Math 216
Differential Equations

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The key to determinants

The determinant for $n \times n$ matrices can be computed by cofactor expansion. See Edwards and Penney, Chapter 5.1, p. 290 (version 3) for details.

♥ The key to determinants.

Theorem

Let $\mathbf{A}$ be an $n \times n$ matrix. The following properties are equivalent.

1. $\det(\mathbf{A}) \neq 0$.
2. $\mathbf{A}$ *has an inverse.*
3. $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution* $\mathbf{x} = \mathbf{0}$.
4. $\mathbf{Ax} = \mathbf{b}$ *has a unique solution.*

So, if $\det(\mathbf{A}) \neq 0$, *then* $\mathbf{A}^{-1}$ *exists and*

$$\mathbf{Ax} = \mathbf{b} \quad \text{has the unique solution } \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

Example 1

Example 1. Find all solutions to the system of equations

$$\begin{aligned} -6 &= -2x_1 + 4x_2 + 2x_3 \\ 12 &= 4x_1 - 2x_2 + 2x_3 \\ 6 &= 2x_1 + 2x_2 + 4x_3 \end{aligned}$$

where a, b are real values.

As a matrix equations:

$$\begin{bmatrix} -6 \\ 12 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 2 \\ 4 & -2 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 1 continued

Compute the determinant (expand along first row)

$$\begin{aligned} \begin{vmatrix} -2 & 4 & 2 \\ 4 & -2 & 2 \\ 2 & 2 & 4 \end{vmatrix} &= -2\begin{vmatrix} -2 & 2 \\ 2 & 4 \end{vmatrix} - 4\begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} + 2\begin{vmatrix} 4 & -2 \\ 2 & 2 \end{vmatrix} \\ &= -2(-8-4) - 4(16-4) + 2(8+4) \\ &= 24 - 48 + 24 = 0 \end{aligned}$$

So, we know the matrix equation does not have a unique solution.

$$\begin{bmatrix} -6 \\ 12 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 2 \\ 4 & -2 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

There could be infinitely many solutions or none.

Example 1 continued

Solve the system of equations.

$$\begin{aligned} -6 &= -2x_1 + 4x_2 + 2x_3 \\ 12 &= 4x_1 - 2x_2 + 2x_3 \\ 6 &= 2x_1 + 2x_2 + 4x_3 \end{aligned}$$

Eliminate x_1 from row 2 (R_2) and row 3 R_3, isolating x_1 to row 1 (R_1).
R_2: add $2 \times R_1$ to R_2:

$$\begin{aligned} -6 &= -2x_1 + 4x_2 + 2x_3 \\ 0 &= 0x_1 + 6x_2 + 6x_3 \\ 6 &= 2x_1 + 2x_2 + 4x_3 \end{aligned}$$

R_3: add $1 \times R_1$ to R_3:

$$\begin{aligned} -6 &= -2x_1 + 4x_2 + 2x_3 \\ 0 &= 0x_1 + 6x_2 + 6x_3 \\ 0 &= 0x_1 + 6x_2 + 6x_3 \end{aligned}$$

Note that the solutions to this new system of equations is the same as the original.

Example 1 continued

Solve the system of equations.

$$\begin{aligned} -6 &= -2x_1 + 4x_2 + 2x_3 \\ 0 &= 6x_2 + 6x_3 \\ 0 &= 6x_2 + 6x_3 \end{aligned}$$

Eliminate x_2 from row 1 and row 3 , isolating x_2 to row 2.
R_1: add$-\frac{2}{3}R_2$ to R_1:

$$\begin{aligned} 6 &= 2x_1 + 0x_2 - 2x_3 \\ 0 &= 6x_2 + 6x_3 \\ 0 &= 6x_2 + 6x_3 \end{aligned}$$

R_3: add $-1 \times R_2$ to R_3:

$$\begin{aligned} 6 &= 2x_1 + 0x_2 - 2x_3 \\ 0 &= 6x_2 + 6x_3 \\ 0 &= 0x_2 + 0x_3 \end{aligned}$$

Note that the solutions to this new system of equations is the same as the original.

