Math 216
Differential Equations

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Types of critical points

Def. (a, b) is a critical point of the autonomous system

$$x' = F(x, y), \quad y' = G(x, y)$$

when $F(a, b) = 0$ and $G(a, b) = 0$.

Classification. There are two dimensions to classifying critical points

- Are trajectories drawn to or repulsed from a critical point?
- How do the trajectories approach the critical point?

There are several types of critical points.

1. Proper node (stable or unstable)
2. Improper node (stable or unstable)
3. Spiral (stable or unstable)
4. Center (always stable)
5. Saddle point (always unstable)

Stability

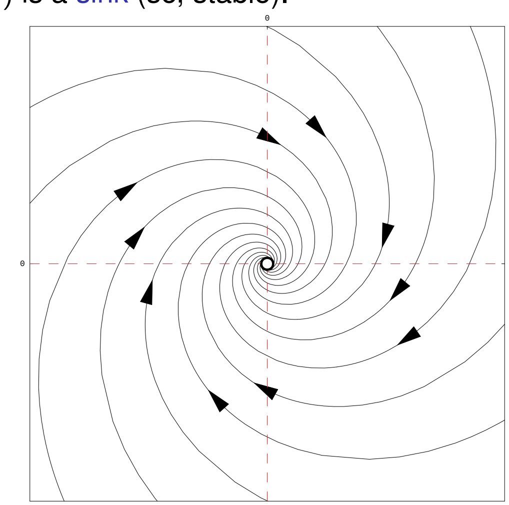
Def. A critical point (a, b) is stable provided all points sufficiently close to (a, b) remain close to it. It is asymptotically stable if all points are drawn to it.

There are four kinds of stability.

- A center is stable, but not asymptotically stable.
- A sink is asymptotically stable.
- A source is unstable and all trajectories recede from the critical point.
- A saddlepoint is unstable, although some trajectories are drawn to the critical point and other trajectories recede.

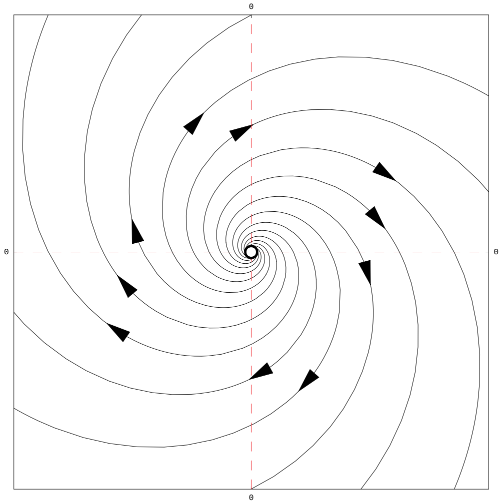
Example of a stable equilibrium

Example. $(0, 0)$ is a sink (so, stable).



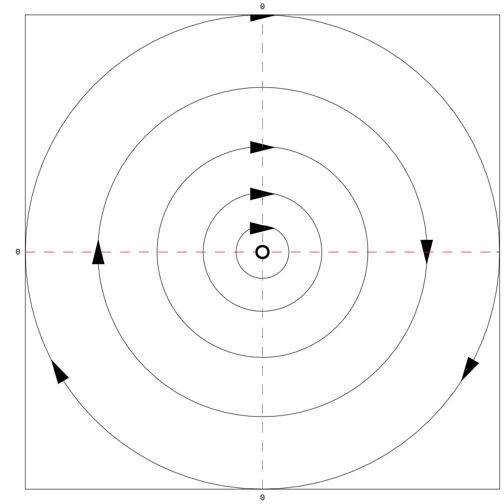
Example of an unstable equilibrium

Example. $(0, 0)$ is a source (so unstable).



