Math 216
Differential Equations

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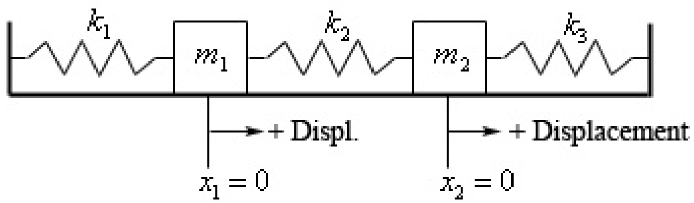
Two mass-three spring system

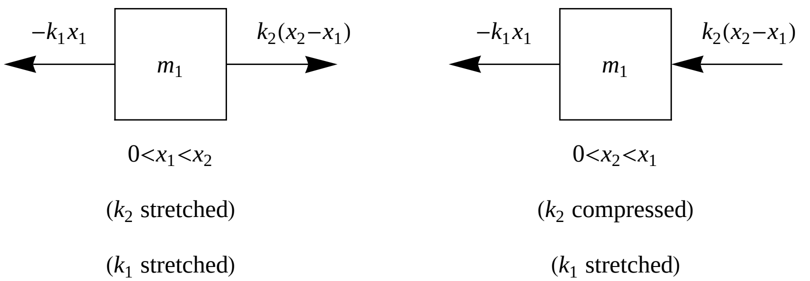
- $m_1, m_2 > 0$, two masses
- $k_1, k_2, k_3 > 0$, spring elasticity
- $x_1(t), x_2(t)$, displacement of m_1, m_2 from equilibrium.
- Positive is right, negative left
- No friction or external forces



Forces on mass m_1

Apply Newton's law to mass m_1.

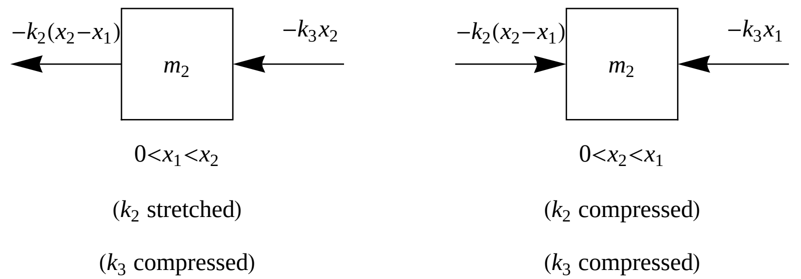
$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) \\ &= -(k_1 + k_2)x_1 + k_2 x_2. \end{aligned}$$



Forces on mass m_2

Apply Newton's law to mass m_2.

$$\begin{aligned} m_2 x_2'' &= -k_2(x_2 - x_1) - k_3 x_2 \\ &= k_2 x_1 - (k_2 + k_3)x_2. \end{aligned}$$



Two mass-three spring system: Equations

Two mass, three spring system as a system of equations:

$$\begin{aligned} m_1 x_1'' &= -(k_1+k_2)x_1 + k_2 x_2 \\ m_2 x_2'' &= k_2 x_1 - (k_2+k_3)x_2. \end{aligned}$$

As a matrix equation (dividing by m_1, m_2):

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{(k_2+k_3)}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The equation has the form

$$\mathbf{x}'' = \mathbf{A}\mathbf{x}.$$

First method for solving second order system

First of three ways to solve the second-order system:

$$\begin{aligned} m_1 x_1'' &= -(k_1+k_2)x_1 + k_2 x_2 \\ m_2 x_2'' &= k_2 x_1 - (k_2+k_3)x_2. \end{aligned}$$

Method 1. Rewrite using differential operators:

$$\begin{aligned} \left(m_1 D^2 + (k_1+k_2)I\right)x_1 - k_2 x_2 &= 0 \\ -k_2 x_1 + \left(m_2 D^2 + (k_2+k_3)I\right)x_2 &= 0. \end{aligned}$$

Solve using Method of Elimination from Section 4.2.

Second method for solving second order system

Second of three ways to solve the second-order system:

$$\begin{aligned} m_1 x_1'' &= -(k_1+k_2)x_1 + k_2 x_2 \\ m_2 x_2'' &= k_2 x_1 - (k_2+k_3)x_2. \end{aligned}$$

Method 2. Rewrite as a system of four equations and four unknowns (Section 4.1):

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{(k_1+k_2)}{m_1} y_1 + \frac{k_2}{m_1} y_3 \\ y_3' &= y_4 \\ y_4' &= \frac{k_2}{m_1} y_1 - \frac{(k_2+k_3)}{m_2} y_3 \end{aligned}$$

Solve using Eigenvalue Method of Section 5.2.

