Math 216
Differential Equations

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Numerical approximation

Goal. To apply Euler's method of approximation to the initial value problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

where t is the independent variable and

$$\mathbf{x} = (x_1(t), x_2(t), \ldots, x_m(t)) \quad \mathbf{f} = (f_1(t, \mathbf{x}), f_2(t, \mathbf{x}), \ldots, f_m(t, \mathbf{x}))$$

If the component functions of $\mathbf{f}$ are continuous in a neighborhood of $(t_0, \mathbf{x}_0)$, then there will be a unique solution to the IVP.

Iteration Method

Iteration method. Fix a step size h. We approximate the solution $\mathbf{x}(t)$ at points $t_0, t_1, t_2, \ldots$ where

- t_0 is the given initial moment.
- $t_{n+1} = t_n + h$ for all $n \geq 0$.

We compute a sequence of approximations $\mathbf{x}_i$ to the actual value $\mathbf{x}(t_i)$:

$$\mathbf{x}(t_0) = \mathbf{x}_0, \ \mathbf{x}(t_1) \approx \mathbf{x}_1, \ \mathbf{x}(t_2) \approx \mathbf{x}_2, \ \ldots, \ \mathbf{x}(t_n) \approx \mathbf{x}_n,$$

We will use Euler's Method to compute the approximations here. (But all the methods of Sections 2.4 to 2.6 can be extended to systems. See Section 4.3.)

Euler's Method

Method. We use the iterative formula

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}(t, \mathbf{x}_n),$$

with step size h.

Concrete case. When $m = 2$, we have an initial value problem

$$\begin{aligned} x' &= f(t, x, y) \quad x(t_0) = x_0 \\ y' &= g(t, x, y) \quad y(t_0) = y_0 \end{aligned}$$

Given the n step value, (x_n, y_n), we compute the $n+1$st approximation:

$$\begin{aligned} x_{n+1} &= x_n + hf(t_n, x_n, y_n) \\ y_{n+1} &= y_n + hg(t_n, x_n, y_n) \end{aligned}$$

where $t_{n+1} = t_n$.

Example of Euler's Method

Example. Approximate the solution to the initial value problem

$$\begin{aligned} x' &= 3x - y \quad x(0) = 2 \\ y' &= x + y \quad y(0) = 1 \end{aligned}$$

with step size 0.1.

Euler's method uses the following formulas

$$\begin{aligned} x_{n+1} &= x_n + 0.1(3x_n - y_n) \\ y_{n+1} &= y_n + 0.1(x_n + y_n) \end{aligned}$$

Example of Euler's Method

$$\begin{aligned} x_{n+1} &= x_n + 0.1(3x_n - y_n) \\ y_{n+1} &= y_n + 0.1(x_n + y_n) \end{aligned}$$

Step 1.

$$\begin{aligned} x_1 &= 2 + 0.1(6 - 1) = 2.5 \\ y_1 &= 1 + 0.1(2 + 1) = 1.3 \end{aligned}$$

Step 2.

$$\begin{aligned} x_2 &= 2.5 + 0.1(7.5 - 1.3) = 3.12 \\ y_2 &= 1.3 + 0.1(2.5 + 1.3) = 1.68 \end{aligned}$$

Approximation. $x(0.2) \approx 3.12, \quad y(0.2) \approx 1.68$

Example of Euler's Method

The actual solution functions are

$$x(t) = (t+2)e^{2t} \quad y(t) = (t+1)e^{2t}$$

Our approximation:

$$x(0.2) \approx x_2 = 3.12 \quad y(0.2) \approx y_2 = 1.68.$$

Actual values:

$$x(0.2) = 2.98 \quad y(0.2) = 1.79.$$

Autonomous systems of equations

Definition. A two-dimensional system of first-order equations in which the dependent variable t does not explicitly appear:

$$\begin{aligned} \frac{dx}{dt} &= F(x, y) \\ \frac{dy}{dt} &= G(x, y) \end{aligned}$$

is called an autonomous system.

Note. We will consider systems where F and G are continuously differentiable in some region R of the xy-plane. This implies

> Given any t_0 and choice point (x_0, y_0) in R, there is a unique solution $x = x(t)$, $y = y(t)$ on some open interval containing t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$.

See Theorem 1 of Section 4.1.

