Math 216
Differential Equations

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Existence and Uniqueness Theorem

Theorem (Theorem 1, Section 1.3)

In any rectangle R in which the function $F(x, y)$ is nicely behaved, the IVP

$$\frac{dy}{dx} = F(x, y), \qquad y(a) = b$$

has exactly one solution on some open interval I containing a.

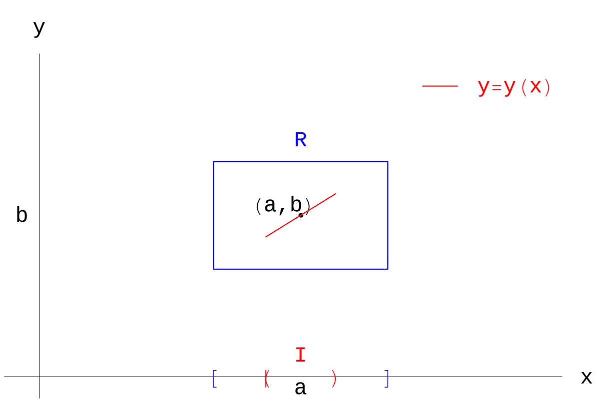
F is nicely behaved on a rectangle R if

(a) *F is continuous on R,*

(b) $\frac{\partial F}{\partial y}$ *is continuous on R.*

Graphic rendition of Theorem

Note. The solution $y = y(x)$ guaranteed by the Theorem may not extend the whole length of the x-side of the rectangle R. We will see an example of this later in the lecture.



Definition: Separable ODE

Definition

A first order ODE

$$\frac{dy}{dx} = F(x, y)$$

is separable provided

$$\frac{dy}{dx} = F(x, y) = g(x)h(y)$$

where g is a function of x only and h is a function of y only.

Definition: Separable ODE

Question. Which of the following ODEs are separable?

(a) $\frac{dy}{dx} = x^2 + y^2$

(b) $\sin x \frac{dy}{dx} = y$

(c) $x^2 \frac{dy}{dx} + xy = \sin x$

(d) $\frac{dy}{dx} = y$

Answer. (b), (d)

Solving separable ODEs

We want to solve the separable ODE

$$\frac{dy}{dx} = g(x)h(y).$$

Divide by $h(y)$

$$\frac{1}{h(y)} \frac{dy}{dx} = g(x).$$

Warning Dividing by $h(y)$ is only legitimate when $y \neq 0$.

Integrate both sides by with respect to x

$$\int \frac{1}{h(y)} \frac{dy}{dx} \, dx = \int g(x) \, dx.$$

We will use this to show that $\int \frac{1}{h(y)} \, dy = \int g(x) \, dx + C$.

Solving separable ODEs

Let

$$H(y) = \int \frac{1}{h(y)} \, dy \qquad \text{and} \qquad G(x) = \int g(x) \, dx$$

By the chain rule $D_x(H(y)) = D_y(H(y)) \frac{dy}{dx}$. So,

$$\begin{aligned} D_x(H(y)) &= D_y\Big(\int \frac{1}{h(y)} \, dy\Big) \frac{dy}{dx} \\ &= \frac{1}{h(y)} \frac{dy}{dx} = g(x) \\ &= D_x\Big(\int g(x) \, dx\Big) \\ &= D_x(G(x)) \end{aligned}$$

Thus, $H(y(x)) = G(x) + C$ (they have the same derivative). So,

$$\int \frac{1}{h(y)} \, dy = \int g(x) \, dx + C$$

Rule for solving separable ODEs

Here is a simple rule for solving separable ODEs. If

$$\frac{dy}{dx} = g(x)h(y).$$

Then,

$$\int \frac{1}{h(y)} \, dy = \int g(x) \, dx + C.$$

ConcepTest

Question. Let $P(t)$ be a continuous function. FInd a general solution for the ODE

$$\frac{dy}{dt} - P(t)y = 0.$$

Answer. $y(t) = \pm e^C e^{\int P(t)\,dt}$.
The equation is separable: $\frac{dy}{dt} = P(t)y$.
We needed $P(t)$ to be continous to guarantee it was integrable.

However, you may have arrived at $Ce^{\int P(t)\,dt}$. Lets look at the more closely.

