Math 216
Differential Equations

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Normal Form

A system of n first-order linear differential equations in n unknowns $x_1(t), x_2(t), \ldots, x_n(t)$ expressed as

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + f_1(t), \\ x_2'(t) &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t), \\ &\vdots \\ x_n'(t) &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + f_n(t), \end{aligned}$$

is said to be in normal form.

Mass-spring equations

Problem. Express the spring-mass equation in unknown $x(t)$

$$mx'' + cx' + kx = f(t)$$

as a system of first-order equations in normal form.

Solution. Let v be velocity, so that $x' = v$ and $x'' = v'$ (acceleration).

$$mv' + cv + kx = f(t) \qquad x' = v$$

or in normal form: where $x_1 = x$ and $x_2 = v$,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{f_1(t)}{m}. \end{aligned}$$

Higher-order equations

Problem. Express the n-order linear equation in unknown $y(t)$

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = f(t).$$

as a system of first-order equations in normal form.

Solution. Introduce unknowns

$$y_1(t) = y(t), \quad y_2(t) = y'(t), \quad y_3(t) = y''(t), \ \ldots, \ y_n(t) = y^{(n-1)}.$$

Then,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= -a_{n-1}y_n - \ldots - a_1y_2 - a_0y_1 - f(t). \end{aligned}$$

Example

Problem. Express the n-order linear equation in unknown $y(t)$

$$\begin{aligned} 2x'' + 6x - 2y &= 2t \sin t \\ y'' + 2y - 2x &= 8 \cos t \end{aligned}$$

as a system of first-order equations in normal form.

Solution. Introduce unknowns

$$\begin{aligned} x_1(t) = x(t), \quad x_2(t) = x'(t) \\ x_3(t) = y(t), \quad x_4(t) = y'(t) \end{aligned}$$

Then,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -3x_1 - x_3 + t \sin t \\ x_3' &= x_4 \\ x_4' &= 2x_1 - 2x_3 + 8 \cos t \end{aligned}$$

Vector notation

Def. A column vector (or simply, vector) is an $n \times 1$ array

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [a_j]$$

where $a_1, a_2, \ldots, a_n$ are real values, called the vector components.

We will write

$$\mathbf{a} = [a_i]$$

where the range of indices $1 \leq i \leq n$ is understood.

An $n \times 1$ column vector represents the vector $(a_1, a_2, \ldots, a_n)$ in $\mathbb{R}^n$.

Def. A scalar value is another name for real number.

Vector equality

Equality. Two $n \times 1$ column vectors are equal when they have the same components. Let

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

then we write

$$\mathbf{a} = \mathbf{b} \quad \text{when } a_i = b_i \text{ for each } i \leq n.$$

Vector operations

Addition. Add vectors componentwise

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

Scalar multiplication. Multiplication of a vector by a scalar is componentwise.

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad c\mathbf{a} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

where c is a real value.

Zero vector

Zero vector. The $n \times 1$ zero vector is $\mathbf{0} = [0]$ and satisfies

$$\mathbf{0} + \mathbf{a} = \mathbf{a} = \mathbf{a} + \mathbf{0}.$$

Properties. Vector addition and scalar multiplication satisfies

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a} \\ c(\mathbf{a} + \mathbf{b}) &= c\mathbf{a} + c\mathbf{b} \\ (c+d)\mathbf{a} &= c\mathbf{a} + d\mathbf{b} \end{aligned}$$

where c, d are real values.

Vector functions

Def. If the components of a vector are functions $a_1(t), a_2(t), \ldots, a_n(t)$ then the column vector

$$\mathbf{a}(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{bmatrix} = [a_i(t)]$$

is a vector valued function, since for input t, $\mathbf{a}(t)$ is a vector.

Differentiate vector functions componentwise

$$\mathbf{a}'(t) = \begin{bmatrix} a_1'(t) \\ a_2'(t) \\ \vdots \\ a_n'(t) \end{bmatrix} = [a_i'(t)]$$

Dot product

Def. The dot product (or scalar product) generalizes the familiar operation in $\mathbb{R}^2$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n.$$

Example. The length of a vector (in $\mathbb{R}^n$) is

$$\begin{aligned} |\mathbf{a}| &= \sqrt{\mathbf{a} \cdot \mathbf{a}} \\ &= \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}. \end{aligned}$$

Dot product and equations

Example. We can re-write equations using vector operations

$$2x_1 + 3x_2 + 4x_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$2x_1 + t^2 x_2 + (4t + e^t)x_4 = \begin{bmatrix} 2 \\ t^2 \\ 0 \\ 4t + e^t \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Dot product properties

Properties.