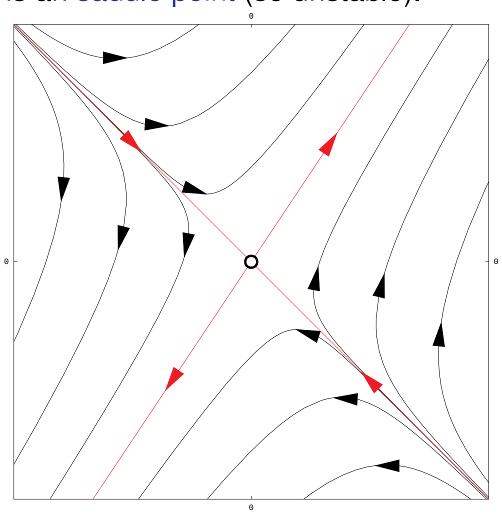
Example of a stable equilibrium

Example. $(0, 0)$ is an center (so stable).



Example: Saddle point

Example. $(0, 0)$ is an saddle point (so unstable).



How do trajectories approach or recede?

Nodes and spiral points

Definition. A critical point (a, b) is a node if

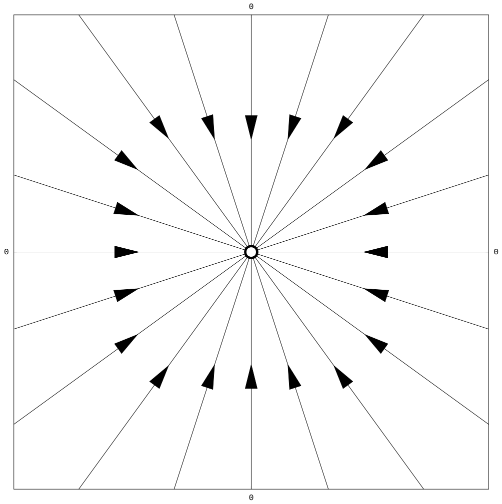
- Every trajectory approaches (a, b) as $t \to \infty$ or every trajectory recedes from (a, b) as $t \to \infty$, and
- Each trajectory approaches (or recedes) from (a, b) in a fixed direction. (That is, every trajectory is tangent to a line through (a, b).)

Three types of approach to critical points .

- A critical point is a proper node if trajectories approach or recede in all directions.
- A critical point is an improper node if all trajectories approach or recede in just two directions.
- A critical point is a spiral point if trajectories spiral around the critical point as they approach or recede. A spiral point **cannot** be a node!!

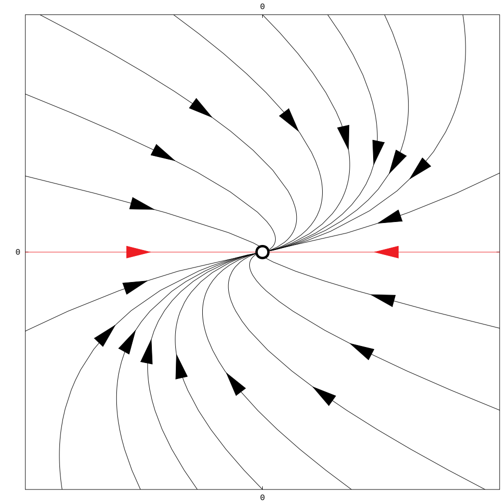
Example: A proper node

Example. $(0, 0)$ is a proper node which is stable.



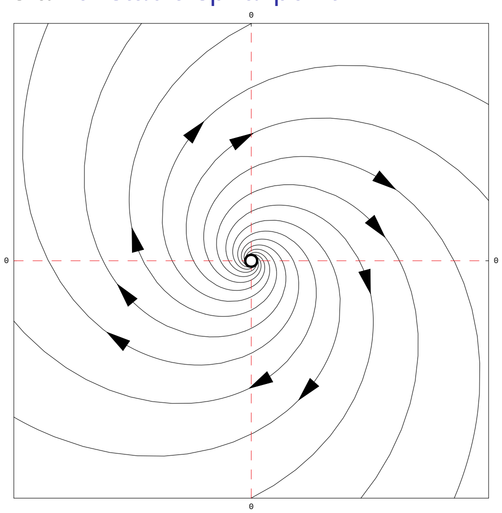
Example: A proper node

Example. $(0, 0)$ is a improper node which is stable. (Trajectories approach in only two directions.) Eigenvector solutions in red.



Example of an unstable spiral point

Example. $(0, 0)$ is an unstable spiral point.



