Math 216
Differential Equations

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Existence and uniqueness

Theorem

Let $\mathbf{A} = [a_{ij}(t)]$ be an $n \times n$ matrix and $\mathbf{f}(t) = [f_i(t)]$ be an $n \times 1$ vector.

Suppose the components $a_{ij}(t)$ and $f_i(t)$ are continuous on an open interval I containing the point t_0.

Then for any choice of an initial $n \times 1$ vector $\mathbf{b}$, there exists a unique solution $\mathbf{x}(t)$ on the interval I to the initial value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{b}.$$

General solutions: Homogeneous case

Theorem

Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be linearly independent solutions to the homogeneous equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ on an open interval I, where $\mathbf{A}$ is continuous.

Any solution $\mathbf{x}(t)$ to the equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ is a linear combination of the independent solutions: there are constants $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t)$$

for all t in I.

Def. *This form is called the general solution to the equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$*

Guess

Problem. Find n linearly independent solutions to the $n \times n$ constant coefficient homogeneous system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

Guess solutions have the form

$$\mathbf{v}e^{\lambda t}$$

and determine the unknowns λ and $\mathbf{v}$:

- λ is a real or complex constant,
- $\mathbf{v} = [v_i]$ is a nonzero $n \times 1$ vector with constant real components.

Check guess

Guess. $\mathbf{v}e^{\lambda t}$ (where λ and $\mathbf{v}$ are unknowns) is a solution to

$$\mathbf{A}\mathbf{x} = \mathbf{x}'$$

Substitute.

$$\mathbf{A}(\mathbf{v}e^{\lambda t}) = (\mathbf{v}e^{\lambda t})'$$

Compute.

$$\mathbf{A}\mathbf{v}e^{\lambda t} = \lambda \mathbf{v}e^{\lambda t}$$

Simplify. Cancel the nonzero $e^{\lambda t}$

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

Conditions on guess

Conditions. We want a real λ and nonzero real vector $\mathbf{v}$ satisfying

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

Equivalently,

$$\begin{aligned} \mathbf{0} &= \mathbf{A}\mathbf{v} - \lambda \mathbf{v} \\ &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{v} \end{aligned}$$

This equation has nontrivial solutions exactly when

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

Eigenvalues and Eigenvectors

Definition

A number λ is called an eigenvalue of the $n \times n$ matrix $\mathbf{A}$ provided that

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

An eigenvector associated with eigenvalue λ is a nonzero solution $\mathbf{v}$ to

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$$

If λ is an eigenvalue for $\mathbf{A}$ with associated eigenvector $\mathbf{v}$, then

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

Eigenvalue solutions

Theorem

Let λ be an eigenvalue of the constant component matrix $\mathbf{A}$, and $\mathbf{v}$ be an associated eigenvector

Then

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

is a nontrivial solution to the equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

Characteristic equation

Eigenvalues. The eigenvalues of a matrix $\mathbf{A}$ are the values λ for which $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

Characteristic polynomial. If $\mathbf{A}$ is an $n \times n$ matrix then

$$|\mathbf{A} - \lambda\mathbf{I}| = p(\lambda)$$

where $p(\lambda)$ is an nth degree polynomial in λ called the characteristic polynomial.

Roots. The roots of the characteristic equation $p(\lambda) = 0$ are the eigenvalues of $\mathbf{A}$. So,

- Finding eigenvalues is equivalent to finding the zeros of the characteristic equation $p(\lambda)$.

Example 1

Problem. Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

Compute the characteristic polynomial of $\mathbf{A}$:

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix} 4-\lambda & 2 \\ -3 & -1-\lambda \end{vmatrix} \\ &= (4-\lambda)(-1-\lambda) + 6 \\ &= \lambda^2 - 3\lambda + 2 \end{aligned}$$

Solve the characteristic equation.

$$0 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

Eigenvalues. The eigenvalues are $\lambda = 1, 2$.

Eigenvectors

Problem. Find eigenvectors associated with each eigenvalue $\lambda = 1, 2$

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}.$$

Solve. An eigenvector for eigenvalue λ is a solution to the equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

Example 1 continued

Problem. Find eigenvector associated with $\lambda = 1$ of

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

Substitute $\lambda = 1$ into $\mathbf{A} - \lambda\mathbf{I}$:

$$\begin{bmatrix} 4-\lambda & 2 \\ -3 & -1-\lambda \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -3 & -2 \end{bmatrix}$$

Solve the matrix equation

$$\begin{bmatrix} 3 & 2 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Equivalently, we want solutions to the system of equations

$$\begin{aligned} 3v_1 + 2v_2 &= 0 \\ -3v_1 - 2v_2 &= 0 \end{aligned}$$

Example 1 continued

The eigenvectors associated with $\lambda = 1$ are solutions to the system

$$\begin{aligned} 3v_1 + 2v_2 &= 0 \\ -3v_1 - 2v_2 &= 0 \end{aligned}$$

Solve. The second equation is redundant, so the system reduces

$$3v_1 + 2v_2 = 0 \qquad \text{or } 3v_1 = -2v_2.$$

There is one degree of freedom:
once we decide upon v_1 (or v_2), the other unknown is determined.