Example 1 concluded

Solve the system of equations.

$$\begin{aligned} 6 &= 2x_1 + \quad -2x_3 \\ 0 &= 6x_2 + 6x_3 \\ 0 &= 0 \end{aligned}$$

Read off the solutions:

$$\begin{aligned} 3 + x_3 &= x_1 \\ -x_3 &= x_2 \end{aligned}$$

There are no constraints placed on x_3, so we are free to let it be any value.
The following are the solutions to the original equation:

$$x_1 = 3 + c, \quad x_2 = -c, \quad x_3 = c \qquad \text{for each real value } c$$

There are infinitely many solutions.

Example 2

Example 2. Find all solutions to the system of equations

$$\begin{aligned} 16 &= 2x_1 + 6x_2 + 8x_3 \\ 6 &= 3x_2 + 3x_3 \\ 6 &= 2x_1 \quad + 3x_3 \end{aligned}$$

where a, b are real values.

As a matrix equations:

$$\begin{bmatrix} 16 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 8 \\ 0 & 3 & 2 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 2 continued

Compute the determinant

$$\begin{vmatrix} 2 & 6 & 8 \\ 0 & 3 & 2 \\ 2 & 0 & 3 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix} + 2\begin{vmatrix} 6 & 8 \\ 3 & 2 \end{vmatrix} = 2(9-0) + 2(12-24) = 18 - 24 = -6$$

So, we know the matrix equation has a unique solution.

$$\begin{bmatrix} 16 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 8 \\ 0 & 3 & 2 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 2 continued

Solve the system of equations.

$$\begin{aligned} 16 &= 2x_1 + 6x_2 + 8x_3 \\ 6 &= 3x_2 + 3x_3 \\ 6 &= 2x_1 + 3x_3 \end{aligned}$$

Eliminate x_1 from row 3 R_3, isolating x_1 to row 1 (R_1).
R_3: add $-1 \times R_1$ to R_3:

$$\begin{aligned} 16 &= 2x_1 + 6x_2 + 8x_3 \\ 6 &= 3x_2 + 3x_3 \\ -10 &= 0x_1 - 6x_2 - 5x_3 \end{aligned}$$

Note that the solutions to this new system of equations is the same as the original.

Example 2 continued

Solve the system of equations.

$$\begin{aligned} 16 &= 2x_1 + 6x_2 + 8x_3 \\ 6 &= 3x_2 + 3x_3 \\ -10 &= -6x_2 - 5x_3 \end{aligned}$$

Eliminate x_2 from row 1 and row 3 , isolating x_2 to row 2.
R_1: add$-2R_2$ to R_1:

$$\begin{aligned} 4 &= 2x_1 + 0x_2 + 2x_3 \\ 6 &= 3x_2 + 3x_3 \\ -10 &= -6x_2 - 5x_3 \end{aligned}$$

R_3: add $2 \times R_2$ to R_3:

$$\begin{aligned} 4 &= 2x_1 + 0x_2 + 2x_3 \\ 6 &= 3x_2 + 3x_3 \\ 2 &= 0x_2 + x_3 \end{aligned}$$

Note that the solutions to this new system of equations is the same as the original.

Example 2 continued

Solve the system of equations.

$$\begin{aligned} 4 &= 2x_1 + 2x_3 \\ 6 &= 3x_2 + 3x_3 \\ 2 &= x_3 \end{aligned}$$

Eliminate x_3 from row 1 and row 2 , isolating x_3 to row 3.
R_1: add$-2R_3$ to R_1:

$$\begin{aligned} 0 &= 2x_1 + 0x_3 \\ 6 &= 3x_2 + 3x_3 \\ 2 &= x_3 \end{aligned}$$

R_2: add $-3 \times R_2$ to R_3:

$$\begin{aligned} 0 &= 2x_1 + 0x_3 \\ 0 &= 3x_2 + 0x_3 \\ 2 &= x_3 \end{aligned}$$

Note that the solutions to this new system of equations is the same as the original.