Types of critical points

Summary. There are several types of critical points based on stability and how solutions approach.

1. Proper node (stable or unstable)
2. Improper node (stable or unstable)
3. Spiral (stable or unstable)
4. Center (always stable, but not asymptotically stable)
5. Saddle point (always unstable)

Critical points of linear systems

Linear systems. We will study the properties at critical points of 2×2 linear systems

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $ad - bc \neq 0$ (so, the matrix is invertible).

Critical point. There is only one critical point at $(0, 0)$.

Let λ_1 and λ_2 be eigenvalues. Five cases.

1. λ_1 and λ_2 are distinct with the same sign.
2. λ_1 and λ_2 are distinct with the different signs.
3. $\lambda_1 = \lambda_2$.
4. λ_1 and λ_2 are complex conjugates with nonzero real parts.
5. λ_1 and λ_2 are pure imaginary.

Real eigenvalues

Real eigenvalues. When λ_1 and λ_2 are real and distinct, solutions take the form

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the associated eigenvectors.

$c_1 = 0$: the solution takes the form

$$\mathbf{x}(t) = c_2\mathbf{v}_2 e^{\lambda_2 t}$$

$\mathbf{x}(t)$ travels along the eigenvector $\mathbf{v}_2$.

$c_2 = 0$: the solution takes the form

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t}$$

$\mathbf{x}(t)$ travels along the eigenvector $\mathbf{v}_1$.

Case 1. Distinct, real, same sign

Case 1. When $0 < \lambda_1 < \lambda_2$: solutions have the form

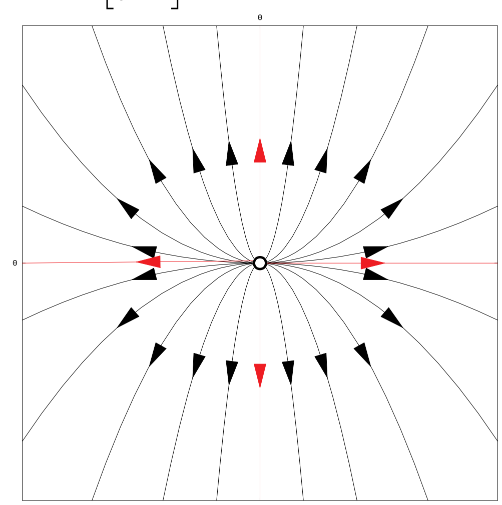
$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 \left(e^{\lambda_1 t}\right)^k \qquad \text{where } k = \frac{\lambda_2}{\lambda_1} > 1$$

Stability. $|\mathbf{x}(t)| \to \infty$ as $t \to \infty$. The origin is a source (so stable).

Type. Trajectories are drawn to $\mathbf{v}_2$. That is, trajectories not starting on $\mathbf{v}_1$ approach $\mathbf{v}_2$ at $t \to \infty$.

Case 1. Distinct, real, same sign

$0 < \lambda_1 < \lambda_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$



Case 1. Distinct, real, same sign

Case 1. When $\lambda_1 < \lambda_2 < 0$: solutions have the form

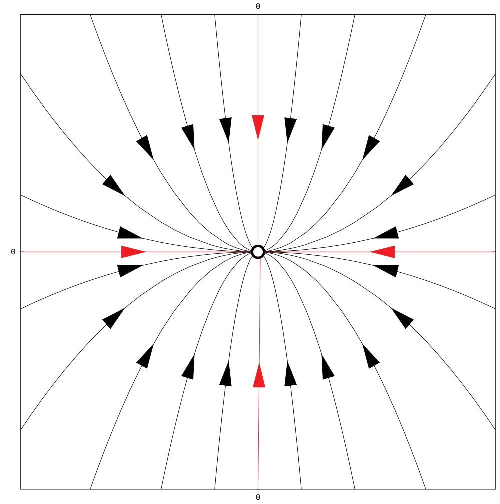
$$\mathbf{x}(t) = c_1 \mathbf{v}_1 \left(e^{\lambda_2 t}\right)^k + c_2 \mathbf{v}_2 e^{\lambda_2 t} \qquad \text{where } k = \frac{\lambda_2}{\lambda_1} > 1$$

Stability. $|\mathbf{x}(t)| \to 0$ as $t \to \infty$. The origin is a sink (so stable).