Examples of autonomous systems

Example 1. An autonomous system:

$$\frac{dx}{dt} = x + 2y$$
$$\frac{dy}{dt} = 3x + 2y$$

Example 2. An autonomous system:

$$\frac{dx}{dt} = \sin x + \cos y + x^3 y^2$$
$$\frac{dy}{dt} = e^{x+y}$$

Example 3. **Not** an autonomous system:

$$\frac{dx}{dt} = 2x - y + e^t$$
$$\frac{dy}{dt} = x - 2y + e^{-t}$$

Phase plane

Plotting solutions. Let $(x(t), y(t))$ be a solution to the autonomous system

$$\frac{dx}{dt} = F(x, y) \qquad \frac{dy}{dt} = G(x, y)$$

As t varies, the solution $(x(t), y(t))$ describes a curve in the xy-plane, called a trajectory.

The xy-plane is called the phase plane. A picture showing several trajectories is called a phase portrait.

In order to plot time we need three dimensions: $(t, x(t), y(t))$. Instead, we use arrows to show the direction of time.

Slope field

We can plot solutions to autonomous systems using a slope field in the phase plane (as in Section 1.3).

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

By the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt},$$

so,

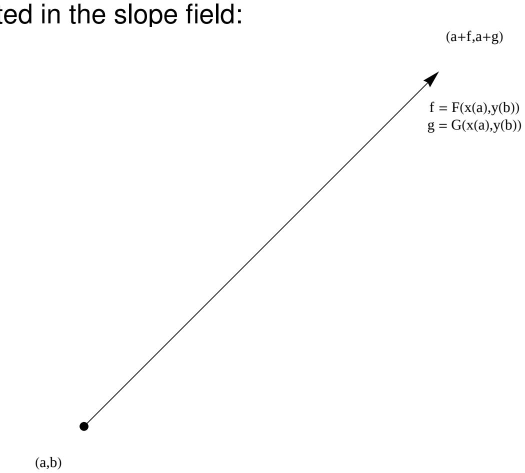
$$\frac{dy}{dx} = \frac{\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{G(x, y)}{F(x, y)}.$$

This is the slope of the tangent line to the point $(x(t), y(t))$ for a solution $x = x(t)$, $y = y(t)$.

Slope field

Problem. The phase plane drops the independent variable t, so knowing the slope alone does not tell you the direction of the solution.

Solution. Place an arrow (not just a slope line) showing the direction at each point (a, b) plotted in the slope field:

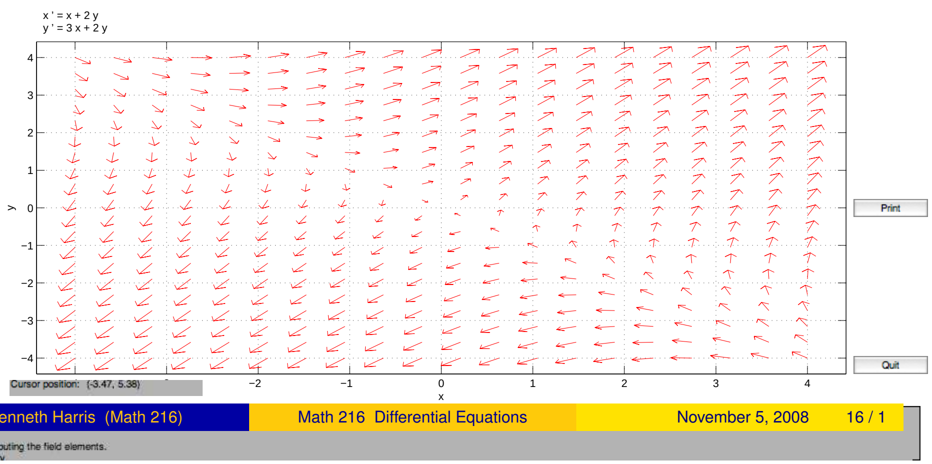


Example

Example. A direction field for the autonomous linear system

$$\frac{dx}{dt} = x + 2y \qquad \frac{dy}{dt} = 3x + 2y.$$

Drawn in Matlab with `pplane`.