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= \mathbf{b}\cdot\mathbf{a} \\ \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} \\ (c\mathbf{a})\cdot\mathbf{b} &= \mathbf{a}\cdot(c\mathbf{b}) = c(\mathbf{a}\cdot\mathbf{b}) \end{aligned}$$

Example. Let $\mathbf{a} = [a_i]$ and $\mathbf{b} = [b_i]$.

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= a_1b_1 + a_2b_2 + \ldots + a_nb_n \\ &= b_1a_1 + b_2a_2 + \ldots + b_na_n \\ &= \mathbf{b}\cdot\mathbf{a}. \end{aligned}$$

Matrix

Def. A matrix is an $m \times n$ rectangular array of numbers

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

the values a_{ij} are the matrix components.

Note. An $m \times n$ matrix has m rows and n columns.

Notation. We will write an $m \times n$ matrix as

$$\mathbf{A} = [a_{ij}]$$

where the range of the indices $1 \le i \le m$ and $1 \le j \le n$ is understood.

Matrix

Def. A matrix function is a matrix of functions $a_{ij}(t)$:

$$\begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{m1}(t) & a_{m2}(t) & \ldots & a_{mn}(t) \end{bmatrix}$$

We write this as $\mathbf{A}(t) = [a_{ij}(t)]$.

Def. A matrix function is continuous on an interval I if each of its components $a_{ij}(t)$ is continuous on I.

Row vectors

Note. A column vector is an $n \times 1$ matrix

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Def. A row vector is an $1 \times n$ matrix

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$$

Matrices as vectors

Notation. It is often convenient to re-write a matrix **A** by either its m row vectors

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{1-} \\ \mathbf{a}_{2-} \\ \vdots \\ \mathbf{a}_{m-} \end{bmatrix} \quad \text{where} \quad \mathbf{a}_{i-} = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix},$$

or its n column vectors

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{-1} & \mathbf{a}_{-2} & \dots & \mathbf{a}_{-n} \end{bmatrix} \quad \text{where} \quad \mathbf{a}_{-j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

Vector product

Def. We define the product of an $1 \times n$ row vector and an $n \times 1$ column vector by dot product:

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Example. We can rewrite equations as vector products:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Vector product: Caution

Warning. Order matters. We have not yet defined the product of a column vector by a row vector.

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}$$

When we do define this, the result is an $n \times n$ matrix

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_n \\ \vdots & \vdots & & \vdots \\ a_n b_1 & a_n b_2 & \dots & a_n b_n \end{bmatrix}$$

Systems of equations as matrix equations

A general first-order linear equation in normal form

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + f_1(t), \\ x_2'(t) &= a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + f_2(t), \\ &\vdots \\ x_n'(t) &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + f_n(t), \end{aligned}$$

can be written as a matrix equation

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

We need to define the product of a matrix and column vector.

Systems of equations as matrix equations

The ith equation of the system of equations is

$$x_i'(t) = a_{i1}(t)x_1 + a_{i2}(t)x_2 + \ldots + a_{in}(t)x_n + f_i(t),$$

$$= \begin{bmatrix} a_{i1}(t) & a_{i2}(t) & \ldots & a_{in}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + f_i(t)$$

and is the product of the ith row of **A** and **x**

$$\mathbf{Ax} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{i1}(t) & a_{i2}(t) & \ldots & a_{in}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_i(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

Matrix and vector multiplication

Def. Product of an $m \times n$ matrix **A** and $n \times 1$ column vector **x** is an $m \times 1$ column vector

$$\mathbf{Ax} = \begin{bmatrix} \mathbf{a}_{1-}\ \mathbf{x} \\ \mathbf{a}_{2-}\ \mathbf{x} \\ \vdots \\ \mathbf{a}_{m-}\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ a_{21}x_1 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{bmatrix}$$

Note. **Ax** is an $m \times 1$ column vector whose ith component is the product of the ith row of **A**, $\mathbf{a}_{i-}$, and the vector **x**.

