Math 216 Differential Equations

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Eigenvalue solutions

Theorem

Let λ be an eigenvalue of the constant component matrix $\mathbf{A}$, and $\mathbf{v}$ be an associated eigenvector

Then

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

is a nontrivial solution to the equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

Eigenvectors and general solutions

Theorem

Suppose the $n \times n$ constant matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Let λ_i be the eigenvalue corresponding to $\mathbf{v}_i$.

Then a general solution to the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \ldots + c_n e^{\lambda_n t}\mathbf{v}_n$$

with parameters $c_1, c_2, \ldots, c_n$.

Distinct eigenvalues

Theorem

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues for the matrix $\mathbf{A}$, and $\mathbf{v}_i$ an eigenvector associated with λ_i.

Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Corollary

Suppose an $n \times n$ constant matrix $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and $\mathbf{v}_i$ is an eigenvector associated with λ_i.

Then

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \ldots + c_n e^{\lambda_n t}\mathbf{v}_n$$

is a general solution for the homogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{x}'$.

Method for finding general solution

Method. To solve an $n \times n$ constant-coefficient system $\mathbf{A}\mathbf{x} = \mathbf{x}'$.

1. Solve the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}|$ for the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $\mathbf{A}$. (Some eigenvalues may have multiplicity greater than one.)
2. Attempt to find n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with these eigenvalues. These are the solutions to $(\mathbf{A} - \lambda_i \mathbf{I})$. (If there are n distinct eigenvalues, there will be n linearly independent eigenvectors.)
3. When Step 2 is possible then the general solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \ldots + c_n e^{\lambda_n t}\mathbf{v}_n$$

Example 0

Problem. Find a general solution of

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix} \mathbf{x}(t).$$

We are looking for two linearly independent solutions.

Step 1. Find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & -5 \\ 2 & -4-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2.$$

So, the eigenvalues are $\lambda = -1 \pm i$.

Example 0 continued

Step 2. Find the eigenvector associated with $\lambda = -1 + i$.
Substitute $-1 + i$ into

$$\begin{bmatrix} 2-\lambda & -5 \\ 2 & -4-\lambda \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We want the solutions to

$$\begin{bmatrix} 3-i & -5 \\ 2 & -3-i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Equivalently, we want the solutions to the system:

$$\begin{aligned} (3-i)z_1 - 5z_2 &= 0 \\ 2z_1 - (-3-i)z_2 &= 0. \end{aligned}$$

Example 0 continued

Solve.

$$\begin{aligned} (3-i)z_1 - 5z_2 &= 0 \\ 2z_1 - (-3-i)z_2 &= 0. \end{aligned}$$

These are redundant: $R_1 = \frac{3-i}{2} \times R_2$. The system reduces to

$$(3-i)z_1 - 5z_2 = 0;$$

equivalently,

$$\frac{(3-i)}{5} z_1 = z_2$$

The eigenvectors are

$$\mathbf{z} = s \begin{bmatrix} 5 \\ 3-i \end{bmatrix} \quad \text{for any complex } s$$

Example 0 continued

Step 2. Find the eigenvector associated with $\lambda = -1 - i$.
Substitute $-1 - i$ into

$$\begin{bmatrix} 2-\lambda & -5 \\ 2 & -4-\lambda \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We want the solutions to

$$\begin{bmatrix} 3+i & -5 \\ 2 & -3+i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Equivalently, we want the solutions to the system:

$$\begin{aligned} (3+i)z_1 - 5z_2 &= 0 \\ 2z_1 - (-3+i)z_2 &= 0. \end{aligned}$$

Example 0 continued

Solve.

$$\begin{aligned} (3+i)z_1 - 5z_2 &= 0 \\ 2z_1 - (-3+i)z_2 &= 0. \end{aligned}$$

These are redundant: $R_1 = \frac{3+i}{2} \times R_2$. The system reduces to

$$(3+i)z_1 - 5z_2 = 0;$$

equivalently,

$$\frac{(3+i)}{5} z_1 = z_2.$$

The eigenvectors are

$$\mathbf{z} = s \begin{bmatrix} 5 \\ 3+i \end{bmatrix} \quad \text{for any complex } s$$

Example 0 continued

Step 3. Find a general solution.