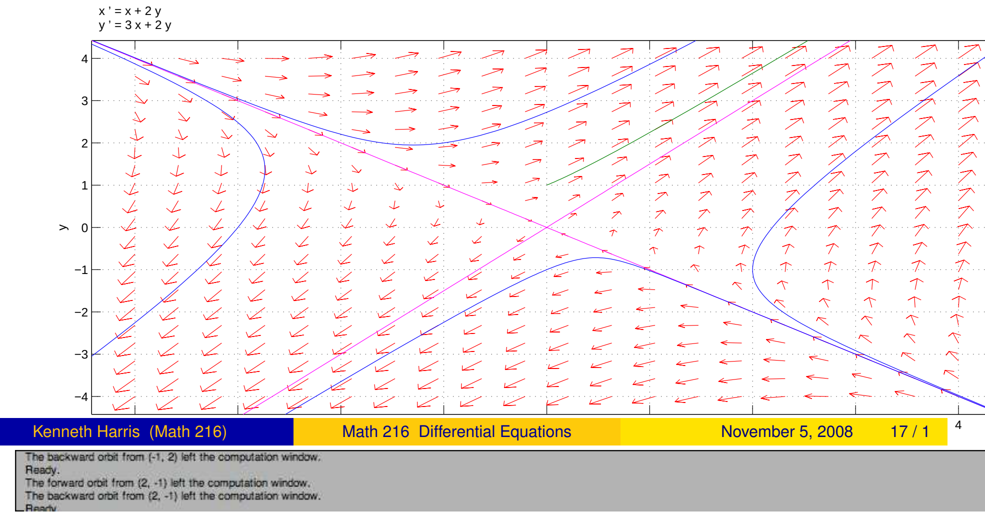
The phase plane

Example

Example. A phase portrait with several trajectories.

$$\frac{dx}{dt} = x + 2y \qquad \frac{dy}{dt} = 3x + 2y.$$

x(0)=0, y(0)=1 is one solution.