Third method for solving second order system

Third of three ways to solve the second-order system:

$$\begin{aligned} m_1 x_1'' &= -(k_1+k_2)x_1 + k_2 x_2 \\ m_2 x_2'' &= k_2 x_1 - (k_2+k_3)x_2. \end{aligned}$$

Method 3. Rewrite as a 2×2 matrix equation (Section 5.1):

$$\mathbf{x}'' = \begin{bmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{(k_2+k_3)}{m_2} \end{bmatrix} \mathbf{x}$$

Solve using the (extended) Eigenvalue Method of Section 5.3.

Extended eigenvalue method

Solve. $\mathbf{x}'' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ has real constant components.

Guess a solution: $\mathbf{x}(t) = \mathbf{v}e^{\alpha t}$.

Substitute into $\mathbf{x}'' = \mathbf{A}\mathbf{x}$: since $\left(\mathbf{v}e^{\alpha t}\right)'' = \alpha^2 \mathbf{v}e^{\alpha t}$,

$$\alpha^2 \mathbf{v}e^{\alpha t} = \mathbf{A}\mathbf{v}e^{\alpha t}$$

Cancel $e^{\alpha t}$

$$\alpha^2 \mathbf{v} = \mathbf{A}\mathbf{v}$$

Answer. $\mathbf{x}(t) = \mathbf{v}e^{\alpha t}$ is a solution when
α^2 is an eigenvalue and
$\mathbf{v}$ an associated eigenvector of $\mathbf{A}$.

Mechanical systems

Solution. If $\lambda = \alpha^2$ is an eigenvalue and $\mathbf{v}$ is an eigenvector, $\mathbf{v}e^{\alpha t}$ is a solution to $\mathbf{x}'' = \mathbf{A}\mathbf{x}$.

Application. Mass-spring problems are of the form $\mathbf{x}'' = \mathbf{A}\mathbf{x}$ where where the eigenvalues are negative.

Let $\alpha^2 = -\omega^2$ and $\mathbf{v}$ an associated (real-valued) eigenvector. Then

$$\mathbf{x}(t) = \mathbf{v}e^{i\omega t} = \mathbf{v}(\cos \omega t + i \sin \omega t)$$

is a solution to $\mathbf{x}'' = \mathbf{A}\mathbf{x}$.

Real Solutions. The real and imaginary parts

$$\mathbf{x}_1(t) = \mathbf{v}\cos \omega t \qquad \mathbf{x}_2(t) = \mathbf{v}\sin \omega t$$

are linearly independent real-valued solutions to $\mathbf{x}'' = \mathbf{A}\mathbf{x}$.

Second-order homogeneous linear systems

The following is useful for mechanical systems.

Theorem

Suppose an $n \times n$ matrix $\mathbf{A}$ has distinct negative eigenvalues $-\omega_1^2, -\omega_2^2, \ldots, -\omega_n^2$ with associated real eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Then a general solution to $\mathbf{x}'' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = \sum_{j=1}^{n} (a_j \cos \omega_j t + b_j \sin \omega_j t)$$

where a_i and b_i are parameters.

Example 1

Problem. Find a general solution to the mass-spring system:

- $m_1 = m_2 = 1$, two masses
- $k_1 = k_3 = 4$, $k_2 = 6$, spring elasticity
- $x_1(t), x_2(t)$, displacement of m_1, m_2 from equilibrium.
- No friction or outside forces

The second-order equations are

$$\begin{aligned} x_1'' &= -10x_1 + 6x_2 \\ x_2'' &= 6x_1 - 10x_2. \end{aligned}$$

Equivalently, as a matrix equation:

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 1 continued

Compute eigenvalues.

$$\begin{vmatrix} -10-\lambda & 6 \\ 6 & -10-\lambda \end{vmatrix} = \lambda^2 + 20\lambda + 64 = (\lambda+16)(\lambda+4)$$

So, $\lambda = -16, -4$.

There are four linearly independent real solutions

$$\mathbf{v}_1 \cos 4t, \quad \mathbf{v}_1 \sin 4t, \quad \mathbf{v}_2 \cos 2t, \quad \mathbf{v}_2 \sin 2t$$

where $\mathbf{v}_1$ is an eigenvector for $\lambda = -16$ and
$\mathbf{v}_2$ is an eigenvector for $\lambda = -4$.