All eigenvectors for $\lambda = 1$ are obtained by

$$\mathbf{v} = s\begin{bmatrix} -2 \\ 3 \end{bmatrix} \qquad \text{for each real } s$$

Example 1 continued

Problem. Find eigenvector associated with $\lambda = 2$ of

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

Substitute $\lambda = 2$ into $\mathbf{A} - \lambda\mathbf{I}$:

$$\begin{bmatrix} 4-\lambda & 2 \\ -3 & -1-\lambda \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix}$$

Solve the matrix equation

$$\begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Equivalently, we want solutions to the system of equations

$$\begin{aligned} 2v_1 + 2v_2 &= 0 \\ -3v_1 - 3v_2 &= 0 \end{aligned}$$

Example 1 continued

The eigenvectors associated with $\lambda = 2$ are solutions to the system

$$\begin{aligned} 2v_1 + 2v_2 &= 0 \\ -3v_1 - 3v_2 &= 0 \end{aligned}$$

Solve. The second equation is redundant, so the solutions are

$$2v_1 + 2v_2 = 0 \qquad \text{or } v_1 = -v_2.$$

There is one degree of freedom:
once we decide upon v_1 (or v_2), the other unknown is determined.

All eigenvectors for $\lambda = 2$ are obtained by

$$\mathbf{v} = s\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{for each real } s.$$

Example 1 concluded

Summary. The eigenvalues and eigenvectors for the matrix

$$\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}:$$

eigenvalue: $\lambda = 1$, eigenvectors: $\mathbf{v}_1 = s\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ for any value s.

eigenvalue: $\lambda = 2$, eigenvectors: $\mathbf{v}_2 = s\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for any value s.

Example 2

Problem. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

Compute the characteristic polynomial of **A**:

$$\begin{aligned} \left|\mathbf{A} - \lambda \mathbf{I}\right| &= \begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix} \\ &= (\lambda-1)(\lambda-2)(\lambda-3) \end{aligned}$$

Solve the characteristic equation.

$$0 = (\lambda-1)(\lambda-2)(\lambda-3)$$

Eigenvalues. The eigenvalues are $\lambda = 1, 2, 3$.

Example 2 continued

Problem. Find eigenvector associated with $\lambda = 1$ of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

These are solutions to the equation

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equivalently, solutions to the system of equations

$$\begin{aligned} 2v_2 - v_3 &= 0 \\ v_1 - v_2 + v_3 &= 0 \\ 4v_1 - 4v_2 + 4v_3 &= 0 \end{aligned}$$

Example 2 continued

The eigenvectors associated with $\lambda = 1$ are solutions to the system

$$\begin{aligned} 2v_2 - v_3 &= 0 \\ v_1 - v_2 + v_3 &= 0 \\ 4v_1 - 4v_2 + 4v_3 &= 0 \end{aligned}$$

The third equation is redundant (it is a multiple of the second). Reduce to

$$\begin{aligned} 2v_2 - v_3 &= 0 \\ v_1 - v_2 + v_3 &= 0 \end{aligned}$$

Once we fix $v_3 = s$, we can determine v_1 and v_2 from this system.
Eigenvectors. The eigenvectors for $\lambda = 1$ are

$$\mathbf{v} = s \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad \text{for each real } s.$$

Example 2 continued

Problem. Find eigenvector associated with $\lambda = 2$ of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

These are solutions to the equation

$$\begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equivalently, solutions to the system of equations

$$\begin{aligned} -v_1 + 2v_2 - v_3 &= 0 \\ v_1 - 2v_2 + v_3 &= 0 \\ 4v_1 - 4v_2 + 3v_3 &= 0 \end{aligned}$$

Example 2 continued

The eigenvectors associated with $\lambda = 2$ are solutions to the system

$$\begin{aligned} -v_1 + 2v_2 - v_3 &= 0 \\ v_1 - 2v_2 + v_3 &= 0 \\ 4v_1 - 4v_2 + 3v_3 &= 0 \end{aligned}$$