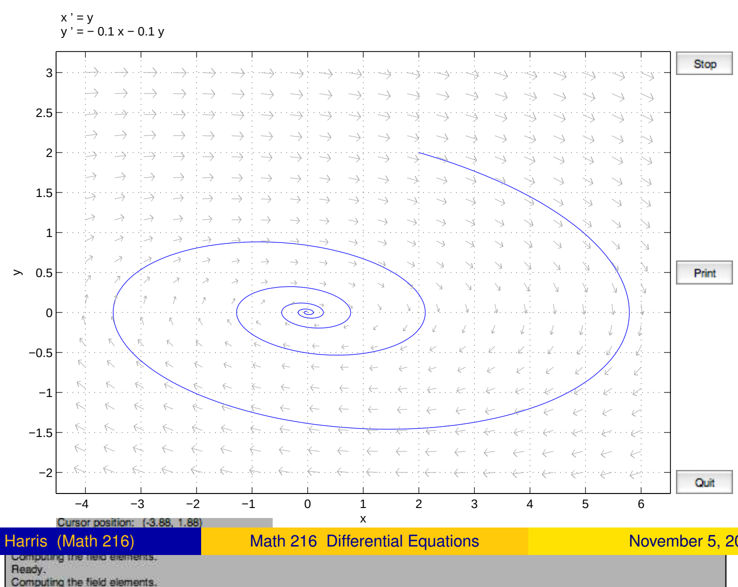


The phase plane

Example: mass-spring

Mass-spring: where $m = 1$, $c = 0.1$ and $k = 0.1$

$$x'' + 0.1x' + 0.1x = 0, \quad x(0) = 2, \; x'(0) = 2.$$



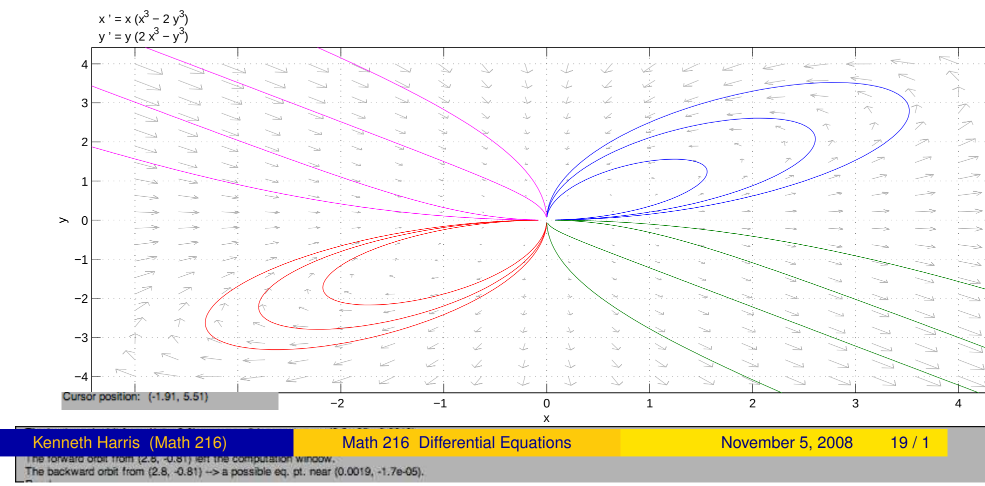
The phase plane

Example

Example. Folia of Descartes.

$$\frac{dx}{dt} = x(x^3 - 2y^3) \qquad \frac{dy}{dt} = y(2x^3 - y^3).$$

x(0)=0, y(0)=1 is one solution.

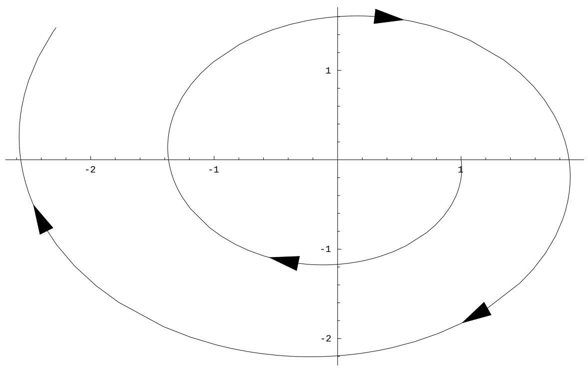


Example phase diagram

Example. The trajectory for the initial value problem

$$\frac{dx}{dt} = \frac{1}{10}x + y, \qquad \frac{dy}{dt} = -x + \frac{1}{10}y, \qquad x(0) = 1, y(0) = 0.$$

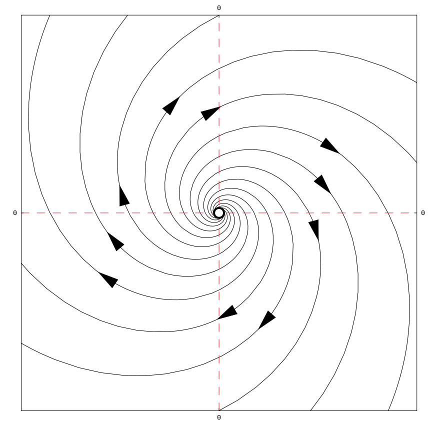
The solution: $\quad x(t) = e^{0.1t}\cos t, \quad y(t) = e^{0.1t}\sin t.$



Example phase portrait

Example. A phase portrait with several solution trajectories of

$$\frac{dx}{dt} = \frac{1}{2}x + y, \qquad \frac{dy}{dt} = -x + \frac{1}{2}y.$$

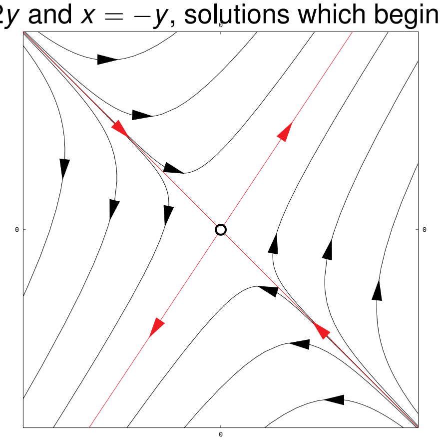


Example phase portrait

Example. A phase portrait with several solution trajectories of

$$\frac{dx}{dt} = x + 2y, \qquad \frac{dy}{dt} = 3x + 2y.$$

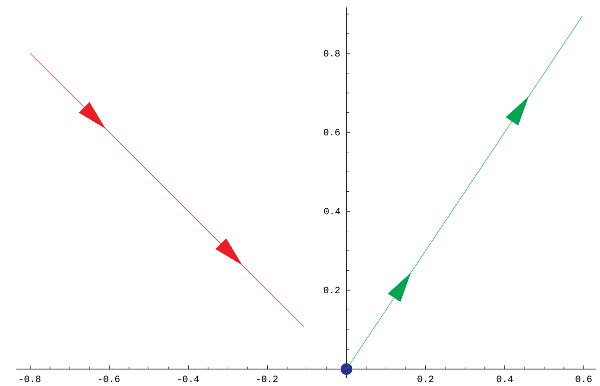
The lines are $3x = 2y$ and $x = -y$, solutions which begin at eigenvectors.