Example 1 continued

Compute eigenvector for $\lambda = -16$. These are solutions to

$$\begin{bmatrix} -10+16 & 6 \\ 6 & -10+16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduces to a single equation

$$6x_1 + 6x_2 = 0 \quad \text{or} \quad x_1 = -x_2.$$

Eigenvector. $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example 1 continued

Compute eigenvector for $\lambda = -4$. These are solutions to

$$\begin{bmatrix} -10+4 & 6 \\ 6 & -10+4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduces to a single equation

$$-6x_1 + 6x_2 = 0 \quad \text{or} \quad x_1 = x_2.$$

Eigenvector. $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example 1 concluded

General Solution. The mass-spring system

$$\begin{aligned} x_1'' &= -10x_1 + 6x_2 \\ x_2'' &= 6x_1 - 10x_2. \end{aligned}$$

has a general solution (as a vector equation)

$$\vec{x}(t) = \left(a_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos 4t + a_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin 4t\right) + \left(b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos 2t + b_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin 2t\right)$$

and as a scalar system:

$$\begin{aligned} x_1(t) &= -(a_1 \cos 4t + a_2 \sin 4t) + (b_1 \cos 2t + b_2 \sin 2t) \\ x_2(t) &= (a_1 \cos 4t + a_2 \sin 4t) + (b_1 \cos 2t + b_2 \sin 2t) \end{aligned}$$

Example 1 analysis

Analysis. The general solution

$$\begin{aligned} x_1(t) &= -(a_1 \cos 4t + a_2 \sin 4t) + (b_1 \cos 2t + b_2 \sin 2t) \\ x_2(t) &= (a_1 \cos 4t + a_2 \sin 4t) + (b_1 \cos 2t + b_2 \sin 2t) \end{aligned}$$

can be simplified to

$$\begin{aligned} x_1(t) &= -c_1 \cos(4t - \alpha_1) + c_2 \cos(2t - \alpha_2) \\ x_2(t) &= c_1 \cos(4t - \alpha_1) + c_2 \cos(2t - \alpha_2) \end{aligned}$$

where

- $c_1 = \sqrt{a_1^2 + a_2^2}$, $\tan \alpha_1 = \frac{a_2}{a_1}$,
- $c_2 = \sqrt{b_1^2 + b_2^2}$, $\tan \alpha_1 = \frac{b_2}{b_1}$

Example 1 analysis

The displacements x_1 of mass m_1 and x_2 of mass m_2:

$$\begin{aligned} x_1(t) &= -c_1 \cos(4t - \alpha_1) + c_2 \cos(2t - \alpha_2) \\ x_2(t) &= c_1 \cos(4t - \alpha_1) + c_2 \cos(2t - \alpha_2) \end{aligned}$$

A linear combination of two natural modes of oscillation:
the natural frequencies $\omega_1 = 4$ and $\omega_2 = 2$.

The natural mode $\omega_2 = 2$

$$\begin{aligned} x_1(t) &= c_2 \cos(2t - \alpha_2) \\ x_2(t) &= c_2 \cos(2t - \alpha_2) \end{aligned}$$

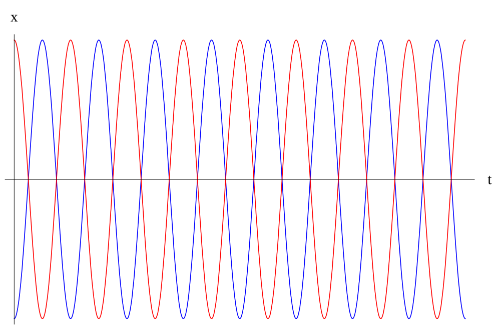
a free oscillation (no damping) in which the masses move in synchrony in the same direction and with the same frequency ($\omega_2 = 2$) and equal amplitudes of oscillation (c_2).

Example 1 analysis

The natural mode $\omega_1 = 4$

$$\begin{aligned} x_1(t) &= -c_1 \cos(4t - \alpha_1) \\ x_2(t) &= c_1 \cos(4 - 1) \end{aligned}$$

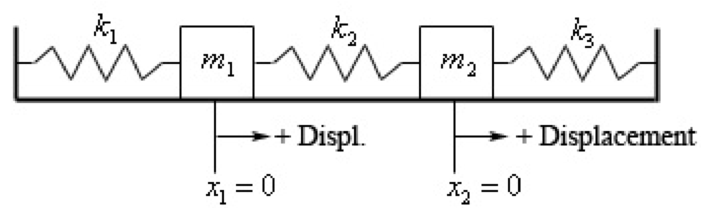
a free oscillation (no damping) in which the masses move in synchrony in the opposite directions and with the same frequency ($\omega_1 = 4$) and equal amplitudes of oscillation (c_1).



