Math 216
Differential Equations

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Computing eigenvalues of 2×2 matrices

The characteristic polynomial for the real-valued matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc) = \lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A})$$

where $\mathrm{tr}(\mathbf{A}) = a + d$ is the sum of the diagonal elements of **A**.
The eigenvalues are

$$\frac{\mathrm{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}.$$

Perturbing a matrix

Perturbations. Consider a small perturbation of the elements of **A**:

$$\mathbf{A}^* = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}$$

where a^*, b^*, c^*, d^* are close to a, b, c, d.

Eigenvalues are

$$\frac{\mathrm{tr}(\mathbf{A}^*)}{2} \pm \frac{1}{2}\sqrt{\mathrm{tr}(\mathbf{A}^*)^2 - 4\det(\mathbf{A}^*)}.$$

where

- $\mathrm{tr}(\mathbf{A}^*) = a^* + d^*$ is close to $\mathrm{tr}(\mathbf{A}) = a + d$
- $\det(\mathbf{A}^*) = a^*d^* - b^*c^*$ is close to $\det(\mathbf{A}) = ad - bc$

Eigenvalues under perturbation

Eigenvalues of **A**: $\frac{\mathrm{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}$.

Assumption 1. Suppose $\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A}) \neq 0$.
Then, $\mathrm{tr}(\mathbf{A}^*)^2 - 4\det(\mathbf{A}^*) \neq 0$, with the same sign.
Assumption 2. Suppose $\mathrm{tr}(\mathbf{A}) \neq 0$.
Then, $\mathrm{tr}(\mathbf{A}^*) \neq 0$, with the same sign.

Stability and Type. Two cases

- $\lambda = p \pm iq$ is a complex eigenvalue of **A**, then $\lambda^* = p^* \pm iq^*$ is an eigenvalue **A** where p, p^* have the same sign.
  A and $\mathbf{A}^*$ have the same qualitative properties at $(0, 0)$.
- λ_1, λ_2 are distinct real eigenvalues for **A**, then the corresponding eigenvalues λ_1^*, λ_2^* for $\mathbf{A}^*$ are both real and have the same signs.
  A and $\mathbf{A}^*$ have the same qualitative properties at $(0, 0)$.

Eigenvalues under perturbation

Eigenvalues of **A**: $\frac{\text{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\text{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}$.

Assumption. Suppose **A** has one eigenvalue.
Then, $\text{tr}(\mathbf{A}) \neq 0$, so $\text{tr}(\mathbf{A}^*) \neq 0$, with the same sign.
However, $\text{tr}(\mathbf{A})^2 - 4\det(\mathbf{A}) = 0$.

Stability and Type. Three possibilities

1. $\text{tr}(\mathbf{A}^*)^2 - 4\det(\mathbf{A}^*) = 0$. Same qualitative properties at $(0,0)$.
2. $\text{tr}(\mathbf{A}^*)^2 - 4\det(\mathbf{A}^*) > 0$. The eigenvalues of $\mathbf{A}^*$ are distinct reals with the same sign.
3. $\text{tr}(\mathbf{A}^*)^2 - 4\det(\mathbf{A}^*) < 0$. Then eigenvalues of $\mathbf{A}^*$ are complex and $(0,0)$ is a spiral point with the same stability as that of **A**.

Eigenvalues under perturbation

Eigenvalues of **A**: $\frac{\text{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\text{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}$.

Assumption. Suppose **A** has imaginary eigenvalues.
Then, $\text{tr}(\mathbf{A})^2 - 4\det(\mathbf{A}) < 0$, and this remains true for $\mathbf{A}^*$.
However, $\text{tr}(\mathbf{A}) = 0$.

Stability and Type. Three possibilities

1. $\text{tr}(\mathbf{A}) = 0$. No change in the qualitative properties at $(0,0)$.
2. $\text{tr}(\mathbf{A}) > 0$. Then $(0,0)$ is now a spiral source.
3. $\text{tr}(\mathbf{A}) < 0$. Then $(0,0)$ is now a spiral sink.