Example phase portrait

Example. Two initial values beginning at eigenvectors:

$$\frac{dx}{dt} = x + 2y, \qquad \frac{dy}{dt} = 3x + 2y \qquad \mathbf{x}_1 = \begin{bmatrix} -0.8 \\ 0.8 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 0.02 \\ 0.03 \end{bmatrix}$$



Types of trajectories

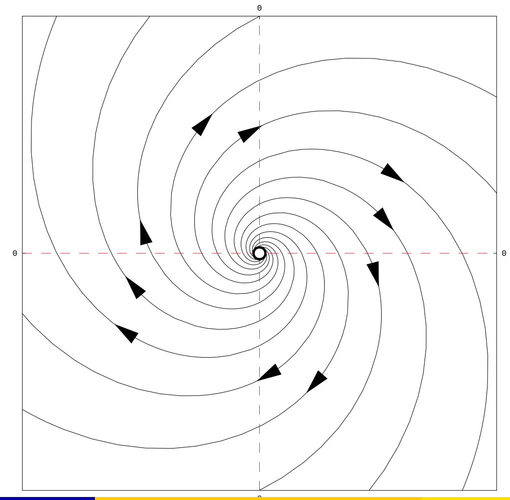
Theorem. There are four possible trajectories for the system

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y)$$

1. Constant (equilibrium solution)
2. Nonintersecting unbounded curve
3. Simple closed curve (periodic solution)
4. Nonintersecting bounded curve (which always either approaches a critical point or a closed trajectory).

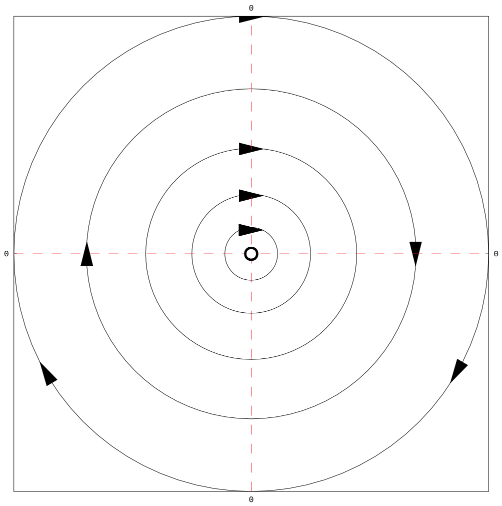
Example

Example. Nonintersecting unbounded curve.



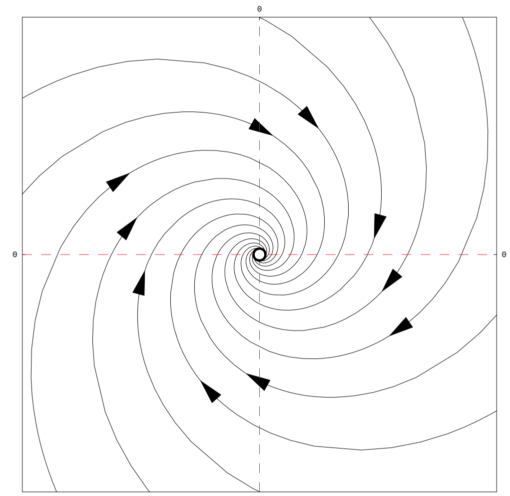
Example periodic solutions

Example. Simple closed curve.