Mass-spring systems with external forces

Suppose we apply external forces F_1 to mass m_1 and F_2 to mass m_2 in the mass-spring system. We now have a nonhomogeneous system

$$\begin{aligned} m_1 x_1''(t) &= -(k_1 + k_2)x_1(t) + k_2 x_2(t) + F_1(t) \\ m_2 x_2''(t) &= k_2 x_1(t) - (k_2 + k_3)x_2(t) + F_2(t) \end{aligned}$$



Mechanical systems with forced oscillations

Forced oscillations. We are interested in periodic external forces applied to the masses ($\mathbf{F}_0$ is a constant vector)

$$\mathbf{F}_0 \cos \omega t = \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} \cos \omega t$$

As a system of equations:

$$\begin{aligned} m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 + F_0 \cos \omega t \\ m_2 x_2'' &= k_2 x_1 - (k_2 + k_3)x_2 + F_1 \cos \omega t \end{aligned}$$

As a matrix equation

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{(k_2+k_3)}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} \cos \omega t,$$

Note the form $\mathbf{x}'' = \mathbf{A}\mathbf{x} + \mathbf{F}_0 \cos \omega t$.

First method for solving second order system

First of two ways to solve the second-order system:

$$\begin{aligned} m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 + F_0 \cos \omega t \\ m_2 x_2'' &= k_2 x_1 - (k_2 + k_3)x_2 + F_1 \cos \omega t \end{aligned}$$

Method 1. Rewrite using differential operators:

$$\begin{aligned} \big(m_1 D + (k_1 + k_2)I\big)x_1 - k_2 x_2 &= F_0 \cos \omega t \\ -k_2 x_1 + \big(m_2 D + (k_2 + k_3)I\big)x_2 &= F_1 \cos \omega t \end{aligned}$$

Solve using Method of Elimination from Section 4.2.

Second method for solving second order system

Second of two ways to solve the second-order system:

$$\begin{aligned} m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 + F_0 \cos \omega t \\ m_2 x_2'' &= k_2 x_1 - (k_2 + k_3)x_2 + F_1 \cos \omega t \end{aligned}$$

Method 2. Rewrite as a 2×2 matrix equation (Section 5.1):

$$\mathbf{x}'' = \begin{bmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{(k_2+k_3)}{m_2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} \cos \omega t$$

which is of the form $\mathbf{x}'' = \mathbf{A}\mathbf{x} + \mathbf{F}_0 \cos \omega t$.

Guess a particular solution of the form $\mathbf{x}_p = \mathbf{c} \cos \omega t$,
and determine the values of the parameter $\mathbf{c}$.

Second method for solving second order system

Solve the second-order system

$$\mathbf{x}'' = \mathbf{A}\mathbf{x} + \mathbf{F}_0 \cos \omega t$$

Guess a particular solution of the form $\mathbf{x}_p = \mathbf{c} \cos \omega t$.
Substitute $\mathbf{x}_p$ where $\mathbf{x}_p'' = -\omega^2 \mathbf{c} \cos \omega t$

$$-\omega^2 \mathbf{c} \cos \omega t = \mathbf{A}\mathbf{c} \cos \omega t + \mathbf{F}_0 \cos \omega t$$

and cancel the common $\cos \omega t$

$$-\omega^2 \mathbf{c} = \mathbf{A}\mathbf{c} + \mathbf{F}_0.$$

New problem. We want a solution $\mathbf{c}$ to

$$(\mathbf{A} + \omega^2 \mathbf{I})\mathbf{c} = -\mathbf{F}_0.$$

Second method for solving second order system

Solve for unknown $\mathbf{c}$

$$(\mathbf{A} + \omega^2\mathbf{I})\mathbf{c} = -\mathbf{F}_0.$$

Observation. As long as $-\omega^2$ is not an eigenvalue for $\mathbf{A}$, then $\mathbf{A} + \omega^2\mathbf{I}$ is invertible.

