Math 216
Differential Equations

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Systems of equations as matrix equations

A general first-order linear equation in normal form

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + f_1(t), \\ x_2'(t) &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t), \\ &\vdots \\ x_n'(t) &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + f_n(t), \end{aligned}$$

can be written as a matrix equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

A is called the coefficient matrix.

Solutions for matrix equations

Let $\mathbf{A} = [a_{ij}(t)]$ be an $n \times n$ matrix and $\mathbf{f}(t) = [f_i(t)]$ be an $n \times 1$ vector.

Def. A solution on interval I to the vector equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

is an $n \times 1$ vector $\mathbf{b}(t) = [b_i(t)]$ satisfying for all t in I

$$\begin{bmatrix} b_1'(t) \\ b_2'(t) \\ \vdots \\ b_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

Solutions for matrix equations

Let $\mathbf{A} = [a_{ij}(t)]$ be an $n \times n$ matrix and $\mathbf{f}(t) = [f_i(t)]$ be an $n \times 1$ vector.

If $\mathbf{b}(t) = [b_i(t)]$ is a solution to the matrix equation

$$\mathbf{b}'(t) = \mathbf{A}(t)\mathbf{b}(t) + \mathbf{f}(t)$$

then $b_1(t), b_2(t), \ldots, b_n(t)$ are solutions to the system of first-order equations

$$\begin{aligned} b_1'(t) &= a_{11}(t)b_1 + a_{12}(t)b_2 + \ldots + a_{1n}(t)b_n + f_1(t), \\ b_2'(t) &= a_{21}(t)b_1 + a_{22}(t)b_2 + \ldots + a_{2n}(t)b_n + f_2(t), \\ &\vdots \\ b_n'(t) &= a_{n1}(t)b_1 + a_{n2}(t)b_2 + \ldots + a_{nn}(t)b_n + f_n(t), \end{aligned}$$

Existence and uniqueness

Theorem

Let $\mathbf{A} = [a_{ij}(t)]$ be an $n \times n$ matrix and $\mathbf{f}(t) = [f_i(t)]$ be an $n \times 1$ vector.

Suppose the components $a_{ij}(t)$ and $f_i(t)$ are continuous on an open interval I containing the point t_0.

Then for any choice of an initial $n \times 1$ vector $\mathbf{b}$, there exists a unique solution $\mathbf{x}(t)$ on the interval I to the initial value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{b}.$$

Matrix operations

Equality. Two $m \times n$ matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are equal when they are equal componentwise:

$$\mathbf{A} = \mathbf{B} \quad \text{when } a_{ij} = b_{ij} \text{ for each } 1 \leq i \leq m, 1 \leq j \leq n.$$

Addition. Add matrices componentwise: when $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices

$$\mathbf{A} + \mathbf{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

Scalar multiplication. Multiplication of a matrix by a scalar is componentwise.

$$c\mathbf{A} = c\,[a_{ij}] = [ca_{ij}]$$

Examples

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Addition.

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{Z} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \mathbf{A}$$

Scalar Multiplication.

$$27\mathbf{A} = 27 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 27 & 54 \\ 81 & 108 \end{bmatrix}$$

Basic properties of addition and scalar multiplication

$\mathbf{0}$ is an $m \times n$ matrix whose entries are all 0.

Let $\mathbf{A}$, $\mathbf{B}$,$\mathbf{C}$ be $m \times n$ matrices and c, d are real numbers.

$$\begin{aligned} \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} &\qquad \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \\ \mathbf{A} + \mathbf{0} = \mathbf{A} &\qquad \mathbf{A} + (-\mathbf{A}) = \mathbf{0} \\ c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B} &\qquad (c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A} \\ c(d\mathbf{A}) = (cd)\mathbf{A} = d(c\mathbf{A}) & \end{aligned}$$

Matrix multiplication

Def. Let $\mathbf{A} = [a_{ik}]$ be an $m \times p$ matrix and $\mathbf{B} = [b_{kj}]$ be an $p \times n$ matrix.

The product **AB** is the $m \times n$ matrix $\mathbf{C} = [c_{ij}]$ where

$$\begin{aligned} c_{ij} &= \mathbf{a}_{i-}\mathbf{b}_{-j} \quad (\text{ } i\text{th row of } \mathbf{A} \text{ times } j\text{th column of } \mathbf{B}\text{ }) \\ &= \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{ip} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \dots \\ b_{pj} \end{bmatrix} \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ip}b_{pj} \end{aligned}$$

Warning . The dimensions of **A**, **B** and **C** are

$$\begin{matrix} \mathbf{A} & \mathbf{B} & & \mathbf{C} \\ (m \times p) & (p \times n) & = & m \times n \end{matrix}$$

Matrix multiplication

$$\mathbf{AB} = [a_{ik}]\,[b_{kj}] = [c_{ij}]$$

where c_{ij} is the product of the ith row of **A** and jth column of **B**:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pj} & \dots & b_{pn} \end{bmatrix}$$

Example

Example.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Warning . Matrix multiplication is not generally commutative.

$$\mathbf{AB} \neq \mathbf{BA} \quad \text{is usually true.}$$

Example

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 5$$

and

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Warning . Order makes a difference.