Example 2 concluded

Solve the system of equations.

$$\begin{aligned} 0 &= 2x_1 \\ 0 &= 3x_2 \\ 2 &= x_3 \end{aligned}$$

Read off the solutions:

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 2.$$

The equations sufficiently constrain the unknowns x_1, x_2 and x_3 to produce a unique solution.

Homogeneous matrix equations

Homogeneous matrix equations are the key to producing general solutions.

Def. The general matrix equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

is nonhomogeneous. It is homogeneous when $\mathbf{f} \equiv \mathbf{0}$.

Def. The associated homogeneous equation is

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}.$$

Superposition Principle I

Theorem

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be solutions on an interval I to the $n \times n$ homogeneous equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}.$$

If $c_1, c_2, \ldots, c_k$ are any constants, then the linear combination

$$\mathbf{x} = c_1\mathbf{x}_2 + c_2\mathbf{x}_2 + \ldots + c_k\mathbf{x}_k$$

is also a solution on I.

Superposition Principle: proof

Proof.

Let $\mathbf{x} = c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k$.

Suppose

$$\mathbf{x}_1' = \mathbf{A}\mathbf{x}_1, \quad \mathbf{x}_2' = \mathbf{A}\mathbf{x}_2, \quad \ldots \quad , \mathbf{x}_k' = \mathbf{A}\mathbf{x}_k.$$

Then

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{A}\big(c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k\big) \\ &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \ldots + c_k\mathbf{A}\mathbf{x}_k \\ &= c_1\mathbf{x}_1' + c_2\mathbf{x}_2' + \ldots + c_k\mathbf{x}_k' \\ &= \big(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_k\mathbf{x}_k\big)' \\ &= \mathbf{x}' \end{aligned}$$

□

Example

Example. The matrix equation

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \mathbf{x}$$

has solutions (on the entire real line)

$$\mathbf{x}_1 = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$$

So, any linear combination is also a solution:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2.$$

Example

Fact. Any solution (on the real line) to the equation

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \mathbf{x}$$

is of the form

$$\begin{aligned} \mathbf{x} &= c_1 \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} \\ &= \begin{bmatrix} 2c_1e^t + c_2e^{2t} \\ -3c_1e^t - c_2e^{2t} \end{bmatrix} \\ &= \begin{bmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

Note. Each column of this final matrix is a solution.

Linear Independence

Definition

Vector functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly dependent on interval I if there are constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t) = \mathbf{0}$$

for all t in I.

Otherwise, the vector functions are linearly independent on I. That is, the only choice of constants is the trivial one: $0 = c_1 = c_2 = \ldots = c_n$.

Linear Independence and matrix equations

The vector functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly dependent on interval I if and only if there are constants $c_1, c_2, \ldots, c_n$, not all zero, such that for all t in I

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_k\mathbf{x}_n(t) \\ &= \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \end{aligned}$$

The vector functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly independent on I if and only if the only solution for all t in I to

$$\mathbf{0} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the trivial solution $\mathbf{0}$.

Wronskian

Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be $n \times 1$ vector functions.
The Wronskian is the $n \times n$ determinant

$$W(t) = \begin{vmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \ldots & \mathbf{x}_n(t) \end{vmatrix}$$

We will sometimes write $W(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ for this determinant.

Wronskian of solutions

Theorem

Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be solutions to the $n \times n$ homogeneous equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ on an open interval I, where $\mathbf{A}$ is continuous.

Let $W = W(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$. Then

- *If $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly dependent on I, then $W(t) = 0$ for all t in I.*
- *If $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly independent on I, then $W(t) \neq 0$ for all t in I.*

Warning: This theorem may only be used to test solutions to homogeneous equations, $\mathbf{x}' = \mathbf{A}x$.

Example

Example. The matrix equation

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \mathbf{x}$$

has the following linearly independent solutions (on the entire real line)

$$\mathbf{x}_1 = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$$

Verify linear independence.

$$\begin{vmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{vmatrix} = -2e^t \cdot e^{2t} + e^{2t} \cdot 3e^t = 3e^{3t} - 2e^{3t} = e^{3t},$$

and e^{3t} is never zero for any t.