- eigenvalue: $-1+i$, eigenvector: $\begin{bmatrix} 5 \\ 3-i \end{bmatrix}$,
- eigenvalue: $-1-i$, eigenvector: $\begin{bmatrix} 5 \\ 3+i \end{bmatrix}$,

Since these are distinct eigenvalues, a general solution is

$$\mathbf{z}(t) = b_1 \begin{bmatrix} 5 \\ 3-i \end{bmatrix} e^{(-1-i)t} + b_2 \begin{bmatrix} 5 \\ 3+i \end{bmatrix} e^{(-1+i)t}.$$

Unfortunately, the solution is complex, and we want real solutions. We will take the same approach as with higher-order equations with complex solutions in Section 3.3.

Example 0 continued

Apply Euler's formula: $e^{(-1\pm i)t} = e^{-t}(\cos t \pm i \sin t)$

$$\mathbf{z}(t) = b_1 \begin{bmatrix} 5 \\ 3-i \end{bmatrix} e^{-t}(\cos t + i \sin t) + b_2 \begin{bmatrix} 5 \\ 3+i \end{bmatrix} e^{-t}(\cos t - i \sin t).$$

Combine the real and imaginary parts

$$\mathbf{z}(t) = (b_1 + b_2)e^{-t} \begin{bmatrix} 5\cos t \\ 3\cos t + \sin t \end{bmatrix} + i(b_1 - b_2)e^{-t} \begin{bmatrix} 5 \sin t \\ -\cos t + 3 \sin t \end{bmatrix})$$

Eliminate i by rewriting $c_1 = b_1 + b_2$ and $c_2 = i(b_1 - b_2)$,

$$\mathbf{z}(t) = c_1 e^{-t} \begin{bmatrix} 5\cos t \\ 3\cos t + \sin t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 5 \sin t \\ -\cos t + 3 \sin t \end{bmatrix}$$

If these real solutions are are linearly independent, we are done.

Example 0 linear independence

Show the following solutions are linearly independent:

$$\mathbf{x}_1(t) = e^{-t}\begin{bmatrix} 5\cos t \\ 3\cos t + \sin t \end{bmatrix}$$
$$\mathbf{x}_2(t) = e^{-t}\begin{bmatrix} 5\sin t \\ -\cos t + 3\sin t \end{bmatrix}$$

They are linearly independent if $W(t) \neq 0$ for any t.

Compute the Wronskian at $t = 0$:

$$W(0) = \left|\mathbf{x}_1(0)\ \mathbf{x}_2(0)\right| = \begin{vmatrix} 5 & 0 \\ 3 & -1 \end{vmatrix} = -5$$

So, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent solutions.

Example 0 concluded

Problem. Find a general solution of

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}\mathbf{x}(t).$$

Answer. The general real solution is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix} 5\cos t \\ 3\cos t + \sin t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 5\sin t \\ -\cos t + 3\sin t \end{bmatrix}$$

or as a scalar equation

$$x_1(t) = e^{-t}\Big(5c_1\cos t + 5c_2\sin t\Big)$$
$$x_2(t) = e^{-t}\Big((2c_1 - c_2)\cos t + (c_1 + 3c_2)\sin t\Big)$$

Complex vectors

A vector with complex components components can always be written as $\mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real vectors:

$$\begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_n + ib_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + i\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The vector complex conjugate is computed componentwise:

$$\mathbf{z} = \mathbf{a} + i\mathbf{b} \qquad \overline{\mathbf{z}} = \mathbf{a} - i\mathbf{b}.$$

Complex conjugates and products

- Let $\mathbf{A}$ be a matrix with real components. Then $\overline{\mathbf{A}} = \mathbf{A}$.
- Complex conjugates distribute over sums and products.

$$\overline{c\mathbf{A}} = \overline{c}\overline{\mathbf{A}}$$
$$\overline{\mathbf{A} + \mathbf{B}} = \overline{\mathbf{A}} + \overline{\mathbf{B}}$$
$$\overline{\mathbf{A}\mathbf{B}} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}$$

where $\mathbf{A}$, $\mathbf{B}$ are any matrices and c is a scalar.

Complex conjugates and eigenvalues

Let λ be a complex number, $\mathbf{A}$ a real matrix, and $\mathbf{z}$ a complex vector.