Type. Trajectories are drawn to $\mathbf{v}_2$. That is, trajectories not starting on $\mathbf{v}_1$ approach $\mathbf{v}_2$ at $t \to \infty$.

Case 1. Distinct, real, same sign

$\lambda_1 < \lambda_2 < 0$: $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$



Case 2. Distinct, real, opposite signs

Case 2. When $\lambda_1 < 0 < \lambda_2$: solutions have the form

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}$$

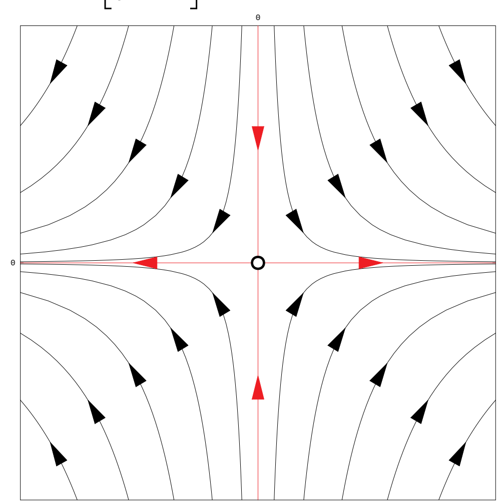
where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the associated eigenvectors.

Stability. The origin is a saddlepoint.
- $|\mathbf{x}(t)| \to \infty$ at $t \to \infty$ when $c_2 \neq 0$.
- $|\mathbf{x}(t)| \to 0$ at $t \to \infty$ when $c_2 = 0$.

Type. Eigenvectors are drawn to $\mathbf{v}_2$. That is, trajectories not starting on $\mathbf{v}_1$ approach $\mathbf{v}_2$ at $t \to \infty$.

Case 2. Distinct, real, opposite signs

$\lambda_1 < 0 < \lambda_2$: $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$



Case 3. Multiplicity two

Case 3. $\lambda_1 = \lambda = \lambda_2$. Two eigenvectors. The solution has the form

$$\mathbf{x}(t) = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2)e^{\lambda t}$$

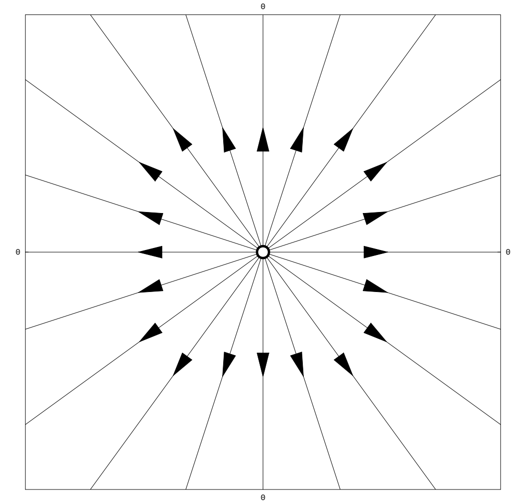
Stability.

- $\lambda > 0$: $|\mathbf{x}(t)| \to \infty$ as $t \to \infty$. The origin is a source.
- $\lambda < 0$: $|\mathbf{x}(t)| \to 0$ as $t \to \infty$. The origin is a sink.

Type. The trajectory is along the line $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. The origin is a proper node.

Case 3. Multiplicity two

Two eigenvectors. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



Case 3. Multiplicity two

Case 3. $\lambda_1 = \lambda = \lambda_2$. One eigenvector. The solution has the form

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{v}_1 e^{\lambda t} + c_2(\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} \\ &= ((c_2 t + c_1)\mathbf{v}_1 + c_2\mathbf{v}_2)e^{\lambda t} \end{aligned}$$

where $\mathbf{v}_1$ is the eigenvector associated with λ, and $\mathbf{v}_2$ is a linearly independent vector satisfying $(\mathbf{A} - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$. (See Section 5.4).

