{1}{P} &= kt + C \\ P(t) &= \frac{1}{C - kt}. \end{aligned}$$

Since $P(0) = P_0$, solving gives $C = \frac{1}{P_0}$. Thus,

$$P(t) = \frac{1}{\frac{1}{P_0} - kt} = \frac{P_0}{1 - kP_0 t}.$$

Qualitative Analysis

$$P(t) = \frac{P_0}{1 - kP_0 t}$$

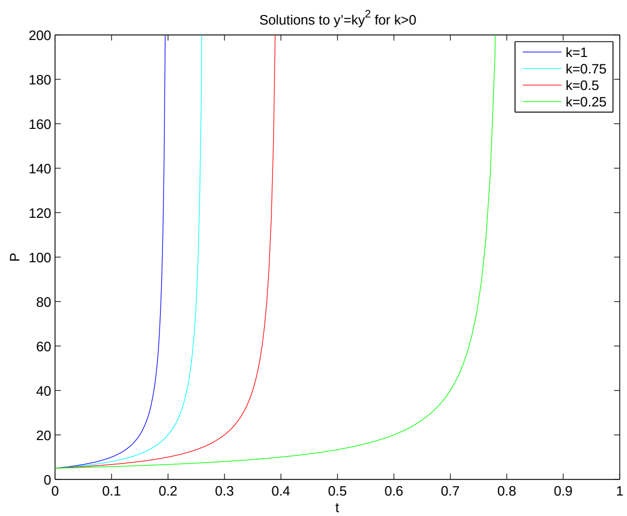
where $k = b - d$ with birth rate b and death rate d.

- If $b < d$, then $k < 0$ and $1 - kP_0 t \to \infty$ as $t \to \infty$. So, $P(t) \to 0$ as $t \to \infty$. The population goes extinct.
- If $b > d$, then $k > 0$ and $1 - kP_0 t \to 0$ as $t \to \frac{1}{kP_0}$. So, $P(t) \to \infty$ as $t \to \frac{1}{kP_0}$. The population explodes, going to infinity in a finite amount of time.
- If $b = d$, then the solution is the singular solution $P \equiv P_0$.

Note. It is a hallmark of nonlinear ODEs that the solutions can exhibit very different behavior with changes in the parameters.

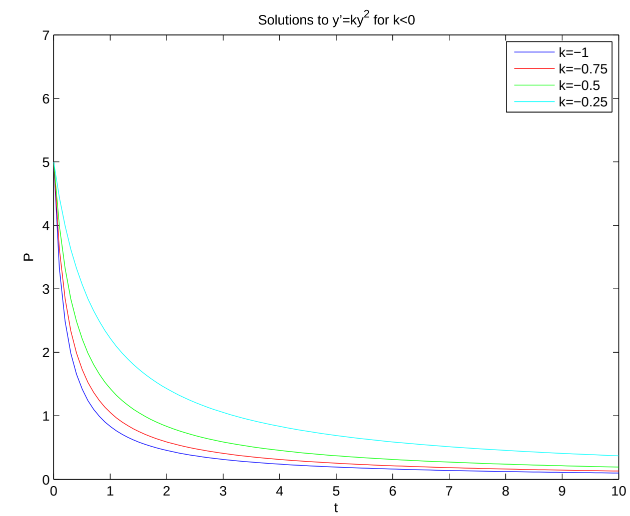
Graphical solution for various k

Solutions of $P(t) = \frac{P_0}{1-kP_0 t}$ where $P_0 = 5$ for some positive k. (see `fig1.m`)



Graphical solution for various k

Solutions of $P(t) = \frac{P_0}{1-kP_0 t}$ where $P_0 = 5$ for some negative k. (see `fig2.m`)



Logistic Population model

Problem. Consider a population where the birth rate is constant, but the death rate is proportional to the population. (For example, as the population grows the ability of the environment to support it decreases, so death rate increases.)

- Let $P = P(t)$ be the population at time t.
- $\beta(t) = b$.
- $\delta(t) = dP(t)$, d is the constant of proportionality.

Then the initial value problem is given by

$$\frac{dP}{dt} = (b - dP)P = bP - dP^2.$$

This equation is called the logistic equation.

Solving the logistic equation

Solve the IVP

$$\frac{dP}{dt} = bP - dP^2, \qquad P(0) = P_0.$$

This equation is separable.

$$\int \frac{dP}{bP - dP^2} = \int dt + C = t + C$$

We can solve the left-hand integral using the method of partial fractions.