Summary

Theorem

*Let λ_1 and λ_2 be the eigenvalues of a matrix **A**, and $\mathbf{A}^*$ a sufficiently small perturbation of **A**. Then, the qualitative properties of the critical point $(0,0)$ for $\mathbf{A}^*$ satisfies*

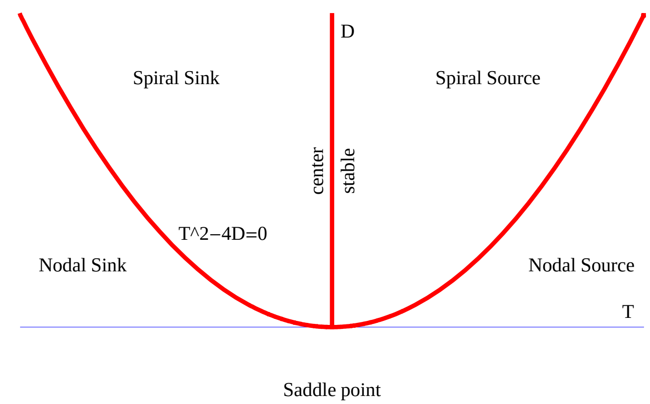
1. *If $\lambda_1 = \lambda_2$, then $(0,0)$ of $\mathbf{A}^*$ is either a node, or a spiral point. It is asymptotically stable if $\lambda_1 < 0$ and unstable if $\lambda_1 > 0$.*
2. *If λ_1 and λ_2 are pure imaginary, then $(0,0)$ is either a center or a spiral. It could be any of asymptotically stable, stable, or unstable.*
3. *Otherwise, $(0,0)$ has the same type and stability at $\mathbf{A}^*$ as at **A**.*

Summary

Distribution of critical points in the Trace-Determinant plane.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad T = \text{tr}(\mathbf{A}) = a + d, \quad D = \det(\mathbf{A}) = ad - bc.$$

Sensitive areas: Places where type of critical point sensitive to perturbations.



Assumption

Assumption. Let $\mathbf{h}(x, y)$ be a vector valued function, so that

$$\mathbf{h}(x,y) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$$

We will assume that $\mathbf{h}$ is continuously differentiable.
This means each of the following are defined and continuous:

$$F_x = \frac{\partial F}{\partial x} \qquad F_y = \frac{\partial F}{\partial y}$$

$$G_x = \frac{\partial G}{\partial x} \qquad G_y = \frac{\partial G}{\partial y}$$

Jacobian

Jacobian. If

$$\mathbf{h}(x,y) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$$

is continuously differentiable, then

$$\mathbf{h}'(x,y) = \mathbf{J}(x,y) = \begin{bmatrix} F_x(x,y) & F_y(x,y) \\ G_x(x,y) & G_y(x,y) \end{bmatrix}$$

where $\mathbf{J}(x, y)$ is the Jacobian at (x, y).

Key fact about Jacobian

Key fact. We can approximate a continuously differentiable function $\mathbf{h}$ at points (x, y) near a point (a, b) using the Jacobian

$$\mathbf{h}(x,y) = \mathbf{h}(a,b) + \mathbf{J}(a,b)\begin{bmatrix} u \\ v \end{bmatrix} + \epsilon(u,v),$$

where $u = x - a$ and $v = y - b$, and

$$\lim_{(u,v)\to(0,0)} \frac{\epsilon(x,y)}{\sqrt{u^2+v^2}} = \mathbf{0}.$$

Summary. A continuously differentiable vector function can be approximated by a matrix.

Almost linear functions

(a, b) is a critical point of a vector function $\mathbf{h}(x, y)$ if $\mathbf{h}(a, b) = \mathbf{0}$.

Definition. A vector function $\mathbf{h}(x, y)$ is almost linear at a critical point $\vec{a} = (a, b)$ provided $\mathbf{h}$ has the form

$$\mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{u} + \mathbf{g}(\mathbf{u}),$$

where $\mathbf{A}$ is an invertible matrix, $\mathbf{u} = \mathbf{x} - \mathbf{a}$, and

$$\lim_{\mathbf{u}\to\mathbf{0}} \frac{\mathbf{g}(\mathbf{u})}{|\mathbf{u}|} = \mathbf{0}.$$

The error $g(\mathbf{u})$ is shrinking faster than the length of $\mathbf{u}$.

Main example of almost linear functions

Our main example of almost linear functions are continuously differentiable ones.