This system can be reduced (using Gauss-Jordan elimination) to

$$\begin{aligned} -v_1 + 2v_2 - v_3 &= 0 \\ 4v_2 - v_3 &= 0 \end{aligned}$$

Once we fix $v_3 = s$, we can determine v_1 and v_2 from this system.
Eigenvectors. The eigenvectors for $\lambda = 2$ are

$$\mathbf{v} = s \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} \quad \text{for each real } s.$$

Example 2 continued

Problem. Find eigenvector associated with $\lambda = 3$ of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

These are solutions to the equation

$$\begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equivalently, solutions to the system of equations

$$\begin{aligned} -2v_1 + 2v_2 - v_3 &= 0 \\ v_1 - 3v_2 + v_3 &= 0 \\ 4v_1 - 4v_2 + 2v_3 &= 0 \end{aligned}$$

Example 2 continued

The eigenvectors associated with $\lambda = 3$ are solutions to the system

$$\begin{aligned} -2v_1 + 2v_2 - v_3 &= 0 \\ v_1 - 3v_2 + v_3 &= 0 \\ 4v_1 - 4v_2 + 2v_3 &= 0 \end{aligned}$$

This system can be reduced (using Gauss-Jordan elimination) to

$$\begin{aligned} -2v_1 + 2v_2 - v_3 &= 0 \\ -4v_2 + v_3 &= 0 \end{aligned}$$

Once we fix $v_3 = s$, we can determine v_1 and v_2 from this system.
Eigenvectors. The eigenvectors for $\lambda = 3$ are

$$\mathbf{v} = s \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \quad \text{for each real } s.$$

Example 2 concluded

Summary. The eigenvalues and eigenvectors for the matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

eigenvalue: $\lambda = 1$, eigenvectors: $\mathbf{v}_1 = s \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ for any value s.

eigenvalue: $\lambda = 2$, eigenvectors: $\mathbf{v}_2 = s \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ for any value s.

eigenvalue: $\lambda = 3$, eigenvectors: $\mathbf{v}_3 = s \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ for any value s.

Eigenvalues and solutions

Let $\mathbf{A}$ be an $n \times n$ matrix, and

- λ be an eigenvalue for $\mathbf{A}$,
- $\mathbf{v}$ an associated eigenvector.

Then $e^{\lambda t}\mathbf{v}$ is a solution to the homogeneous equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

Question. Can we obtain n linearly independent solutions to the homogeneous system by finding all eigenvalues and eigenvectors?

Eigenvectors and general solutions

Theorem

Suppose the $n \times n$ constant matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Let λ_i be the eigenvalue corresponding to $\mathbf{v}_i$. Then

$$e^{\lambda_1 t}\mathbf{v}_1, \quad e^{\lambda_2 t}\mathbf{v}_2, \quad \ldots, \quad e^{\lambda_n t}\mathbf{v}_n$$

is a linearly independent set of solutions on the entire real line to the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

A general solution to the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \ldots + c_n e^{\lambda_n t}\mathbf{v}_n$$

with parameters $c_1, c_2, \ldots, c_n$.

Distinct eigenvalues

Theorem

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues for the matrix $\mathbf{A}$, and $\mathbf{v}_i$ an eigenvector associated with λ_i.

Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Corollary

Suppose an $n \times n$ constant matrix $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and $\mathbf{v}_i$ is an eigenvector associated with λ_i.

Then

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \ldots + c_n e^{\lambda_n t}\mathbf{v}_n$$

is a general solution for the homogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{x}'$.

Example 1: vector solution

Problem. Find a general solution to the homogeneous equation

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \mathbf{x}$$

Answer. We saw earlier

- eigenvalue: $\lambda = 1$, eigenvectors: $s \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ for any value s.
- eigenvalue: $\lambda = 2$, eigenvectors: $s \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for any value s.

Since the 2×2 matrix has two distinct eigenvalues, we can choose any eigenvector for each eigenvalue to get a general solution

$$\mathbf{x} = c_1 e^{t} \begin{bmatrix} -2 \\ 3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(I chose $s = 1$ here.)

Example 1: scalar solution

Problem. Find a general scalar solution to the system of equations

$$\begin{aligned} x_1' &= 4x_1 + 2x_2 \\ x_2' &= -3x_1 - x_2. \end{aligned}$$

Answer. The general solution to the associated matrix was

$$\mathbf{x} = c_1 e^t \begin{bmatrix} -2 \\ 3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The scalar solution is

$$\begin{aligned} x_1 &= -2c_1 e^t + c_2 e^{2t} \\ x_2 &= 3c_1 e^t - c_2 e^{2t}. \end{aligned}$$

Example 2: vector solution

Problem. Find a general solution to the homogeneous equation

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \mathbf{x}$$

Answer. From earlier

- eigenvalue: $\lambda = 1$, eigenvectors: $s \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ for any value s.
- eigenvalue: $\lambda = 2$, eigenvectors: $s \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ for any value s.
- eigenvalue: $\lambda = 3$, eigenvectors: $s \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ for any value s.