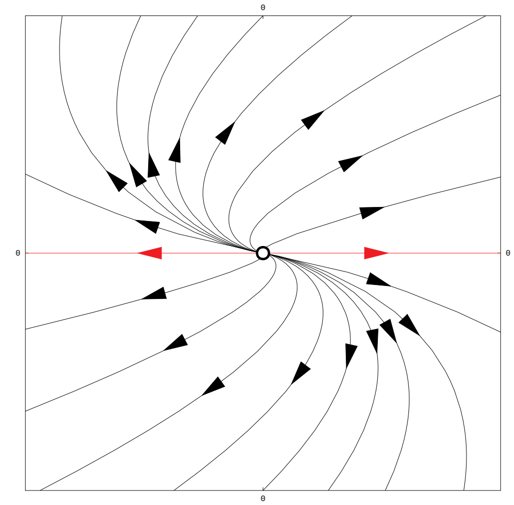
Stability.

- $\lambda > 0$: $|\mathbf{x}(t)| \to \infty$ as $t \to \infty$. The origin is a source.
- $\lambda < 0$ $|\mathbf{x}(t)| \to 0$ as $t \to \infty$. The origin is a sink.

Type. The trajectory is along $(c_2 t + c_1)\mathbf{v}_1 + c_2\mathbf{v}_2$, so is drawn to $\mathbf{v}_1$. The origin is an improper node.

Case 3. Multiplicity two

One eigenvector. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



Case 4. Complex

Case 4. λ_1 and λ_2 are complex conjugates with nonzero real parts. Let the eigenvalues be $p \pm qi$ and eigenvectors $\mathbf{a} \pm i\mathbf{b}$. A general solution:

$$\mathbf{x}(t) = e^{pt}(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t))$$

where

$$\mathbf{x}_1(t) = \mathbf{a}\cos qt + \mathbf{b}\sin qt \quad \text{and} \quad \mathbf{x}_2(t) = \mathbf{b}\cos qt - \mathbf{a}\sin qt.$$

So, $\mathbf{x}_1$ and $\mathbf{x}_2$ are periodic:

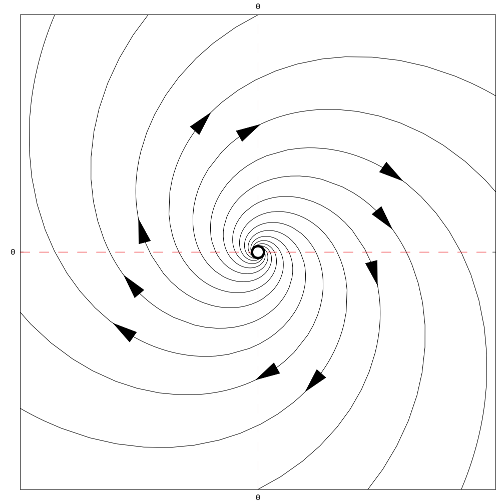
$$\mathbf{x}_1(t) = \mathbf{x}_1(t + \frac{2\pi}{q}) \quad \text{and} \quad \mathbf{x}_2(t) = \mathbf{x}_2(t + \frac{2\pi}{q})$$

Stability and Type.

- $p > 0$: $|\mathbf{x}(t)| \to \infty$ as $t \to \infty$. Origin is a spiral source,
- $p < 0$: $|\mathbf{x}(t)| \to 0$ as $t \to \infty$. Origin is a spiral sink.

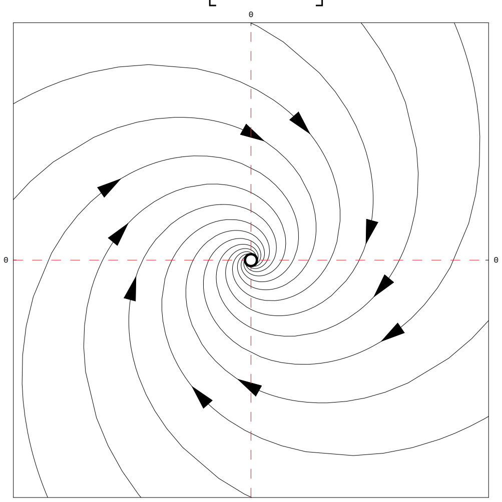
Case 4. Complex

$\lambda_1, \lambda_2 = p \pm qi, p > 0.$ $\quad \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ $\quad \lambda = 1 \pm i2$



Case 4. Complex

$\lambda_1, \lambda_2 = p \pm qi, p < 0.$ $\quad \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$ $\quad \lambda = -1 \pm 2i$



Case 5. Pure Imaginary

Case 5. λ_1 and λ_2 are pure imaginary.