If $\mathbf{z}$ is an eigenvector with eigenvalue λ, then

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{z} = \mathbf{0}.$$

Furthermore, $\bar{\mathbf{z}}$ is also an eigenvector with eigenvalue $\bar{\lambda}$:

$$\begin{aligned} \mathbf{0} &= \overline{(\mathbf{A} - \lambda\mathbf{I})\mathbf{z}} \\ &= (\mathbf{A} - \bar{\lambda}\mathbf{I})\bar{\mathbf{z}}. \end{aligned}$$

(since $\bar{\mathbf{A}} = \mathbf{A}$ and $\bar{\mathbf{I}} = \mathbf{I}$).

Complex solutions

Fact. If $\mathbf{x}_1 + i\mathbf{x}_2$ is a solution ($\mathbf{x}_1, \mathbf{x}_2$ are real vectors) to the equation $\mathbf{A}\mathbf{x} = \mathbf{x}'$, then so are $\mathbf{x}_1$ and $\mathbf{x}_2$.

Reason. Since $\mathbf{x}_1 + i\mathbf{x}_2$ is a solution,

$$\begin{aligned} (\mathbf{x}_1 + i\mathbf{x}_2)' &= \mathbf{A}(\mathbf{x}_1 + i\mathbf{x}_2) \\ \mathbf{x}_1' + i\mathbf{x}_2' &= \mathbf{A}\mathbf{x}_1 + i\mathbf{A}\mathbf{x}_2 \end{aligned}$$

Equate real and imaginary parts

$$\mathbf{x}_1' = \mathbf{A}\mathbf{x}_1 \quad \text{and} \quad \mathbf{x}_2' = \mathbf{A}\mathbf{x}_2.$$

So, $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions as well.

Complex conjugates

Compute. By Euler's formula, $e^{(p \pm iq)t} = e^{pt}(\cos qt \pm i \sin qt)$

$$\begin{aligned} (\mathbf{a} + i\mathbf{b})e^{(p+iq)t} &= (\mathbf{a} + i\mathbf{b})e^{pt}(\cos qt + i \sin qt) \\ &= e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right) + ie^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right) \\ (\mathbf{a} - i\mathbf{b})e^{(p-iq)t} &= \overline{(\mathbf{a} + i\mathbf{b})e^{p+iq}} \\ &= e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right) - ie^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right) \end{aligned}$$

So, the real and imaginary parts of these products are the same:

$$\begin{aligned} \mathbf{x}_1(t) &= e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right) \\ \mathbf{x}_2(t) &= e^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right). \end{aligned}$$

Real solutions

Let $\mathbf{A}$ be a matrix with real components.

Let $p + iq$ ($q \neq 0$) be an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{a}$+i$\mathbf{b}$. Then,

$$(\mathbf{a} \pm i\mathbf{b})e^{(p \pm iq)t}$$

is a complex solution to $\mathbf{A}\mathbf{x} = \mathbf{x}'$.

The complex solutions from conjugate eigenvalues are linearly independent:

$$\begin{aligned} (\mathbf{a} + i\mathbf{b})e^{(p+iq)t} &= e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right) + ie^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right) \\ (\mathbf{a} - i\mathbf{b})e^{(p-iq)t} &= e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right) - ie^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right) \end{aligned}$$

Their real constituents are also linearly independent solutions to $\mathbf{A}\mathbf{x} = \mathbf{x}'$:

$$\begin{aligned} \mathbf{x}_1(t) &= e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right) \\ \mathbf{x}_2(t) &= e^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right). \end{aligned}$$

Real solutions

Theorem

Suppose $\mathbf{A}$ *is a real matrix with complex conjugate eigenvalues* $p \pm iq$ *and corresponding eigenvectors* $\mathbf{a} \pm i\mathbf{b}$.

Then two linearly independent real vector solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *are*

$$\begin{aligned} \mathbf{x}_1(t) &= e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt) \\ \mathbf{x}_2(t) &= e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt). \end{aligned}$$

Example 1

Problem. Find a general solution of

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \mathbf{x}(t).$$

We are looking for two linearly independent solutions.

Step 1. Find the eigenvalues:

$$\begin{vmatrix} -1-\lambda & 2 \\ -1 & -3-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 5.$$

So, $\lambda = 2 \pm i$.