Finding all solutions

Problem. Let $P(t)$ be a continuous function. Find all solutions for the ODE

$$\frac{dy}{dt} = P(t)y.$$

This equation is separable.

$$\frac{1}{y}\frac{dy}{dt} = P(t).$$

Integrating,

$$\int \frac{1}{y}\,dy = \ln|y| = \int P(t) + C.$$

Take exponentials,

$$|y| = e^C e^{\int P(t)}.$$

So,

$$y_C(t) = \pm e^C e^{\int P(t)}.$$

Finding all solutions

However, $y_C(t) = \pm e^C e^{\int P(t)}$ are not all solutions to the equation

$$\frac{dy}{dt} = P(t)y;$$

there is also the constant zero function, $y \equiv 0$ ($y(t) = 0$ for every t).

Remember the ☠ Warning!: We divided both sides by y; our general solution missed this function.

All solutions to the equation are given by

$$\begin{cases} y_C(t) = \pm e^C e^{\int P(t)} & \text{the general solution} \\ y \equiv 0 & \text{singular solution} \end{cases}$$

More compactly, $y(t) = Ce^{\int P(t)}$.

Finding all solutions for separable equations

Consider the separable ODE

$$\frac{dy}{dx} = g(x)h(y);$$

- If $h(b) = 0$ (b is a root of $h(y)$), then the constant function $y \equiv b$ ($y(t) = b$ for every t) is a solution to the equation.
- $y \equiv b$ is a solution that is excluded by the general solution obtained by the method of separation

$$\int \frac{1}{h(y)} = \int g(x) + C.$$

- Solutions of the form $y \equiv b$ where $h(b) = 0$ are called singular solutions, because they do not appear in the general solution obtained by the method of separation.

ConcepTest

Question. Find all solutions to the ODE

$$x^2 \frac{dy}{dx} = (y+1)^2.$$

Answer. Method of separation of variables:

$$\int (y+1)^{-2}\, dy = \int x^{-2}\, dx + C$$

$$\begin{cases} \frac{x}{1-Cx} - 1 & \text{the general solution} \\ y \equiv -1 & \text{singular solution} \end{cases}$$

Equivalently,

$$\begin{cases} \frac{x}{1+Cx} - 1 & \text{the general solution} \\ y \equiv -1 & \text{singular solution} \end{cases}$$

ConcepTest

Problem. Find a solution for the IVP

$$\frac{dy}{dx} = 4xy^3, \qquad y(0) = 1.$$

Answer. The general solution by separation of variables:

$$y = \pm \frac{1}{\sqrt{C - x^2}}.$$

Since $y(0) = 1$, we are interested in the positive branch.

Solving for $y(0) = 1$ gives $C = 1$. So, the solution to the original equation is

$$y(x) = \frac{1}{\sqrt{1 - x^2}}.$$

Return to Theorem 1

Let R be the rectangle: $-2 \leq x \leq 2$ and $0 \leq y \leq 2$.
Consider solutions to the ODE on R of

$$\frac{dy}{dx} = 4xy^3.$$

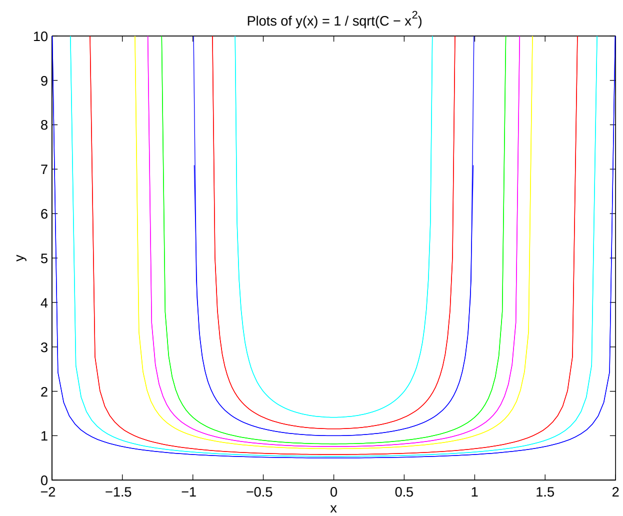
- $F(x,y) = 4xy^3$ is nice (F is continuous and $\frac{\partial F}{\partial y} = 12xy^2$ is continuous.
- By Theorem 1 (Section 1.3) there exists a unique solution through $(0,1) \in R$, on some x-interval I contained in $-2 \leq x \leq 2$. This is

$$y(x) = \frac{1}{\sqrt{1 - x^2}}.$$

- However, this is **not** a solution containing the point $(1,0)$. Why?
- The constant zero function is the unique solution at $(1,0)$. Why?