Example

Example. The matrix equation

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}$$

has the following linearly independent solutions (on the entire real line)

$$\mathbf{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}$$

Verify linear independence. (Expand along first row.)

$$\begin{vmatrix} e^{2t} & -e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & -e^{-t} \end{vmatrix} = e^{2t} \begin{vmatrix} 0 & e^{-t} \\ e^{-t} & -e^{-t} \end{vmatrix} + e^{-t} \begin{vmatrix} e^{2t} & e^{-t} \\ e^{2t} & -e^{-t} \end{vmatrix}$$
$$= e^{2t}(0 - e^{-2t}) + e^{-t}(-e^t - e^t) = -3$$

So, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent.

General solutions: Homogeneous case

Theorem

Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be linearly independent solutions to the homogeneous equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ on an open interval I, where $\mathbf{A}$ is continuous.

Any solution $\mathbf{x}(t)$ to the equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ is a linear combination of the independent solutions: there are constants $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t)$$

for all t in I.

***Def**. This form is called the general solution to the equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$*

Example

Example. The matrix equation

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \mathbf{x}$$

has a general solution on all numbers, of the form

$$\mathbf{x} = c_1 \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$$

Superposition Principle II

Theorem

If $\mathbf{x}_p(t)$ is a solution on interval I to the nonhomogeneous equation

$$\mathbf{x}_p' = \mathbf{A}(t)\mathbf{x}_p + \mathbf{f}(t).$$

and $\mathbf{x}_h$ is any solution on I to the associated homogeneous equation

$$\mathbf{x}_h' = \mathbf{A}(t)\mathbf{x}_h,$$

then $\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_h(t)$ is also a solution on I to

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t).$$

Proof of Superposition theorem II

Proof.

Let $\mathbf{x}_p(t)$ be a solution on interval I to the nonhomogeneous equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t).$$

Let $\mathbf{x}_h(t)$ be a solution on interval I to the associated homogeneous equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}.$$

Then

$$\begin{aligned} \mathbf{A}(\mathbf{x}_p + \mathbf{x}_h) + \mathbf{f} &= (\mathbf{A}\mathbf{x}_p + \mathbf{f}) + (\mathbf{A}\mathbf{x}_h) \\ &= \mathbf{x}_p' + \mathbf{x}_h' \\ &= (\mathbf{x}_p + \mathbf{x}_h)' \end{aligned}$$

So, $\mathbf{x}_p + \mathbf{x}_h$ is a solution to the homogeneous equation. □

General solutions: Nonhomogeneous case

Theorem

Let $\mathbf{x}_p$ be a particular solution to the nonhomogeneous equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}$$

on an open interval I on which $\mathbf{A}$ and $\mathbf{f}$ are continuous.

Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be linearly independent solutions to the homogeneous equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ on the interval I.

Then any solution $\mathbf{x}(t)$ to the nonhomogeneous equation can be expressed in the form

$$\mathbf{x}(t) = \mathbf{x}_p + c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t)$$

for some constants $c_1, c_2, \ldots, c_n$.

Def. *This form is called the general solution to the equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$*

Example

Example. The nonhomogeneous equation

$$\mathbf{x}' = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -9t \\ 0 \\ -18t \end{bmatrix}$$

and a particular solution

$$\mathbf{x}_p = \begin{bmatrix} 5t+1 \\ 2t \\ 4t+2 \end{bmatrix}$$

The following are linearly independent solutions to the associated homogeneous equation

$$\mathbf{x}_1 = \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -e^{3t} \\ e^{3t} \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{bmatrix},$$

So, any solution to the nonhomogeneous equation is of the form

$$\mathbf{x} = \mathbf{x}_p + c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$$