Solution. If $-\omega^2$ is not an eigenvalue for $\mathbf{A}$, then

$$(\mathbf{A} + \omega^2\mathbf{I})\mathbf{c} = -\mathbf{F}_0.$$

has a unique solution.

Observation. If $-\omega^2$ is an eigenvalue for $\mathbf{A}$, then we have resonance. In this case, we can use techniques of Section 5.6 which generalize the Method of Undetermined Coefficients and Method of Variation of Parameters to vector systems.

Example 2

Problem. Find a general solution to the mass-spring system of Example 1 when:

- External forces: $F_1 = 30\cos t$, $F_2 = 60\cos t$

The second-order equations are

$$\begin{aligned} x_1'' &= -10x_1 + 6x_2 + 30\cos t \\ x_2'' &= 6x_1 - 10x_2 + 60\cos t \end{aligned}$$

Equivalently, as a matrix equation:

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 30\cos t \\ 60\cos t \end{bmatrix}$$

Example 2 continued

Compute a particular solution for

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 30\cos t \\ 60\cos t \end{bmatrix}$$

Guess. $\mathbf{x}_p = \mathbf{c}\cos t$.
Let $\omega = 1$. Since $-\omega^2 = -1$ is not an eigenvalue for the system $(\lambda = -16, -4)$, there is no resonance.
Substitute the guess $\mathbf{x}_p = \mathbf{c}\cos t$ into the equation.

$$-\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\cos t = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\cos t + \begin{bmatrix} 30 \\ 60 \end{bmatrix}\cos t$$

This reduces to

$$\begin{bmatrix} -10+1 & 6 \\ 6 & -10+1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\begin{bmatrix} 30 \\ 60 \end{bmatrix}$$

Example 2 continued

Solve the system of equations

$$\begin{aligned} -10 &= -3c_1 + 2c_2 \\ -20 &= 2c_1 - 3c_2 \end{aligned}$$

So, $c_1 = 14$ and $c_2 = 16$.

Particular solution

$$\mathbf{x}_p = \begin{bmatrix} 14 \\ 16 \end{bmatrix}\cos t$$

General solution. $\mathbf{x}_c + \mathbf{x}_p$, where $\mathbf{x}_c$ is from Example 1:

$$\begin{aligned} x_1(t) &= -(a_1\cos 4t + a_2\sin 4t) + (b_1\cos 2t + b_2\sin 2t) + 14\cos t \\ x_2(t) &= (a_1\cos 4t + a_2\sin 4t) + (b_1\cos 2t + b_2\sin 2t) + 16\cos t \end{aligned}$$

Example 2 continued

Problem. Find the motion when the two masses begin at the equillibrium position at rest. The initial conditions are

$$x_1(0) = 0, \quad x_1'(0) = 0, \quad x_2(0) = 0, \quad x_2'(0) = 0$$

The general solution and derivatives are

$$\begin{aligned}
x_1(t) &= -(a_1 \cos 4t + a_2 \sin 4t) + (b_1 \cos 2t + b_2 \sin 2t) + 14 \cos t \\
x_1'(t) &= -(-4a_1 \sin 4t + 4a_2 \cos t) + (-2b_1 \sin 2t + 2b_2 \cos 2t) - 16 \sin t \\
x_2(t) &= (a_1 \cos 4t + a_2 \sin 4t) + (b_1 \cos 2t + b_2 \sin 2t) + 16 \cos t \\
x_2'(t) &= (-4a_1 \sin 4t + 4a_2 \cos 4t) + (-2b_1 \sin 2t + 2b_2 \cos 2t) - 14 \sin t
\end{aligned}$$

Substitute.

$$\begin{aligned}
0 &= -a_1 + b_1 + 14 \\
0 &= -4a_2 + 2b_2 \\
0 &= a_1 + b_1 + 16 \\
0 &= 4a_2 + 2b_2
\end{aligned}$$

Example 2

Example 2 continued

Solve.

$$\begin{aligned}
0 &= -a_1 + b_1 + 14 \\
0 &= -4a_2 + 2b_2 \\
0 &= a_1 + b_1 + 16 \\
0 &= 4a_2 + 2b_2
\end{aligned}$$

The solutions are

$$a_1 = -1 \quad b_1 = -15 \quad a_2 = b_2 = 0$$

Solution. The general solution and derivatives are

$$\begin{aligned}
x_1(t) &= \cos 4t - 15 \cos 2t + 14 \cos t \\
x_2(t) &= -\cos 4t - 15 \cos 2t + 16 \cos t
\end{aligned}$$