Basic properites of matrix multiplication

A, **B**, and **C** are matrices of the right dimensions, and c is a real number.

$$\begin{aligned} (\mathbf{AB})\mathbf{C} &= \mathbf{A}(\mathbf{BC}) && \text{Associativity}\\ (\mathbf{A}+\mathbf{B})\mathbf{C} &= \mathbf{AC}+\mathbf{BC} && \text{Distributivity}\\ \mathbf{A}(\mathbf{B}+\mathbf{C}) &= \mathbf{AB}+\mathbf{AC} && \text{Distributivity}\\ c(\mathbf{A})\mathbf{B} = c(\mathbf{AB}) &= \mathbf{A}(c\mathbf{B}) && \text{Associativity}\\ \mathbf{A0} = \mathbf{0} &= \mathbf{0A} && \text{Zero} \end{aligned}$$

Example

Example.

$$\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}\begin{bmatrix}-2 & 6\\ 1 & -3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

☠ **Warning** . It is possible that $\mathbf{AB} = \mathbf{0}$ with $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{A} \neq \mathbf{0}$.

Matrix Identity

Def. The $n \times n$ identity matrix is the square matrix

$$\mathbf{I}_n = \begin{bmatrix}1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\end{bmatrix}$$

having ones down the main diagonal and zeros everywhere else.

Example.

$$\mathbf{I}_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

We write **I** leaving the dimension to be implicitly determined.

Matrix identity is multiplicative identity

Example. $\mathbf{I}_3$ is a multiplicative identity:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ 1 & 3 & 2\\ 1 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2 & 1\\ 1 & 3 & 2\\ 1 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2 & 1\\ 1 & 3 & 2\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

If **A** is any $n \times n$ matrix and $\mathbf{I} = \mathbf{I}_3$.

$$\mathbf{AI} = \mathbf{A} = \mathbf{IA}.$$

Matrix inverses

Def. A square matrices ($n \times n$) is said to have an inverse if there is a square matrix $\mathbf{B}$ with the property

$$\mathbf{AB} = \mathbf{I}.$$

When this happens it is also true that

- $\mathbf{BA} = \mathbf{I}$, and
- $\mathbf{B}$ is the unique matrix with $\mathbf{AB} = \mathbf{I}$.

We write $\mathbf{A}^{-1}$ for the unique inverse to $\mathbf{A}$ (when it exists!!)

Def. A matrix is invertible (or nonsingular) when it has an inverse.

Example

Example.

$$\begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also, verify that

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

Warning . Not all square matrices are invertible.

Example. Recall,

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, none of these three matrices has inverse.

Having inverses is useful

Fact. If $\mathbf{A}$ is an invertible matrix, then the matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

always has a unique solution, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Reason.

$$\mathbf{x} = \mathbf{Ix} = \mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{b};$$

the solution is unique since if $\mathbf{x}$ and $\mathbf{y}$ are solutions, then

$$\mathbf{Ax} = \mathbf{b} = \mathbf{Ay}$$

so multiply both sides by $\mathbf{A}^{-1}$ to get $\mathbf{x} = \mathbf{y}$.

When do inverses exist?

Question. When does an $n \times n$ matrix have a solution?

Answer. When its determinant is nonzero.

(It is more useful to know a matrix is invertible then to compute its inverse.)

2×2 determinants

♥ Determinants hold the key to the existence of inverses.

Def. For a 2×2 matrix $\mathbf{A}$, the determinant of $\mathbf{A}$, denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$ is defined by

$$\det(\mathbf{A}) := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Inverses from determinants

♥ If $\det(\mathbf{A}) \neq 0$ then we can compute $\mathbf{A}^{-1}$ by

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}, \quad \det(\mathbf{A}) = 9, \quad \mathbf{A}^{-1} = \frac{1}{9} \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

Verify by multiplying the matrices.