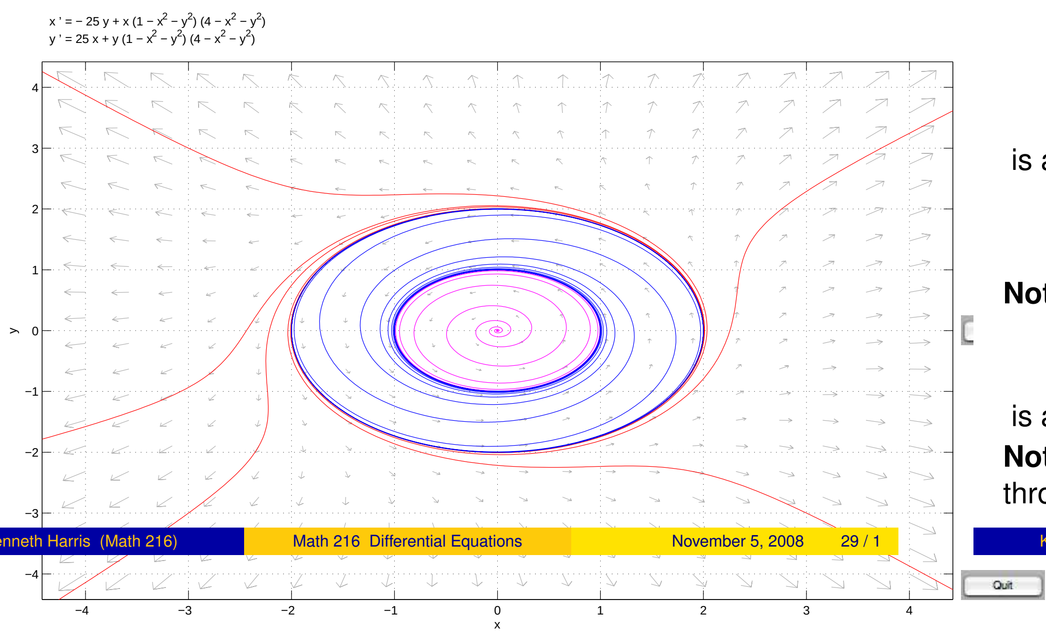
Example

Example. Bounded curve approaching a critical point.



Example

Example. Bounded curve approaching a closed trajectory.



Critical points

Definition. A critical point of an autonomous system

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y)$$

is a point (a,b) of the xy-plane such that

$$F(a,b) = G(a,b) = 0$$

Note. If (a,b) is a critical point, then the constant valued functions:

$$x(t) \equiv a \qquad y(t) \equiv b$$

is a solution, called the equilibrium solution.

Note. The equilibrium solutions are the only solutions which pass through (a,b) (by the uniqueness theorem).

Example of critical points

Example. Find the critical points of the autonomous system

$$\frac{dx}{dt} = -y - x^2$$
$$\frac{dy}{dt} = x - y - x^2 + xy$$

Answer. The critical points are solutions to the following conditions

$$y = -x^2 \qquad x - y - x^2 + xy = 0$$

Substitute the first equation into the second:

$$0 = x - x^3 = x(1 - x^2)$$

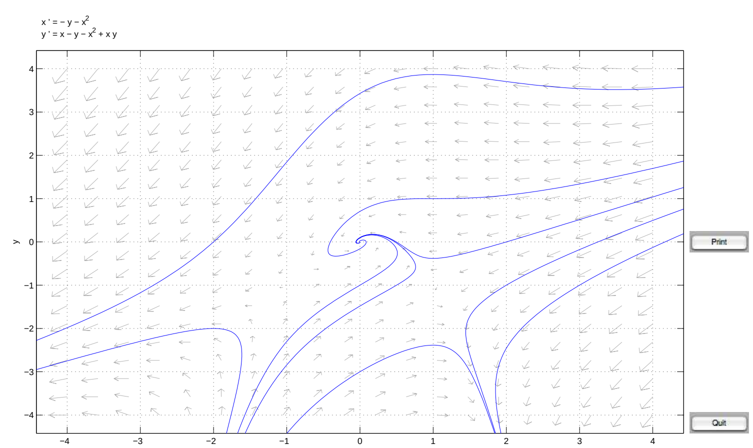
So, the critical points occur at

$$(0,0), \quad (1,-1), \quad (-1,-1)$$

Example: critical points

Example. Critical points: $(0,0), \quad (1,-1), \quad (-1,-1)$

$$\frac{dx}{dt} = -y - x^2 \qquad \frac{dy}{dt} = x - y - x^2 + xy$$



The backward orbit from (2, -1) left the computation window.
Ready.

Types of critical points

Classification. There are two dimensions to classifying critical points

- Are trajectories drawn to or repulsed from a critical point?
- How do the trajectories approach the critical point?

There are several types of critical points.

1. Proper node (stable or unstable)
2. Improper node (stable or unstable)
3. Spiral (stable or unstable)
4. Center (always stable)
5. Saddle point (always unstable)

Stability

Informal. A critical point (a, b) is stable provided all points sufficiently close to (a, b) remain close to it.