Method of partial fractions

Solve the integral $\int \frac{dP}{bP - dP^2}$. We want to find A and B satisfying

$$\frac{A}{P} + \frac{B}{b - dP} = \frac{1}{P(b - dP)}.$$

Cross-multiplying leads to two equations

$$\begin{aligned} bA &= 1 \\ -dA + B &= 0 \end{aligned}$$

So, $A = \frac{1}{b}$ and $B = \frac{d}{b}$. Thus,

$$\begin{aligned} \int \frac{dP}{P(b - dP)} &= \frac{1}{b}\Big[\int \frac{1}{P} + \frac{d}{b - dP}\, dP\Big] \\ &= \frac{1}{b}\Big[\ln|P| - \ln|b - dP|\Big] \\ &= \frac{1}{b}\ln\Big|\frac{P}{b - dP}\Big|. \end{aligned}$$

A word on absolute values

Absolute value arises when solving ODEs since

$$\int \frac{dx}{x} = \ln|x|$$

The presence of absolute values can lead to messy case analysis. Here, we need to consider two cases (i) $x > 0$ and (ii) $x < 0$. The rule of thumb here is

- If you only want a general solution then you can drop the absolute value.
- If you want a particular solution you will need to decide which of the two cases, (i) $x > 0$ or (ii) $x < 0$, apply to the problem at hand.

See Example 1, Section 1.4 for an application of this rule of thumb.

Solving the logistic equation

Simplifying our solution to the logistic equation was

$$\begin{aligned} \int \frac{dP}{bP - dP^2} &= t + C \\ \frac{1}{b} \ln \left| \frac{P}{b - dP} \right| &= t + C \\ \left| \frac{P}{b - dP} \right| &= e^C e^{bt} \\ \frac{P}{b - dP} &= Ke^{bt} \qquad \text{where } K = \pm e^C \end{aligned}$$

Since $P(0) = P_0$, $K = \frac{P_0}{b - dP_0}$. Solving for P

$$\begin{aligned} P(t) &= \frac{bKe^{bt}}{1 + dKe^{bt}} \\ &= \frac{b}{K^{-1}e^{-bt} + d} \end{aligned}$$

Qualitative analysis

The solution to the logistic equation

$$\frac{dP}{dt} = bP - dP^2, \qquad P(0) = P_0$$

is

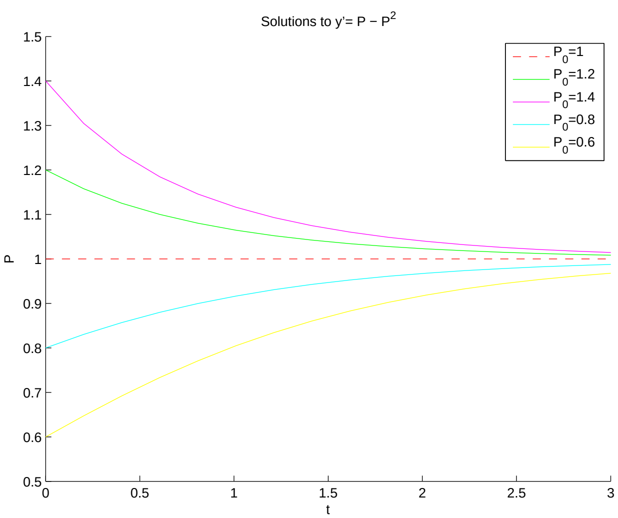
$$P(t) = \frac{b}{Ke^{-bt} + d} \qquad \text{where } K = \frac{b - dP_0}{P_0}$$

Now, $Ke^{-bt} \to 0$ as $t \to \infty$.

So, $P(t) \to \frac{b}{d}$ as $t \to \infty$ regardless of the initial population. Note, that $P \equiv \frac{b}{d}$ is a solution to the equation. The value $\frac{b}{d}$ is called a stable equilibria point – see Section 2.2.

Qualitative analysis

Solutions of $P(t) = \frac{1}{Ke^{-t}+1}$ for $b = 1, d = 1$, and various initial P_0. $K = \frac{1-P_0}{P_0}$ (see `fig3.m`)



General solution to Logistic equation

All solutions to the logistic equation

$$\begin{aligned} \frac{dP}{dt} &= bP - dP^2, \qquad P(0) = P \\ P(t) &= \frac{b}{Ke^{-bt} + d} \quad \text{where } K = \frac{b - dP_0}{P_0} \end{aligned}$$

are drawn to a limiting population $\frac{b}{d}$.

Show. We can re-write the logistic equation and its solution in terms of the limiting population as

$$\begin{aligned} \frac{dP}{dt} &= dP(M - P), \qquad P(0) = P \\ P(t) &= \frac{MP_0}{P_0 + (M - P_0)e^{-dMt}}, \end{aligned}$$

where $M = \frac{b}{d}$.

Applications of the logistic equation

Limited Environment. An environment has a capacity to support a maximal population M. In this case, the population growth rate, $\beta - \delta$, is proportional to $(M - P)$. So,

$$\begin{aligned} \beta - \delta &= k(M - P) \\ \frac{dP}{dt} &= (\beta - \delta)P = kP(M - P) \end{aligned}$$

Spread of infection. Let M be the (fixed) total population, and $P(t)$ the amount of the population infected. Infection spread from infected to non-infected population due to chance encounter, with virulency rate k. Then

$$\frac{dP}{dt} = kP(M - P)$$

Extinction/Explosion equation

Problem. Consider a population where the death rate is constant, but the birth rate is proportional to the population. (For example, procreation is due to chance encounters between males and females in the population.)