Figure1

Graph of $\frac{1}{\sqrt{C - x^2}}$ (see `fig1.m`)
Note that the solutions approach $y \equiv 0$ as $C \to \infty$.



ConcepTest

A linear first-order equation is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Question. Which of the following ODEs are linear first-order equations?

(a) $\frac{dy}{dx} + (x^4 + 2y) = 0$

(b) $\frac{dy}{dx} + x\cos y = \sin x$

(c) $\frac{dy}{dx} = x^2 + y^2$

(d) $\frac{dy}{dx} - 2y\tan x = e^x$

Answer. (a), (d)

Method of Integrating factors

We can explicitly solve any linear first-order equation

$$\frac{dy}{dx} + P(x)y = Q(x).$$

using the method of integrating factors.

✳ **Wish list.** We want a function $\rho(x)$ satisfying

$$D_x\big(\rho(x)y(x)\big) = \rho(x)\frac{dy}{dx} + \rho(x)P(x)y(x).$$

$\rho(x)$ is called an integrating factor.

Strategy for solution

Suppose it is X-mas (when all wishes come true!) So

$$D_x\big(\rho(x)y(x)\big) = \rho(x)\frac{dy}{dx} + \rho(x)P(x)y(x)$$

Now we can solve the original equation.

- Multiply both sides of the original equation by $\rho(x)$.

$$\rho(x)\frac{dy}{dx} + \rho(x)P(x)y(x) = D_x\big(\rho(x)y(x)\big) = \rho(x)Q(x).$$

- Integrate both sides of the equation

$$\int D_x\big(\rho(x)y(x)\big)\,dx = \rho(x)y(x) = \int \rho(x)Q(x)\,dx + C.$$

- Solve for y to obtain the general solution of the original equation.

X-mas time

We want an integrating factor $\rho(x)$ satisfying

$$D_x\big(\rho(x)y(x)\big) = \rho(x)\frac{dy}{dx} + \rho(x)P(x)y(x).$$

Let

$$\rho(x) = e^{\int P(x)\,dx}$$

Then, (apply product rule for differentiation)

$$\begin{aligned} D_x\big(e^{\int P(x)\,dx}y(x)\big) &= e^{\int P(x)\,dx}\frac{dy}{dx} + D_x\big(e^{\int P(x)\,dx}\big)y(x) \\ &= e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}D_x\Big(\int P(x)\,dx\Big)y(x) \\ &= e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}P(x)y(x) \end{aligned}$$

Method of solution

To solve a linear first-order equation with unknown $y = y(x)$

1. Put the equation in the form
$$\frac{dy}{dx} + P(x)y = Q(x).$$
2. Calculate the integrating factor, $\rho(x) = e^{\int P(x)\,dx}$.
3. Multiply both sides of the equation by $\rho(x)$. The new left-hand side of the equation is a derivative.
$$\rho(x)\frac{dy}{dx} + \rho(x)P(x)y = D_x(\rho(x)y(x)) = \rho(x)Q(x).$$
4. Integrate both sides of the equation
$$\int D_x(\rho(x)y(x))\,dx = \rho(x)y(x) = \int \rho(x)Q(x)\,dx + C.$$
5. Solve for y to obtain the general solution of the original equation.
$$y(x) = \frac{1}{\rho(x)}\Big[\int \rho(x)Q(x)\,dx + C\Big].$$

ConcepTest

Problem. Find the general solution to the ODE
$$2xy' - 3y = 9x^3$$
Put the equation in standard form:
$$y' - \frac{3}{2x}y = \frac{9}{2}x^2.$$
☠ Warning Dividing by $2x$ means that $x = 0$ is no longer possible.