Method for solving normal systems

1. To determine a general solution to an $n \times n$ homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
   a) Find n linearly independent solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$.
   b) Form the linear combination

   $$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$$

   for the general solution.
2. To determine a general solution to an $n \times n$ nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$.
   a) Find a particular solution $\mathbf{x}_p$ to the nonhomogeneous system.
   b) Find a general solution $\mathbf{x}_c$ to the associated homogeneous system $\mathbf{x} = \mathbf{A}\mathbf{x}$.
   c) The sum of the solutions from (a) and (b) is the desired general solution

   $$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_c.$$

Example

Example. Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 5 \\ 6 \\ -2 \end{bmatrix}$$

We saw earlier that the following are linearly independent solutions:

$$\mathbf{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = e^{-t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = e^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

A general solution to the equation is given by

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3.$$

Example continued

We want values of c_1, c_2 and c_3 satisfying the initial values

$$\mathbf{x}(0) = \begin{bmatrix} 5 \\ 6 \\ -2 \end{bmatrix} = c_1 e^{2\cdot 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-0} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} c_3 e^{-0} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Solve the system of equations in unknowns c_1, c_2 and c_3:

$$\begin{aligned} 5 &= c_1 - c_2 \\ 6 &= c_1 + \quad + c_3 \\ -2 &= c_1 + c_2 - c_3. \end{aligned}$$

Example continued

Solve the system of equations.

$$\begin{aligned} 5 &= c_1 - c_2 \\ 6 &= c_1 + \quad + c_3 \\ -2 &= c_1 + c_2 - c_3 \end{aligned}$$

Eliminate c_1 from row 2 and row 3 R_3, isolating c_1 to row 1 (R_1).
R_2: add $-1 \times R_1$ to R_2:

$$\begin{aligned} 5 &= c_1 - c_2 \\ 1 &= 0c_1 + c_2 + c_3 \\ -2 &= c_1 + c_2 - c_3 \end{aligned}$$

R_3: add $-1 \times R_1$ to R_3:

$$\begin{aligned} 5 &= c_1 - c_2 \\ 1 &= 0c_1 + c_2 + c_3 \\ -7 &= 0c_1 + 2c_2 - c_3 \end{aligned}$$

Example continued

Solve the system of equations.

$$\begin{aligned} 5 &= c_1 - c_2 \\ 1 &= \quad c_2 + c_3 \\ -7 &= \quad +2c_2 - c_3 \end{aligned}$$

Eliminate c_2 from row 1 and row 3 , isolating c_2 to row 2.
R_1: addR_2 to R_1:

$$\begin{aligned} 6 &= c_1 + 0c_2 + c_3 \\ 1 &= \quad c_2 + c_3 \\ -7 &= \quad +2c_2 - c_3 \end{aligned}$$

R_3: add $-2 \times R_2$ to R_3:

$$\begin{aligned} 6 &= c_1 + 0c_2 + c_3 \\ 1 &= \quad c_2 + c_3 \\ -9 &= \quad +0c_2 - 3c_3 \end{aligned}$$

Example continued

Solve the system of equations.

$$\begin{aligned} 6 &= c_1 + \quad + c_3 \\ 1 &= \quad c_2 + c_3 \\ -9 &= \quad -3c_3 \end{aligned}$$

Eliminate c_3 from row 1 and row 2 , isolating c_3 to row 3.
R_1: add $\frac{1}{3} \times R_3$ to R_1:

$$\begin{aligned} 6 &= c_1 + \quad + c_3 \\ 1 &= \quad c_2 + c_3 \\ -9 &= \quad -3c_3 \end{aligned}$$

R_2: add $\frac{1}{3} \times R_3$ to R_2:

$$\begin{aligned} 3 &= c_1 + \quad + 0c_3 \\ -2 &= \quad c_2 + 0c_3 \\ -9 &= \quad -3c_3 \end{aligned}$$

Example concluded

Solve the system of equations.

$$\begin{aligned} 3 &= c_1 \\ -2 &= c_2 \\ -9 &= -3c_3 \end{aligned}$$

Read off the solutions

$$c_1 = 3, \quad c_2 = -2, \quad c_3 = 3.$$

The solution to the initial value problem is

$$\mathbf{x} = 3e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2e^{-t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 3e^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$