Example 1 continued

Step 2. Find the eigenvector associated with $\lambda = -2 + i$.
Substitute $-2 + i$ into

$$\begin{bmatrix} -1-\lambda & 2 \\ -1 & -3-\lambda \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

to obtain the equation

$$\begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We are looking for a solution to the system:

$$\begin{aligned} (1-i)z_1 + 2z_2 &= 0 \\ -2z_1 - (1+i)z_2 &= 0. \end{aligned}$$

Example 1 continued

We are looking for a solution to the system:

$$\begin{aligned} (1-i)z_1 + 2z_2 &= 0 \\ -2z_1 - (1+i)z_2 &= 0. \end{aligned}$$

The second equation **must be** a multiple of the first. So,

$$(1-i)z_1 = -2z_2$$

The solutions can be expressed as

$$\mathbf{z} = s \begin{bmatrix} -2 \\ 1-i \end{bmatrix} \qquad \text{for complex } s.$$

So, a complex eigenvector for $\lambda = -2 + i$ is

$$\mathbf{z} = \begin{bmatrix} -2 \\ 1-i \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Example 1 continued

Step 2. Find two real linearly independent solutions.

$$\begin{aligned}
\mathbf{z}e^{(-2+i)t} &= \begin{bmatrix} -2 \\ 1-i \end{bmatrix} e^{-2t}(\cos t + i \sin t) \\
&= e^{-2t}\begin{bmatrix} -2(\cos t + i\sin t) \\ (1-i)(\cos t + i \sin t) \end{bmatrix} \\
&= e^{-2t}\begin{bmatrix} -2\cos t - i2\sin t \\ (\cos t + \sin t) + i(-\cos t + \sin t) \end{bmatrix} \\
&= e^{-2t}\begin{bmatrix} -2\cos t \\ \cos t + \sin t \end{bmatrix} + ie^{-2t}\begin{bmatrix} -2\sin t \\ -\cos t + \sin t \end{bmatrix}
\end{aligned}$$

Example 1 concluded

Step 3. Express as a vector solution.

$$\mathbf{x} = c_1 e^{-2t}\begin{bmatrix} -2\cos t \\ \cos t + \sin t \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -2\sin t \\ -\cos t + \sin t \end{bmatrix}$$

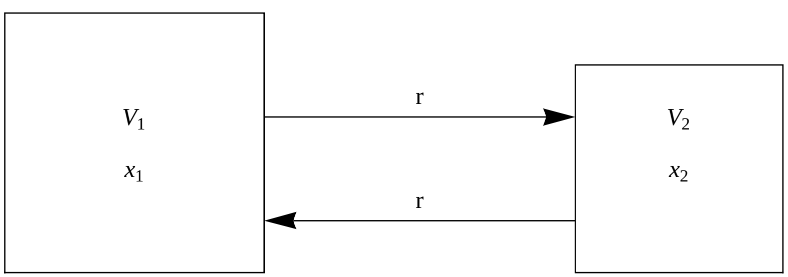
Express as a scalar solution.

$$\begin{aligned}
z_1 &= e^{-2t}(-2c_1\cos t - 2c_2 \sin t) \\
z_2 &= e^{-2t}((c_1 - c_2)\cos t + (c_1 + c_2)\sin t)
\end{aligned}$$

Mixing: two tanks

Problem. Two brine tanks with volumes V_1, V_2 liters of fluid and $x_1(t), x_2(t)$ kg of salt. Flow rates are r liters/minute.

$$\begin{aligned}
x_1'(t) &= -\frac{r}{V_1}x_1 + \frac{r}{V_2}x_2 \\
x_2'(t) &= \frac{r}{V_1}x_1 - \frac{r}{V_2}x_2
\end{aligned}$$



Mixing: two tanks

Problem.

- $V_1 = 25$ liters, $V_2 = 40$ liters
- $x_1(0) = 13$ kg, $x_2 = 0$ kg
- $r = 10$ liters/minute

The matrix equation is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{10}{25} & \frac{10}{40} \\ \frac{10}{25} & -\frac{10}{40} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \end{bmatrix}$$

We must first find a general solution, before we can solve the initial value problem.