Continuing, $P(x) = \frac{3}{2x}$ and $Q(x) = \frac{9}{2}x^2$. $P(x)$ is not continuous at $x = 0$. We suppose that $x > 0$. Compute $\rho(x)$:
$$\rho(x) = \exp\Big(\int -\frac{3}{2x}\Big) = \exp\Big(-\frac{3}{2}\ln x\Big) = x^{-\frac{3}{2}}.$$
(We dropped the absolute values since x is positive.) From here the solution is computed using the algorithm
$$3x^3 + Cx^{\frac{3}{2}}$$

Absolute values

- The indefinite integral
$$\int \frac{1}{x}\,dx = \ln|x|.$$
When we are simply looking for a general solution of an equation the absolute value sign is ignored (thus avoiding nasty case analysis).
- However, when we are solving an IVP we need to be sensitive to the absolute value. See Example 1, Section 1.4 of E+P.
- ☠ Warning We must be careful when putting an ODE into normal form when dividing by a possibly zero term. If we are looking for the general solution, this can usually be ignored; but, if we are solving an IVP we need to be sensitive about the possibility of of harm, as in the last example where $x = 0$ is no longer possible.

Existence and Completeness

Theorem

If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point a, then the IVP
$$\frac{dx}{dy} + P(x)y = Q(x), \qquad y(a) = b$$
has a unique solution $y = y(x)$ on I.

The unique solution is given by
$$y(x) = e^{-\int P(x)\,dx}\Big[\int \big(Q(x)e^{\int P(x)\,dx}\big)\,dx + C\Big]$$
for some choice of C.

Existence and Completeness

- The proof of the Theorem is the same as our argument that there exists an integrating factor for first-order linear equation. That argument only required that $P(x)$ and $Q(x)$ be continuous.
- The solution is uniquely determined on the entire interval. This contrasts with Theorem 1 of section 1.3 where the solution for the case $y(a) = b$ may only hold on a smaller subinterval. (Recall, the earlier example where this arose in applying the method of separation.)
- Every solution is included in the general solution. (In the proof we divided by the integrating factor $\rho(x)$, but our choice of $\rho(x)$ is never zero.) This contrasts with the case of separable equations.

Computing IVP

Problem. Solve the following IVP

$$xy' + y = 3xy, \qquad y(1) = 0$$

Answer. $y \equiv 0$ (y is the constant zero function). We can rewrite this equation as

$$y' = \frac{3x-1}{x}y.$$

This equation is both linear and separable. Using the separation method the general solution is

$$y(x) = e^C\frac{e^{3x}}{x},$$

so that $y \equiv 0$ is a singular solution. Using the integrating factor method the general solution is

$$y(x) = C\frac{e^{3x}}{x},$$

so that $y \equiv 0$ is part of the general solution.

Computing IVP

Problem. Suppose $P(x)$ and $Q(x)$ are continuous on an interval I containing a. Find the unique $y = y(x)$ satisfying the IVP

$$\frac{dx}{dy} + P(x)y = Q(x), \qquad y(a) = b$$

Let

$$\begin{aligned} \rho(x) &= \exp\Big(\int_a^x P(t)\,dt\Big) \\ y(x) &= \frac{1}{\rho(x)}\Big[b + \int_a^x \rho(t)Q(t)\,dt\Big]. \end{aligned}$$

This choice of ρ and y guarantees that

(a) $\rho(a) = 1$, so

(b) $y(a) = b$

ConcepTest

Problem. Solve the IVP

$$2x\frac{dy}{dx} = y + 2x\cos x, \qquad y(1) = 0.$$

Answer.We can put the equation in normal form by dividing by $2x$ (☠)

$$\frac{dy}{dx} - \frac{1}{2x}y = \cos x, \qquad y(1) = 0.$$

We will assume that x is positive (which is consistent with $y(1) = 0$.) Now, $P(x) = -\frac{1}{2x}$ and $Q(x) = \cos x$, so we can apply the IVP method for linear equations to get a unique solution.

$$y(x) = x^{\frac{1}{2}}\Big[\int_1^x x^{-\frac{1}{2}}\cos x\,dx\Big]$$

The integral on the right-hand side cannot be simplified. (Compare to Example 3 of Section 1.5.)