Formal. For each $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\mathbf{x}(0) - (a, b)\right| < \delta \quad \text{implies} \quad \left|\mathbf{x}(t) - (a, b)\right| < \epsilon \quad \text{for all } t > 0.$$

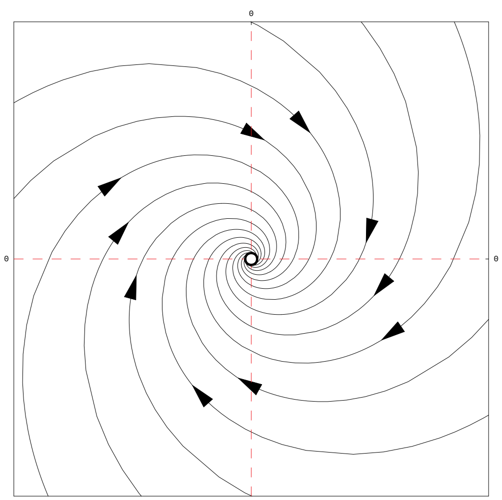
where $\mathbf{x}$ is any possible trajectory.

Where $|\cdot|$ is length:

$$\left|(c_1, c_2) - (d_1, d_2)\right| = \sqrt{(c_1 - d_1)^2 + (c_2 - d_2)^2}.$$

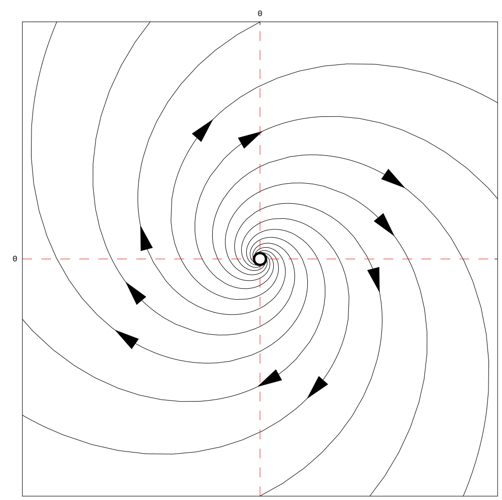
Example of a stable equilibrium

Example. $(0, 0)$ is a stable equillibrium. This is a sink, because all trajectories are drawn to it.



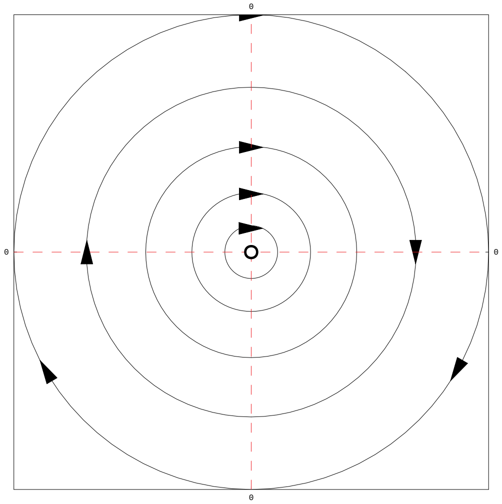
Example of an unstable equilibrium

Example. $(0, 0)$ is an unstable equillibrium. This is a source, because all trajectories emanate from it.



Example of a stable equilibrium

Example. $(0,0)$ is an stable equillibrium. This is a center, because the trajectories are simple closed curves representing periodic solutions.



Asymptotic Stability

Informal. A critical point (a,b) is asymptotically stable provided all points sufficiently close to (a,b) are drawn to it.

Formal. There is a $\delta > 0$ such that

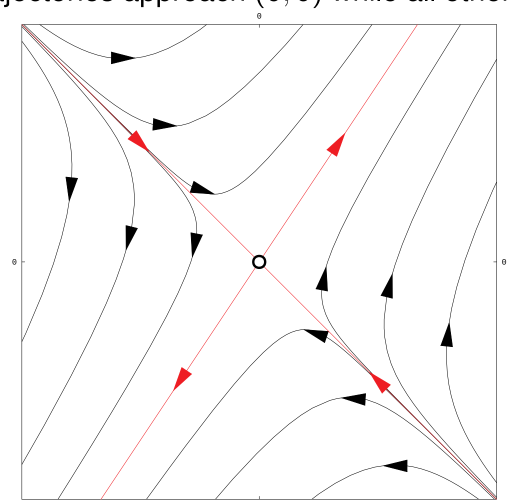
$$\left|\mathbf{x}(0) - \begin{bmatrix} a \\ b \end{bmatrix}\right| < \delta \quad \text{implies} \quad \lim_{t\to\infty} \mathbf{x}(t) = \begin{bmatrix} a \\ b \end{bmatrix}$$

where $\mathbf{x}$ is any possible trajectory.