Since the 3×3 matrix has three distinct eigenvalues the general solution is

$$\mathbf{x} = c_1 e^t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

Example 2: scalar solution

Problem. Find a general scalar solution to the system of equations

$$\begin{aligned} x_1' &= x_1 + 2x_2 - x_3 \\ x_2' &= x_1 + x_3 \\ x_3' &= 4x_1 - 4x_2 + 5x_3 \end{aligned}$$

Answer. The general solution to the associated matrix was

$$\mathbf{x} = c_1 e^t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

The scalar solution is

$$\begin{aligned} x_1 &= -c_1 e^t - 2c_2 e^{2t} - c_3 e^{3t} \\ x_2 &= c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ x_3 &= 2c_1 e^t + 4c_2 e^{2t} + 4c_3 e^{3t} \end{aligned}$$

Method for finding general solution

Method. To solve an $n \times n$ constant-coefficient system $\mathbf{Ax} = \mathbf{x}'$.

1. Solve the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}|$ for the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $\mathbf{A}$. (Some eigenvalues may have multiplicity greater than one.)
2. Attempt to find n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with these eigenvalues. These are the solutions to $(\mathbf{A} - \lambda_i \mathbf{I})$. (If there are n distinct eigenvalues, there will be n linearly independent eigenvectors.)
3. When Step 2 is possible then the general solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 + \ldots + c_n e^{\lambda_n t} \mathbf{v}_n$$

Multiplicity and eigenvectors

- If an eigenvalue has multiplicity 1, then it will have exactly one linearly independent eigenvector.
- If an eigenvalue has multiplicity k, then it will have at least one eigenvector, and no more than k linearly independent eigenvectors.
- An eigenvalue has multiplicity $k > 1$ may have fewer than k linearly independent eigenvectors. See Section 5.5 for how to produce k linearly independent solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Example 3

Problem. Find a general solution, if possible, for

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}$$

Step 1. Find eigenvalues using the characteristic equation.

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2 = -(\lambda - 2)(\lambda + 1)^2$$

Eigenvalues. The eigenvalues are $\lambda = 2, -1$.
Eigenvalue -1 has multiplicity two.

Example 3

Step 2. Find an eigenvector for $\lambda = 2$.

Solve.

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Use Gauss-Jordan elimination to this system to

$$\begin{aligned} -2v_1 \quad + 2v_3 &= 0 \\ -v_2 + v_3 &= 0 \end{aligned}$$

The eigenvalues for $\lambda = 2$ are

$$\mathbf{v} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{for any real } s$$

Example 3

Step 2. Find an eigenvector for $\lambda = -1$.

Solve.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Use Gauss-Jordan elimination to this system to

$$v_1 + v_2 + v_3 = 0$$

We have two degrees of freedom in choosing v_2 and v_3.
The eigenvalues for $\lambda = -1$ are

$$\mathbf{v} = s \begin{bmatrix} -s - t \\ s \\ t \end{bmatrix} \qquad \text{for any reals } s, t$$

Example 3 - concluded

Step 3.

- Eigenvalue: $\lambda = 2$, eigenvector: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,
- Eigenvalue: $\lambda = -1$, eigenvectors are

$$\mathbf{v} = s \begin{bmatrix} -s - t \\ s \\ t \end{bmatrix} \qquad \text{for any reals } s, t$$

We get two linearly independent vectors by choosing $s = 1, t = 0$ and $s = 1, t = 0$:

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

A general solution is

$$\mathbf{x}(t) = c_1 e^{2t}\mathbf{v}_1 + c_2 e^{-t}\mathbf{v}_2 + c_3 e^{-t}\mathbf{v}_3.$$

Example 3: scalar solution

Problem. Find a general scalar solution to the system of equations

$$\begin{aligned} x_1' &= x_2 + x_3 \\ x_2' &= x_1 + x_3 \\ x_3' &= x_1 + x_2 \end{aligned}$$

Answer. The general solution to the associated matrix was

$$\mathbf{x}(t) = c_1 e^{2t}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The scalar solution is

$$\begin{aligned} x_1 &= c_1 e^{2t} - c_2 e^{-t} - c_3 e^{-t} \\ x_2 &= c_1 e^{2t} + c_2 e^{-t} \\ x_3 &= c_1 e^{2t} + c_3 e^{-t} \end{aligned}$$