Theorem

If $\mathbf{h}$ *is a continuously differentiable function and the Jacobian* $\mathbf{J}(a,b)$ *is invertible then* $\mathbf{h}$ *is almost linear.*

$\mathbf{J}(a,b)$ *is invertible if and only if*

$$\begin{vmatrix} F_x(a,b) & F_y(a,b) \\ G_x(a,b) & G_y(a,b) \end{vmatrix} \neq 0.$$

Example of almost linear function

Example. Let

$$\mathbf{h}(x,y) = \begin{bmatrix} -y - x(1-x^2-y^2) \\ -x - y(1-x^2-y^2) \end{bmatrix}$$

The Jacobian of $\mathbf{h}$ is

$$\mathbf{J}(x,y) = \begin{bmatrix} 3x^2+y^2-1 & 2xy+1 \\ 2xy-1 & x^2+3y^2-1 \end{bmatrix},$$

so, $\mathbf{h}$ is continuously differentiable.

Critical point. $\mathbf{h}$ has a critical point at $(0,0)$, since $\mathbf{h}(0,0)=0$, and

$$\mathbf{J}(0,0) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},$$

so, $\det(\mathbf{J}(0,0)) = 2$. Thus, $\mathbf{J}(0,0)$ is invertible.

Conclusion. $\mathbf{h}$ is almost linear.

Linearization

Let $\mathbf{a} = (a,b)$ be a critical point of the autonomous system

$$x' = F(x,y), \qquad y' = G(x,y)$$

where F and G are continuously differentiable. In vector form,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{h}(x,y) \qquad \text{where } \mathbf{h}(x,y) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$$

Approximate $\mathbf{h}$ by the Jacobian at $\mathbf{x}$ near $\mathbf{a}$:

$$\mathbf{h}(\mathbf{x}) \approx \mathbf{h}(\mathbf{a}) + \begin{bmatrix} F_x(a,b) & F_y(a,b) \\ G_x(a,b) & G_y(a,b) \end{bmatrix} \qquad \text{where } \mathbf{u} = \mathbf{x} - \mathbf{a}$$

Since $\mathbf{a}$ is a critical point, $\mathbf{h}(\mathbf{a}) = \mathbf{0}$, so

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{h}(\mathbf{x}) \approx \begin{bmatrix} F_x(a,b) & F_y(a,b) \\ G_x(a,b) & G_y(a,b) \end{bmatrix} \mathbf{u} \qquad \text{where } \mathbf{u} = \mathbf{x} - \mathbf{a}$$

Linearization

Definition. Let F, G be continuously differentiable, and (a,b) a critical point of the system:

$$x' = F(x,y), \qquad y' = G(x,y).$$

The linearization is the system of equations

$$\begin{aligned} u' &= F_x(a,b)u + F_y(a,b)v \\ v' &= G_x(a,b)u + G_y(a,b)v, \end{aligned}$$

In matrix notation, the linearization is

$$\mathbf{u}' = \mathbf{J}(a,b)\mathbf{u} \qquad \text{where } \mathbf{J}(a,b) = \begin{bmatrix} F_x(a,b) & F_y(a,b) \\ G_x(a,b) & G_y(a,b) \end{bmatrix}$$

and has a critical point at the origin.

Key theorem for linearization

Theorem

Suppose F, G are continuously differentiable, and (a, b) is a critical point (so, $F(a, b) = G(a, b) = 0$) where $\det(\mathbf{J}(a, b)) \neq 0$.

Then, near (a, b), the solutions to

$$x' = F(x, y), \quad y' = G(x, y)$$

have the same qualitative properties that a perturbation of $\mathbf{J}(a, b)$ has at $(0, 0)$.

So, if the eigenvalues of $\mathbf{J}(a, b)$ are distinct and not imaginary, then the solutions to the original equation look like slightly distorted versions of solutions to the linear system at $(0, 0)$

$$\mathbf{u}' = \mathbf{J}(a, b)\mathbf{u}$$

Summary

λ_1, λ_2 are eigenvalues for the linearization.
Type and stability of critical point of the almost linear system.

- $\lambda_1 < \lambda_2 < 0$: Stable improper node.
- $\lambda_1 = \lambda_2 < 0$: Stable node or spiral sink
- $\lambda_1 < 0 < \lambda_2$: Unstable saddle point
- $\lambda_1 = \lambda_2 > 0$: Unstable node or spiral source
- $\lambda_1 > \lambda_2 > 0$: Unstable improper node
- $\lambda_1, \lambda_2 = p \pm iq, a < 0$: Spiral sink
- $\lambda_1, \lambda_2 = p \pm iq, a > 0$: Spiral source
- $\lambda_1, \lambda_2 = \pm iq$: stable or unstable, center or spiral point.