- Let $P(t)$ be the population at time t.
- $\beta(t) = bP(t)$, where b is the constant of proportionality.
- $\delta(t) = d$.

Then the initial value problem is given by

$$\frac{dP}{dt} = \big(bP - d\big)P = bP^2 - dP$$

This equation is called the extinction/explosion equation.

Solving the extinction/explosion equation

Solve the IVP

$$\frac{dP}{dt} = bP^2 - dP, \qquad P(0) = P_0$$

This equation is separable.

$$\int \frac{dP}{bP^2 - dP} = \int dt + C = t + C$$

We can solve the left-hand integral using the method of partial fractions.

ConcepTest

Problem. Solve the integral

$$\int \frac{dP}{bP^2 - dP}$$

using the method of partial fractions.

Answer.

$$\int \frac{dP}{bP^2 - dP} = -\frac{1}{d} \ln \Big| \frac{P}{bP - d} \Big|.$$

You can get more practice solving by partial fractions in the exercises 1-8 of Section 2.1.

Solving the extinction/explosion equation

Simplifying our solution to the extinction/explosion equation was

$$\begin{aligned}
\int \frac{dP}{bP^2 - dP} &= t + C \\
-\frac{1}{d}\ln\left|\frac{P}{bP-d}\right| &= t + C \\
\left|\frac{P}{bP-d}\right| &= e^C e^{-dt} \\
\frac{P}{bP-d} &= Ke^{-dt} \qquad \text{where } K = \pm e^C
\end{aligned}$$

Since $P(0) = P_0$, $K = \frac{P_0}{bP_0 - d}$. Solving for P

$$\begin{aligned}
P(t) &= \frac{dKe^{-dt}}{bKe^{-dt} - 1} \\
&= \frac{d}{b - K^{-1}e^{dt}}
\end{aligned}$$

Qualitative analysis

The solution to the extinction/explosion equation

$$\frac{dP}{dt} = bP^2 - dP, \qquad P(0) = P_0$$

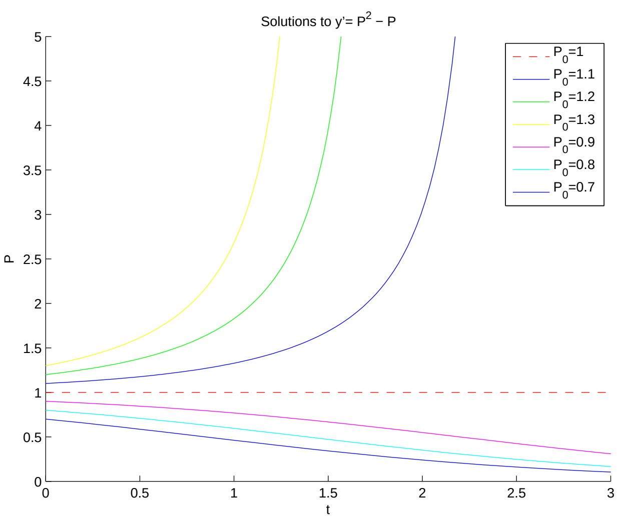
is

$$P(t) = \frac{d}{b - Ke^{dt}} \qquad \text{where } K = \frac{bP_0 - d}{P_0}$$

- $P_0 = \frac{d}{b}$: Then $P \equiv \frac{d}{b}$, the singular solution. ($\frac{d}{b}$ is an unstable critical point – see Section 2.2).
- $P_0 > \frac{d}{b}$: then $K > 0$. So, $b - K^{-1}e^{dt} \to 0$ and $P(t) \to \infty$ as $t \to \frac{\ln(bK^{-1})}{d}$. The population explodes. .
- $P_0 < \frac{d}{b}$: then $K < 0$. So, $b - K^{-1}e^{dt} \to \infty$ and $P(t) \to 0$ as $t \to \infty$. The population goes to extinction.

Qualitative analysis

Solutions of $P(t) = \frac{1}{1 - Ke^t}$ for $b = 1, d = 1$, and various initial P_0. $K = \frac{P_0 - 1}{P_0}$ (see `fig3.m`)



General solution to extinction/explosion equation

All solutions to the extinction/explosion equation

$$\begin{aligned}
\frac{dP}{dt} &= bP^2 - dP, \quad P(0) = P_0 \\
P(t) &= \frac{d}{b - Ke^{dt}} \quad \text{where } K = \frac{bP_0 - d}{P_0}
\end{aligned}$$

are repulsed by the threshold population $\frac{d}{b}$. If the $P_0 > \frac{d}{b}$ the population explodes, and if $P_0 < \frac{d}{b}$ the population goes extinct.

Show. We can re-write the extinction/explosion equation and its solution in terms of the limiting population as

$$\begin{aligned}
\frac{dP}{dt} &= bP(P - M), \quad P(0) = P_0 \\
P(t) &= \frac{MP_0}{P_0 + (M - P_0)e^{kMt}} \quad \text{where } M = \frac{d}{b}.
\end{aligned}$$