Mixing: two tanks, eigenvalues

Step 1. Find the eigenvalues of

$$\begin{bmatrix} -\frac{2}{5} & \frac{1}{4} \\ \frac{2}{5} & -\frac{1}{4} \end{bmatrix}$$

Compute.

$$\begin{vmatrix} -\frac{2}{5}-\lambda & \frac{1}{4} \\ \frac{2}{5} & -\frac{1}{4}-\lambda \end{vmatrix} = \lambda^2 + \frac{13}{20}\lambda$$

Eigenvalues. $\lambda = 0, -\frac{13}{20}$

Mixing: two tanks, eigenvectors

Step 2. Find the eigenvector for $\lambda = 0$. These are solutions to

$$\begin{bmatrix} -\frac{2}{5} & \frac{1}{4} \\ \frac{2}{5} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve. This reduces to solutions for the equation

$$-\frac{2}{5}x_1 + \frac{1}{4}x_2 = 0 \quad \text{or} \quad 5x_2 = 8x_1$$

The eigenvectors are

$$\mathbf{x}_1 = s\begin{bmatrix} 5 \\ 8 \end{bmatrix} \quad \text{for any complex } s.$$

Mixing: two tanks, eigenvectors

Step 2. Find the eigenvector for $\lambda = \frac{13}{20}$. These are solutions to

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve. This reduces to solutions for the equation

$$\frac{1}{4}x_1 + \frac{1}{4}x_2 = 0 \quad \text{or} \quad x_2 = -x_1$$

The eigenvectors are

$$\mathbf{x}_1 = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Mixing: two tanks, general solution

Step 3. Produce a general solution.

- eigenvalue: 0 , eigenvector: $\begin{bmatrix} 5 \\ 8 \end{bmatrix}$
- eigenvalue: $-\frac{13}{20} = -0.65$, eigenvector: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

General solution, vector equation :

$$\mathbf{x}(t) = c_1\begin{bmatrix} 5 \\ 8 \end{bmatrix} + c_2 e^{-0.65t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

scalar equation:

$$\begin{aligned} x_1(t) &= 5c_1 - c_2 e^{-0.65t} \\ x_2(t) &= 8c_1 + c_2 e^{-0.65t} \end{aligned}$$

Mixing: two tanks, concluded

Step 4. Solve the initial value problem.

$$\begin{aligned} x_1(t) &= 5c_1 - c_2 e^{-0.65t} \\ x_2(t) &= 8c_1 + c_2 e^{-0.65t} \end{aligned}$$

where $x_1(0) = 13$ and $x_2(0) = 0$.

$$\begin{aligned} x_1(0) = 13 &= 5c_1 - c_2 \\ x_2(0) = 0 &= 8c_1 + c_2 \end{aligned}$$

So, $c_1 = 1$ and $c_2 = -8$.

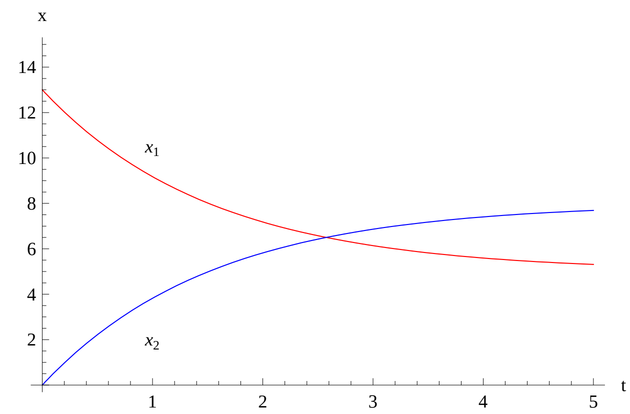
The salt amounts in each tank over time is given by

$$\begin{aligned} x_1(t) &= 5 + 8e^{-0.65t} \\ x_2(t) &= 8 - 8e^{-0.65t} \end{aligned}$$

Mixing: Two tanks, plotted

Solution. Two salt amounts in each brine tank over time:

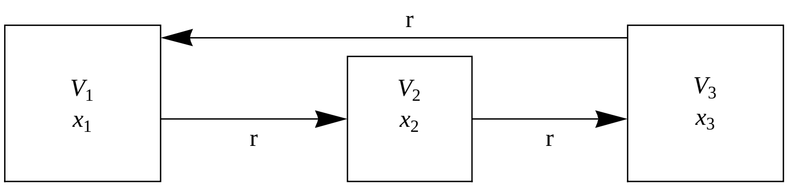
$$\begin{aligned} x_1(t) &= 5 + 8e^{-0.65t} \\ x_2(t) &= 8 - 8e^{-0.65t} \end{aligned}$$