Qualitative properties of linearization

Example.

$$\begin{aligned} x' &= 1 - y \\ y' &= x^2 - y^2 \end{aligned}$$

Critical points. $(1, 1), (-1, 1)$.
Jacobian. Critical point: $(1, 1)$.

$$\mathbf{J}(x, y) = \begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix} \quad \mathbf{J}(1, 1) = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}$$

Eigenvalues. At (1,1): $\lambda = -1 \pm i$

Analysis. $(1, 1)$ is a spiral sink in the linearization; so it is a spiral sink in the autonomous system.

Qualitative properties of linearization

Example.

$$\begin{aligned} x' &= 1 - y \\ y' &= x^2 - y^2 \end{aligned}$$

Jacobian. Critical point: $(-1, 1)$.

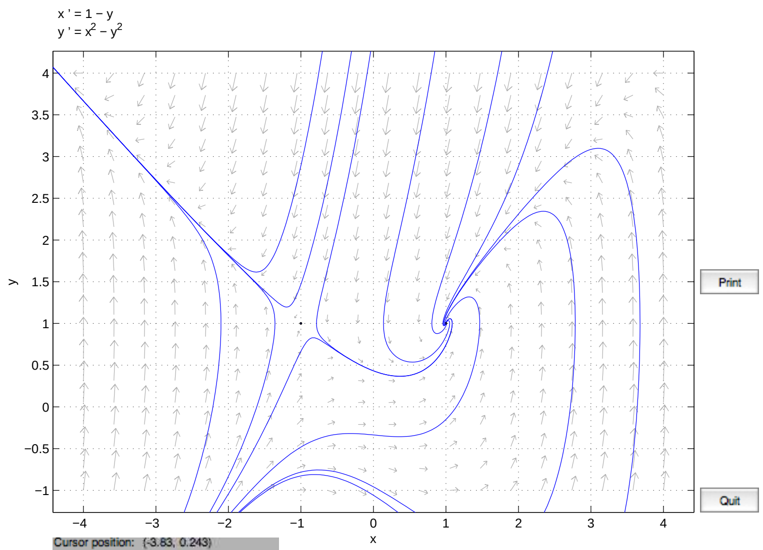
$$\mathbf{J}(x, y) = \begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix} \quad \mathbf{J}(-1, 1) = \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$$

Eigenvalues. At (-1,1): $\lambda = 1 \pm \sqrt{3}$.

Analysis. $(-1, 1)$ is a saddlepoint in the linearization; so it is a saddlepoint in the autonomous system.

Direction field

$(-1,1)$ is a saddlepoint, $(1,1)$ is a spiral sink.



Predator-prey model

Model. The Lotka-Volterra equations (or predator-prey equations, are often used to describe the dynamics of biological systems in which two species interact, one as predator and the other as prey.

Use.

- Volterra (1926) proposed the model to explain population dynamics in sharks (Predator) and prey fish.
- Lotka (1925) proposed the model to describe a hypothetical chemical reaction in which chemical concentrations oscillate.
- Models used to study many types of relationships in biosystems which involve competitive interactions: parasite-host, tumor cells (virus)-immune system, susceptible-infectious interactions.
- The model deals with the general loss-win interactions and have been applied in chemistry and economics (a populace and and a predator institution).

Simple predator-prey model

Lotka-Volterra equations. $x(t)$ (prey population), $y(t)$ (predator population)

$$\begin{aligned} \frac{dx}{dt} &= ax - pxy = x(a - py) \qquad a, p \geq 0 \\ \frac{dy}{dt} &= -by + qxy = -y(b - qx) \qquad b, q \geq 0 \end{aligned}$$

Assumptions.

1. Prey ($x(t)$) has an unlimited food supply and would grow at the natural growth rate $x' = ax$ unless subject to predation.
2. Predator ($y(t)$) has no other food source than $x(t)$, so would starve at the natural growth rate $y' = -by$ unless prey present.
3. Rate of predation upon the prey is proportional to the rate at which the predators and the prey meet (xy). The interaction leads to
   - Decline in prey population $-pxy$,
   - Increase in predator population qxy.

Example of Lotka-Volterra equation

Equation.

$$\begin{aligned} \frac{dx}{dt} &= 4x - xy = x(4 - y) \\ \frac{dy}{dt} &= -16y + 2xy = -y(16 - 2x) \end{aligned}$$

Critical points. $(0,0)$, $(8,4)$.