Alternative view: Matrix and vector multiplication

Alternative. Product of an $m \times n$ matrix **A** and $n \times 1$ column vector **x**

$$\mathbf{Ax} = \begin{bmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ a_{21}x_1 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{bmatrix} = x_1\mathbf{a}_{-1} + \ldots + x_n\mathbf{a}_{-n}$$

Ax is a linear combination of column vectors of **A**.

Alternative view: Matrix and vector multiplication

Example.

$$\begin{bmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 2e^t + c_2 2e^{3t} + c_3 2e^{5t} \\ c_1 2e^t - c_3 2e^{5t} \\ c_1 e^t - c_2 e^{3t} + c_3 e^{5t} \end{bmatrix}$$

$$= c_1 \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$$

Systems of equations as matrix equations

A general first-order linear equation in normal form

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + f_1(t), \\ x_2'(t) &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t), \\ &\vdots \\ x_n'(t) &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + f_n(t), \end{aligned}$$

can be written as a matrix equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

A is called the coefficient matrix.

Example

Example. Write the third-order linear equation as a matrix equation.

$$x''' + a_2x'' + a_1x' + a_0x = f(t).$$

As a system of first-order linear equations in normal form:
set $x_1 = x$

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -a_0x_1 - a_1x_2 - a_2x_3 + f(t) \end{aligned}$$

As a matrix equation:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f(t) \end{bmatrix}$$

Solutions for matrix equations

Let $\mathbf{A} = [a_{ij}(t)]$ be an $n \times n$ matrix function and $\mathbf{f}(t) = [f_i(t)]$ be an $n \times 1$ vector function.

Def. A solution on interval I to the vector equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

is an $n \times 1$ vector function $\mathbf{b}(t) = [b_i(t)]$ satisfying for all t in I the substitution

$$\begin{bmatrix} b_1'(t) \\ b_2'(t) \\ \vdots \\ b_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

Existence and uniqueness

Theorem

Let $\mathbf{A} = [a_{ij}(t)]$ be an $n \times n$ matrix function and $\mathbf{f}(t) = [f_i(t)]$ be an $n \times 1$ vector function.

Suppose the components $a_{ij}(t)$ and $f_i(t)$ are continuous on an open interval I containing the point t_0.

Then for any choice of an initial $n \times 1$ vector $\mathbf{b}$, there exists a unique solution $\mathbf{x}(t)$ on the interval I to the initial value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{b}.$$

Example

Example. The matrix equation initial value problem

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

has the unique solution $\mathbf{x}_1 = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}$.

Example. The matrix equation initial value problem

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

has the unique solution $\mathbf{x}_1 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$.

Homogeneous matrix equations

Def. The general matrix equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

is nonhomogeneous. It is homogeneous when $\mathbf{f} \equiv \mathbf{0}$.

Def. The associated homogeneous equation is

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}.$$

Superposition Principle

Theorem

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n solutions on an interval I to the homogeneous matrix equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}.$$

If $c_1, c_2, \ldots, c_n$ are any constants, then the linear combination

$$\mathbf{x} = c_1\mathbf{x}_2 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$$

is also a solution on I.

Example

Example. The matrix equation

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \mathbf{x}$$

has solutions (on the entire real line)

$$\mathbf{x}_1 = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$$

So, any linear combination is also a solution:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2.$$

Question. Is this a general solution? That is, are all solutions to the equation linear combinations of $\mathbf{x}_1$ and $\mathbf{x}_2$?

Example

Example. The general solution (on the entire real line) to the equation

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \mathbf{x}$$

is

$$\begin{aligned} \mathbf{x} &= c_1 \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} \\ &= \begin{bmatrix} 2c_1 e^t + c_2 e^{2t} \\ -3c_1 e^t - c_2 e^{2t} \end{bmatrix} \end{aligned}$$

Example

Example. Solve the initial value problem,

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \mathbf{x}, \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Substitute $t = 0$ into the general solution

$$\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2e^0 \\ -3e^0 \end{bmatrix} + c_2 \begin{bmatrix} e^{2\cdot 0} \\ -e^{2\cdot 0} \end{bmatrix}$$

The problem is reduced to solving the equations

$$\begin{aligned} 3 &= 2c_1 + c_2 \\ 1 &= -3c_1 - c_2 \end{aligned}$$

so $c_1 = -4$ and $c_2 = 11$.