Mixing: Three tanks

Problem. Two brine tanks three tanks with volumes V_1, V_2, V_3 liters and $x_1(t), x_2(t), x_3(t)$ kg of salt. Flow rate is r liters/minute.

$$\begin{aligned} x_1'(t) &= -\frac{r}{V_1}x_1 + \qquad + \frac{r}{V_3}x_3 \\ x_2'(t) &= \frac{r}{V_1}x_1 - \frac{r}{V_2}x_2 \\ x_3'(t) &= \qquad + \frac{r}{V_2}x_2 - \frac{r}{V_3}x_3 \end{aligned}$$



Mixing: three tanks

Problem.

- $V_1 = V_3 = 20$ liters, $V_2 = 50$ liters
- $x_1(0) = 18$ kg, $x_2 = x_3 = 0$ kg
- $r = 10$ liters/minute

The matrix equation is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -\frac{10}{20} & 0 & \frac{10}{20} \\ \frac{10}{20} & -\frac{10}{50} & 0 \\ 0 & \frac{10}{50} & -\frac{10}{20} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$$

We must first find a general solution, before we can solve the initial value problem.

Mixing: three tanks, eigenvalues

Step 1. Find the eigenvalues of

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix}$$

Compute.

$$\begin{vmatrix} -\frac{1}{2}-\lambda & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5}-\lambda & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2}-\lambda \end{vmatrix} = -\lambda(\lambda^2 + \frac{6}{5}\lambda + \frac{9}{20})$$

Eigenvalues. $\lambda = 0, -\frac{3}{5} \pm i\frac{3}{10}$.

Mixing: three tanks, eigenvectors

Step 2. Find the eigenvector for $\lambda = 0$. These are solutions to

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve. This reduces to the system

$$\begin{aligned} -x_1 \qquad + x_3 &= 0 \\ -2x_2 + 5x_3 &= 0 \end{aligned}$$

The eigenvectors are

$$\mathbf{x}_1 = s \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \qquad \text{for any complex } s.$$

Mixing: three tanks, eigenvectors

Step 2. Find the eigenvector for $\lambda = -\frac{3}{5} + i\frac{3}{10}$. These are solutions to

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve. This reduces to the system

$$\begin{aligned} (1-3i)x_1 \qquad + 5x_3 &= 0 \\ (1+3i)x_2 + 5x_3 &= 0 \end{aligned}$$

The eigenvectors are

$$\mathbf{z} = s \begin{bmatrix} 1+3i \\ 1-3i \\ -2 \end{bmatrix} \qquad \text{for any real } s.$$

Mixing: three tanks, general solution

Step 3. Produce a general solution.

- eigenvalue: 0 , eigenvector: $\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$
- eigenvalue: $-\frac{3}{5} \pm i\frac{3}{10}$, eigenvector: $\begin{bmatrix} 1+3i \\ 1-3i \\ -2 \end{bmatrix}$

We need two linearly independent real solutions for the complex conjugate eigenvalues $-\frac{3}{5} \pm i\frac{3}{10}$.

Mixing: three tanks, 2 real solutions

Step 3. Find two real linearly independent solutions (for the two conjugate eigenvalues).