3×3 determinants

Def. For a 3×3 matrix $\mathbf{A}$, the determinant of $\mathbf{A}$, denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$ is defined by

Expansion along the first row:

$$\det(\mathbf{A}) := \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Equivalently, you can compute $\det(\mathbf{A})$ as follows:

Expansion along the first column:

$$\det(\mathbf{A}) := \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Example of 3 × 3 determinants

Example.
Expansion along the first row:

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 0 & 5 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix}$$
$$= 1(-3-5) - 2(0-10) + 1(0-6) = 6$$

Expansion along the first column:

$$\det(\mathbf{A}) := \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix}$$
$$= 1(-3-5) - 0(-2-1) + 2(10-3) = 6.$$

The key to determinants

The determinant for $n \times n$ matrices can be computed by cofactor expansion. See Edwards and Penney, Chapter 5.1, p. 290 (version 3) for details.

♥ The key to determinants.

Theorem

Let $\mathbf{A}$ *be an* $n \times n$ *matrix. The following properties are equivalent.*

1. $\det(\mathbf{A}) \neq 0$.
2. $\mathbf{A}$ *has an inverse.*
3. $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution* $\mathbf{x} = \mathbf{0}$.
4. $\mathbf{Ax} = \mathbf{b}$ *has a unique solution.*

So, if $\det(\mathbf{A}) \neq 0$, *then* $\mathbf{A}^{-1}$ *exists and*

$$\mathbf{Ax} = \mathbf{b} \quad \text{has the unique solution } \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

Example

Example. Find all solutions to the system of equations

$$\begin{aligned} 8 &= 2x_1 + 4x_2 + x_3 \\ 6 &= 2x_1 + 4x_2 \\ -10 &= -4x_1 - 8x_2 + x_3 \end{aligned}$$

As a matrix equations:

$$\begin{bmatrix} 8 \\ 6 \\ -10 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 2 & 4 & 0 \\ -4 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example continued

Compute the determinant

$$\begin{vmatrix} 2 & 4 & 1 \\ 2 & 4 & 0 \\ -4 & -8 & 1 \end{vmatrix} = 2\begin{vmatrix} 4 & 0 \\ -8 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ -8 & 1 \end{vmatrix} - 4\begin{vmatrix} 4 & 0 \\ 4 & 1 \end{vmatrix}$$
$$= 2(4-0) - 2(4+8) - 4(0-4)$$
$$= 0.$$

So, we know the matrix equation does not have a unique solution.

$$\begin{bmatrix} 8 \\ 6 \\ -10 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 2 & 4 & 0 \\ -4 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example continued

Solve the system of equations.

$$\begin{aligned} 8 &= 2x_1 + 4x_2 + x_3 \\ 6 &= 2x_1 + 4x_2 \\ -10 &= -4x_1 - 8x_2 + x_3 \end{aligned}$$

Eliminate x_1 from row 2 (R_2) and row 3 R_3, isolating x_1 to row 1 (R_1).
R_2: by adding $-1 \times R_1$ to R_2:

$$\begin{aligned} 8 &= 2x_1 + 4x_2 + x_3 \\ -2 &= 0x_1 + 0x_2 - x_3 \\ -10 &= -4x_1 - 8x_2 + x_3 \end{aligned}$$

R_3: by adding $2 \times R_1$ to R_3:

$$\begin{aligned} 8 &= 2x_1 + 4x_2 + x_3 \\ -2 &= 0x_1 + 0x_2 - x_3 \\ 6 &= 0x_1 + 0x_2 + 3x_3 \end{aligned}$$

Note that the solutions to this new system of equations is the same as the

Example continued

Solve the system of equations.

$$\begin{aligned} 8 &= 2x_1 + 4x_2 + x_3 \\ -2 &= -x_3 \\ 6 &= 3x_3 \end{aligned}$$

Eliminate x_3 from row 1 and row 3 , isolating x_3 to row 2.
R_1: by adding R_2 to R_1:

$$\begin{aligned} 6 &= 2x_1 + 4x_2 + 0x_3 \\ -2 &= x_3 \\ 6 &= 3x_3 \end{aligned}$$

R_3: by adding $3 \times R_2$ to R_3:

$$\begin{aligned} 8 &= 2x_1 + 4x_2 + 0x_3 \\ -2 &= -x_3 \\ 0 &= 0x_3 \end{aligned}$$

Note that the solutions to this new system of equations is the same as the

Example continued

Solve the system of equations.

$$\begin{aligned} 8 &= 2x_1 + 4x_2 \\ -2 &= -x_3 \\ 0 &= 0 \end{aligned}$$

Read off the solutions:

$$\begin{aligned} 4 - 2x_2 &= x_1 \\ 2 &= x_3 \end{aligned}$$

There are no constraints placed on x_2, so we are free to let it be any value.
The following are the solutions to the original equation:

$$x_1 = 4 - 2c, \quad x_2 = c, \quad x_3 = 2 \qquad \text{for each real value } c$$

There are infinitely many solution.