Qualitative properties of example

$$\frac{dx}{dt} = 4x - xy = x(4-y)$$
$$\frac{dy}{dt} = -16y + 2xy = -y(16-2x)$$

Jacobian. Critical point: $(0,0)$. The stable solution $x \equiv 0, y \equiv 0$ is one where both populations go extinct.

$$\mathbf{J}(x,y) = \begin{bmatrix} 4-y & -x \\ 2y & -16+2x \end{bmatrix} \quad \mathbf{J}(-1,1) = \begin{bmatrix} 4 & 0 \\ 0 & -16 \end{bmatrix}$$

Eigenvalues. At (0,0): $\lambda = 4, -16$

Analysis. $(0,0)$ is a unstable saddlepoint in the linearization; so it is an unstable saddlepoint in the autonomous system.

Qualitative properties of example

$$\frac{dx}{dt} = 4x - xy = x(4-y)$$
$$\frac{dy}{dt} = -16y + 2xy = -y(16-2x)$$

Jacobian. Critical point: $(8,4)$. The stable solution $x \equiv 8, y \equiv 4$ is one where both populations coexist permanently.

$$\mathbf{J}(x,y) = \begin{bmatrix} 4-y & -x \\ 2y & -16+2x \end{bmatrix} \quad \mathbf{J}(8,4) = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$$

Eigenvalues. At (8,4): $\lambda = \pm 8i$

Analysis. $(8,4)$ is a stable center in the linearization; we can draw **no conclusions about stability** in the autonomous system:
it could be a stable center, stable spiral sink or unstable spiral source.

Finding implicit solutions

Since $(8,4)$ is a center, we cannot determine the stability of

$$\frac{dx}{dt} = 4x - xy = x(4-y)$$
$$\frac{dy}{dt} = -16y + 2xy = -y(16-2x)$$

without looking at solutions.
By the Chain rule

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-y(16-2x)}{x(4-y)}$$

We want solutions x, y to this ordinary first order equation.

Finding implicit solutions

Solve.

$$\frac{dy}{dx} = \frac{-y(16-2x)}{x(4-y)}$$

Answer. Use separation of variables:

$$\frac{y-4}{y}dy = \frac{16-2x}{x}dx$$

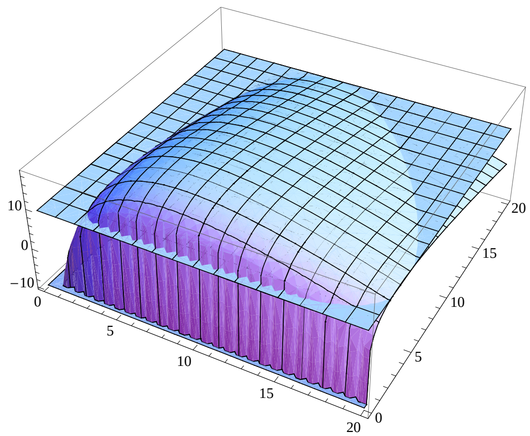
So, an implicit solution for x, y is given (for each constant C) by

$$y - 4\ln y + 2x - 16\ln x = C.$$

We can determine C from an initial value $x(0), y(0)$ (population at $t = 0$).

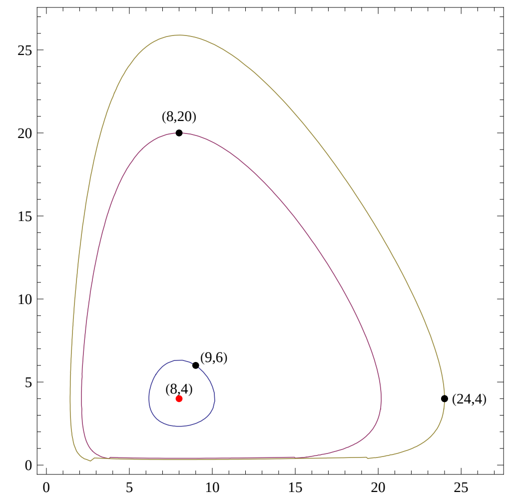
Three dimensional plot

Implicit plot of $y - 4 \ln y + 2x - 16 \ln x$. Solutions are planes $z = C$.
Here: $z = 9.139$ corresponding to $x(0) = 8, y(0) = 20$.