$$\begin{aligned}
\mathbf{z}e^{(-\frac{3}{5}+i\frac{3}{10})t} &= 5\begin{bmatrix}1+3i\\1-3i\\-2\end{bmatrix} e^{-\frac{3}{5}t}(\cos\frac{3}{10}t + i\sin\frac{3}{10}t)\\
&= e^{-\frac{3}{5}t}\begin{bmatrix}(1+3i)(\cos\frac{3}{10}t+i\sin\frac{3}{10}t)\\(1-3i)(\cos\frac{3}{10}t+i\sin\frac{3}{10}t)\\-2(\cos\frac{3}{10}t+i\sin\frac{3}{10}t)\end{bmatrix}\\
&= e^{-\frac{3}{5}t}\begin{bmatrix}(\cos\frac{3}{10}t-3\sin\frac{3}{10}t)+i(3\cos\frac{3}{10}t+\sin\frac{3}{10}t)\\(\cos\frac{3}{10}t-3\sin\frac{3}{10}t)+i(-3\cos\frac{3}{10}t+\sin\frac{3}{10}t)\\-2\cos\frac{3}{10}t-i2\sin\frac{3}{10}t\end{bmatrix}\\
&= e^{-\frac{3}{5}t}\begin{bmatrix}(\cos\frac{3}{10}t-3\sin\frac{3}{10}t)\\(\cos\frac{3}{10}t+3\sin\frac{3}{10}t)\\-2\cos\frac{3}{10}t\end{bmatrix} + ie^{-\frac{3}{5}t}\begin{bmatrix}(3\cos\frac{3}{10}t+\sin\frac{3}{10}t)\\(-3\cos\frac{3}{10}t+\sin\frac{3}{10}t)\\-2\sin\frac{3}{10}t\end{bmatrix}
\end{aligned}$$

Mixing: three tanks, general solution

Step 3. Produce a general solution.
General solution, vector equation :

$$\mathbf{x}(t) = c_1\begin{bmatrix}2\\5\\2\end{bmatrix} + c_2e^{-\frac{3}{5}t}\begin{bmatrix}(\cos\frac{3}{10}t+3\sin\frac{3}{10}t)\\(\cos\frac{3}{10}t-3\sin\frac{3}{10}t)\\-2\cos\frac{3}{10}t\end{bmatrix} + c_3e^{-\frac{3}{5}t}\begin{bmatrix}(3\cos\frac{3}{10}t+\sin\frac{3}{10}t)\\(-3\cos\frac{3}{10}t+\sin\frac{3}{10}t)\\-2\sin\frac{3}{10}t\end{bmatrix}$$

scalar equation:

$$\begin{aligned}
x_1(t) &= 2c_1 + (c_2+3c_3)e^{-\frac{3}{5}t}\cos\frac{3}{10}t + (-3c_2+c_3)e^{-\frac{3}{5}t}\sin\frac{3}{10}t\\
x_2(t) &= 5c_1 + (c_2-3c_3)e^{-\frac{3}{5}t}\cos\frac{3}{10}t + (3c_2+c_3)e^{-\frac{3}{5}t}\sin\frac{3}{10}t\\
x_3(t) &= 2c_1 - 2c_2e^{-\frac{3}{5}t}\cos\frac{3}{10}t - 2c_3e^{-\frac{3}{5}t}\sin\frac{3}{10}t
\end{aligned}$$

Mixing: two tanks, concluded

Step 4. Solve the initial value problem.

$$\begin{aligned}
x_1(t) &= 2c_1 + (c_2+3c_3)e^{-\frac{3}{5}t}\cos\frac{3}{10}t + (-3c_2+c_3)e^{-\frac{3}{5}t}\sin\frac{3}{10}t\\
x_2(t) &= 5c_1 + (c_2-3c_3)e^{-\frac{3}{5}t}\cos\frac{3}{10}t + (3c_2+c_3)e^{-\frac{3}{5}t}\sin\frac{3}{10}t\\
x_3(t) &= 2c_1 - 2c_2e^{-\frac{3}{5}t}\cos\frac{3}{10}t - 2c_3e^{-\frac{3}{5}t}\sin\frac{3}{10}t
\end{aligned}$$

where $x_1(0) = 18$ and $x_2(0) = x_3(0) = 0$.
Substitute.

$$\begin{aligned}
18 &= 2c_1 + c_2 + 3c_3\\
0 &= 5c_1 + c_2 - 3c_2\\
0 &= 2c_1 - 2c_2
\end{aligned}$$

So, $c_1 = c_2 = 2$ and $c_3 = 4$.

Mixing: two tanks, plotted

The salt amounts in each tank over time is given by

$$\begin{aligned}
x_1(t) &= 4 + e^{-\frac{3}{5}t}(14\cos\frac{3}{10}t - 2\sin\frac{3}{10}t)\\
x_2(t) &= 10 + e^{-\frac{3}{5}t}(-10\cos\frac{3}{10}t + 10\sin\frac{3}{10}t)\\
x_3(t) &= 4 - e^{-\frac{3}{5}t}(4\cos\frac{3}{10}t + +8\sin\frac{3}{10}t)
\end{aligned}$$