Note. A sink is asymptotically stable, while a center is stable but not asymptotically stable.

Example: Saddle point

Example. $(0,0)$ is an unstable equillibrium. This is a saddle point, because two trajectories approach $(0,0)$ while all others are repulsed.



How do trajectories approach or recede?

Nodes and spiral points

Definition. A critical point (a,b) is a node if

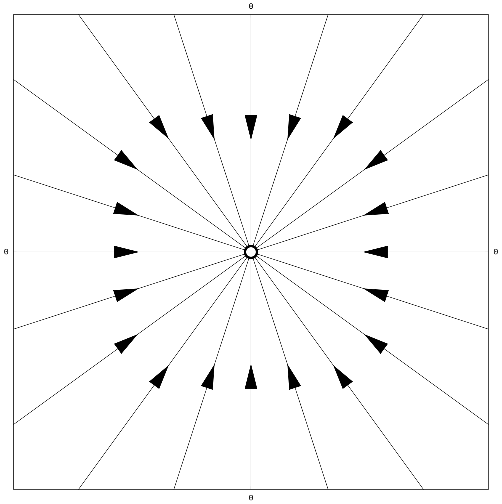
- Every trajectory approaches (a,b) as $t\to\infty$ or every trajectory recedes from (a,b) as $t\to\infty$, and
- Each trajectory approaches (or recedes) from (a,b) in a fixed direction. (That is, every trajectory is tangent to a line through (a,b)).

A node is proper if trajectories approach or recede in all directions; a node is improper if all trajectories approach or recede in just two directions.

A critical point (a,b) is a spiral point if trajectories spiral around the critical point as the approach or recede. A spiral point **cannot** be a node!!

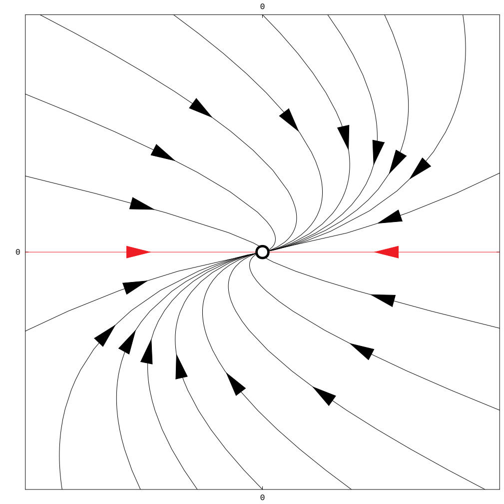
Example: A proper node

Example. $(0, 0)$ is a proper node which is stable. (Trajectories approach in all directions.)



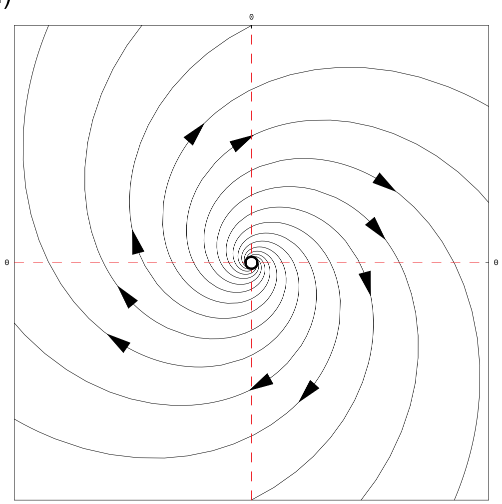
Example: A proper node

Example. $(0, 0)$ is a improper node which is stable. (Trajectories approach in only two directions.) Eigenvector solutions in red.



Example of an unstable spiral point

Example. $(0, 0)$ is an unstable spiral point. (Trajectories sprial around the critical point.)



Types of critical points

Summary. There are several types of critical points based on stability and how solutions approach.

1. Proper node (stable or unstable)
2. Improper node (stable or unstable)
3. Spiral (stable or unstable)
4. Center (always stable, but not asymptotically stable)
5. Saddle point (always unstable)