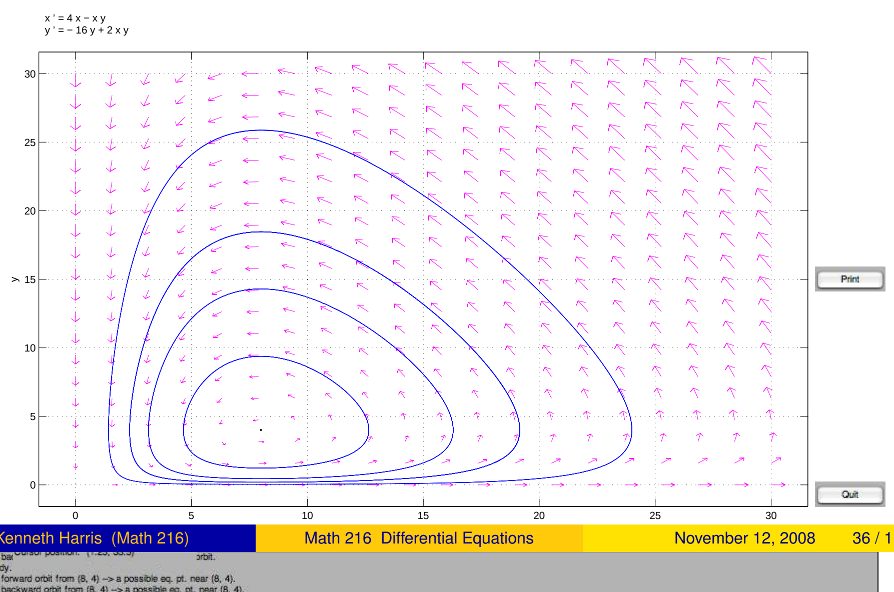
Three trajectories

Three trajectories with initial values: $(9, 6), (8, 20), (24, 4)$.



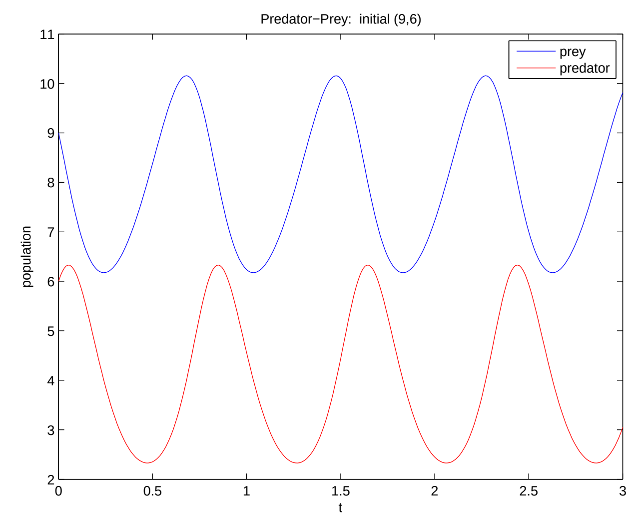
Direction field

Direction field. I had to change the solver for `pplane` to Runge-Kutta and step size to 0.005 to get accurate renderings of trajectories.



Solution through $(9, 6)$

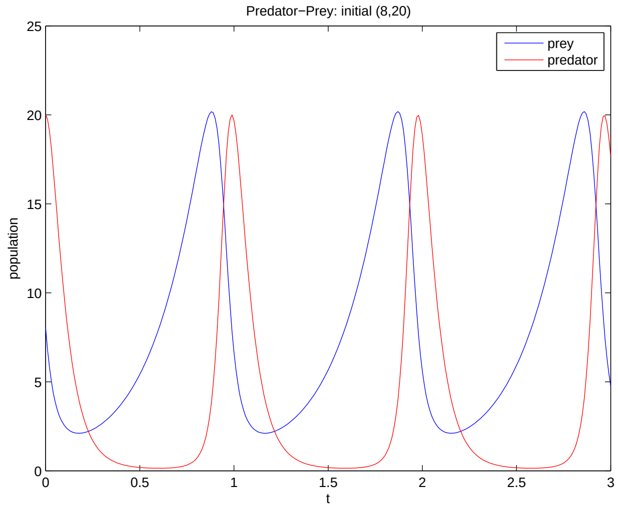
Solution trajectories for $x(t), y(t)$ for initial populations: $x(0) = 9$, $y(0) = 6$.
Generated using Runge-Kutta approximation, `rk2.m`.



Solution through $(8, 20)$

Solution trajectories for $x(t), y(t)$ for initial populations: $x(0) = 8$, $y(0) = 20$.

Generated using Runge-Kutta approximation, `rk2.m`.



Solution through $(24, 4)$

Solution trajectories for $x(t), y(t)$ for initial populations: $x(0) = 20$, $y(0) = 4$.

Generated using Runge-Kutta approximation, `rk2.m`.