Let the eigenvalues be $\pm qi$ and eigenvectors $\mathbf{a} \pm i\mathbf{b}$. A general solution:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

where

$$\mathbf{x}_1(t) = \mathbf{a}\cos qt + \mathbf{b}\sin qt \quad \text{and} \quad \mathbf{x}_2(t) = \mathbf{b}\cos qt - \mathbf{a}\sin qt.$$

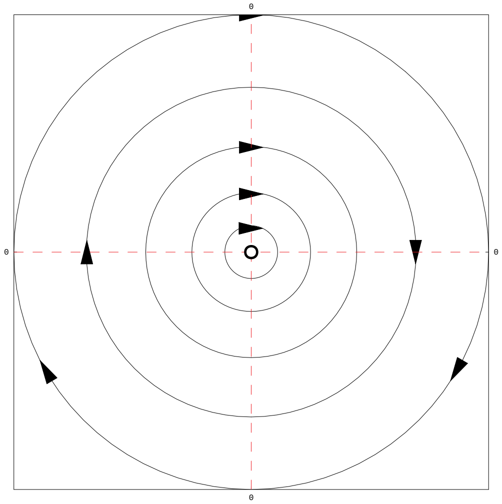
So, $\mathbf{x}_1$ and $\mathbf{x}_2$ are periodic:

$$\mathbf{x}_1(t) = \mathbf{x}_1(t + \frac{2\pi}{q}) \quad \text{and} \quad \mathbf{x}_2(t) = \mathbf{x}_2(t + \frac{2\pi}{q})$$

Stability and Type. The origin is a center. Trajectories are ellipses.

Case 5. Imaginary

$\lambda_1, \lambda_2 = p \pm qi, p > 0.$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\lambda = \pm i$



Summary

λ_1, λ_2 are eigenvalues of the linear system.
Type and stability of critical point.

- $\lambda_1 < \lambda_2 < 0$: Stable improper node.
- $\lambda_1 = \lambda_2 < 0$: Stable node
- $\lambda_1 < 0 < \lambda_2$: Unstable saddle point
- $\lambda_1 = \lambda_2 > 0$: Unstable node
- $\lambda_1 > \lambda_2 > 0$: Unstable improper node
- $\lambda_1, \lambda_2 = p \pm iq, p < 0$: Spiral sink
- $\lambda_1, \lambda_2 = p \pm iq, p > 0$: Spiral source
- $\lambda_1, \lambda_2 = \pm iq$: center

Computing eigenvalues of 2 × 2 matrices

The characteristic polynomial for the real-valued matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$\begin{aligned} \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) \\ &= \lambda^2 - \mathrm{tr}(\mathbf{A}) + \det(\mathbf{A}) \end{aligned}$$

where $\mathrm{tr}(\mathbf{A}) = a + d$ is the sum of the diagonal elements of $\mathbf{A}$.
The eigenvalues are

$$\frac{\mathrm{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}.$$

Perturbing a matrix

Perturbations. Consider a small perturbation of the elements of $\mathbf{A}$:

$$\mathbf{A}^* = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}$$

where a^*, b^*, c^*, d^* are close to a, b, c, d.

Eigenvalues are

$$\frac{\mathrm{tr}(\mathbf{A}^*)}{2} \pm \frac{1}{2}\sqrt{\mathrm{tr}(\mathbf{A}^*)^2 - 4\det(\mathbf{A}^*)}.$$

where

- $\mathrm{tr}(\mathbf{A}^*) = a^* + d^*$ is close to $\mathrm{tr}(\mathbf{A}) = a + d$
- $\det(\mathbf{A}^*) = a^*d^* - b^*c^*$ is close to $\det(\mathbf{A}) = ad - bc$

Eigenvalues under perturbation

Eigenvalues of **A**: $\frac{\mathrm{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}$.

Assumption 1. Suppose $\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A}) \neq 0$.
Then, $\mathrm{tr}(\mathbf{A}^*)^2 - 4\det(\mathbf{A}^*) \neq 0$, with the same sign.
Assumption 2. Suppose $\mathrm{tr}(\mathbf{A}) \neq 0$.
Then, $\mathrm{tr}(\mathbf{A}^*) \neq 0$, with the same sign.

Stability and Type. Two cases

- $\lambda = p \pm iq$ is a complex eigenvalue of **A**, then $\lambda^* = p^* \pm iq^*$ is an eigenvalue **A** where p, p^* have the same sign.
  A and $\mathbf{A}^*$ have the same qualitative properties at $(0,0)$.
- λ_1, λ_2 are distinct real eigenvalues for **A**, then the corresponding eigenvalues λ_1^*, λ_2^* for $\mathbf{A}^*$ are both real and have the same signs.
  A and $\mathbf{A}^*$ have the same qualitative properties at $(0,0)$.

Eigenvalues under perturbation

Eigenvalues of **A**: $\frac{\mathrm{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}$.

Assumption. Suppose **A** has one eigenvalue.
Then, $\mathrm{tr}(\mathbf{A}) \neq 0$, so $\mathrm{tr}(\mathbf{A}^*) \neq 0$, with the same sign.
However, $\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A}) = 0$.

Stability and Type. Three possibilities

1. $\mathrm{tr}(\mathbf{A}^*)^2 - 4\det(\mathbf{A}^*) = 0$. Same qualitative properties at $(0,0)$.
2. $\mathrm{tr}(\mathbf{A}^*)^2 - 4\det(\mathbf{A}^*) > 0$. The eigenvalues of $\mathbf{A}^*$ are distinct reals with the same sign.
3. $\mathrm{tr}(\mathbf{A}^*)^2 - 4\det(\mathbf{A}^*) < 0$. Then eigenvalues of $\mathbf{A}^*$ are complex and $(0,0)$ is a spiral point with the same stability as that of **A**.

Eigenvalues under perturbation

Eigenvalues of **A**: $\frac{\mathrm{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}$.

Assumption. Suppose **A** has imaginary eigenvalues.
Then, $\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A}) < 0$, and this remains true for $\mathbf{A}^*$.
However, $\mathrm{tr}(\mathbf{A}) = 0$.

Stability and Type. Three possibilities

1. $\mathrm{tr}(\mathbf{A}) = 0$. No change in the qualitative properties at $(0,0)$.
2. $\mathrm{tr}(\mathbf{A}) > 0$. Then $(0,0)$ is now a spiral source.
3. $\mathrm{tr}(\mathbf{A}) < 0$. Then $(0,0)$ is now a spiral sink.

Summary

Theorem

*Let λ_1 and λ_2 be the eigenvalues of a matrix **A**, and $\mathbf{A}^*$ a sufficiently small perturbation of **A**. Then, the qualitative properties of the critical point $(0,0)$ for $\mathbf{A}^*$ satisfies*

1. *If $\lambda_1 = \lambda_2$, then $(0,0)$ of $\mathbf{A}^*$ is either a node, or a spiral point. It is asymptotically stable if $\lambda_1 < 0$ and unstable if $\lambda_1 > 0$.*
2. *If λ_1 and λ_2 are pure imaginary, then $(0,0)$ is either a center or a spiral. It could be any of asymptotically stable, stable, or unstable.*
3. *Otherwise, $(0,0)$ has the same type and stability at $\mathbf{A}^*$ as at **A**.*

Summary

Distribution of critical points in the Trace-Determinant plane.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad T = \mathrm{tr}(\mathbf{A}) = a + d, \quad D = \det(\mathbf{A}) = ad - bc.$$

Sensitive areas: Places where type of critical point sensitive to perturbations.