Math 216
Differential Equations

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

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Example 1

Find a particular solution for

$$y'' + 4y = \sin^2 x.$$

- The obvious choice is the method of variation of parameters.
- We can use the trig identities to re-write the right-side to one we can use the method of undetermined coefficients.

Sum of two angles law

$$\begin{aligned} \cos 2x &= 1 - 2\sin^2 x = 2\cos^2 x - 1 \\ \sin 2x &= 2\cos x \sin x \end{aligned}$$

Use Euler's formula.

$$\begin{aligned} e^{2ix} &= \cos 2x + i \sin 2x \\ \left(e^{ix}\right)^2 &= \left(\cos x + i \sin x\right)^2 \\ &= \left(\cos^2 x - \sin^2 x\right) + i2\cos x \sin x. \end{aligned}$$

Equate coefficients

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \\ \sin 2x &= 2\cos x \sin x. \end{aligned}$$

Example 1 continued

Find a particular solution for

$$y'' + 4y = \sin^2 x.$$

Re-write the right-side

$$y'' + 4y = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

A particular solution for

$$y'' + 4y = \frac{1}{2}.$$

is $\frac{1}{8}$.

Example 1 continued

Find a particular solution for

$$y'' + 4y = -\frac{1}{2}\cos 2x.$$

There complementary solution is

$$y_c = c_1 \cos 2x + c_2 \sin 2x.$$

Guess.

$$y_p = x(A\cos 2x + B\sin 2x).$$

Substitute.

$$y_p'' + 4y_p = 4B\cos 2x - 4A\sin 2x.$$

Match to $-\frac{1}{2}\cos 2x$. So, $B = -2$ and $A = 0$. A particular solution is

$$y_p = -2x\sin 2x.$$

Example 1 continued

A particular solution for

$$y'' + 4y = \sin^2 x.$$

is

$$y_p = \frac{1}{8} - 2x\sin 2x.$$

Example 2

Find a particular solution $y = y(\theta)$ for

$$y'' + 16y = \sec(4\theta).$$

Step 1. Two linearly independent solutions to the homogeneous equation are $y_1 = \cos 4\theta$ and $y_2 = \sin 4\theta$.

Step 2. Solve the system of equations

$$\begin{aligned} 0 &= u_1'\cos 4\theta + u_2'\sin 4\theta \\ \sec 4\theta &= -4u_1'\sin 4\theta + 4u_2'\cos 4\theta \end{aligned}$$

From the first equation

$$u_1' = -u_2'\frac{\sin 4\theta}{\cos 4\theta} = -u_2'\tan 4\theta$$

Example 2 continued

Step 2, continued. Solve for u_2' in the second equation

$$\begin{aligned} \sec 4\theta &= -4u_1'\sin 4\theta + 4u_2'\cos 4\theta \\ \sec 4\theta &= 4u_2'\tan 4\theta\sin 4\theta + 4u_2'\cos 4\theta \\ &= 4u_2'\left(\frac{\sin^2 4\theta + \cos^2 4\theta}{\cos 4\theta}\right) \\ &= 4u_2'\frac{1}{\cos 4\theta} \\ &= 4u_2'\sec 4\theta. \end{aligned}$$

So, $u_2' = \frac{1}{4}$.

Since $u_1' = -u_2'\tan 4\theta$:

$$u_1' = -\frac{\tan(4\theta)}{4}.$$

Example 2 continued

Step 2, continued. Integrate $u_2' = \frac{1}{4}$ and $u_1' = -\frac{\tan 4\theta}{4}$

$$u_1 = \frac{\ln|\cos 4\theta|}{16}, \quad u_2 = \frac{\theta}{4}.$$

Note that (use integration by substitution, $v = \cos 4\theta$)

$$\int \tan 4\theta \, d\theta = \int \frac{\sin 4\theta}{\cos 4\theta} \, d\theta = -\frac{1}{4} \ln|\cos 4\theta|$$

Step 3. A particular solution is

$$y_p = u_1 y_1 + u_2 y_2 = \frac{\cos 4\theta \ln|\cos 4\theta|}{16} + \frac{\theta \sin 4\theta}{4}.$$

A general solution is

$$y(x) = c_1 \cos(4\theta) + c_2 \sin(4\theta) + \frac{\cos(4\theta) \ln|\cos(4\theta)|}{16} + \frac{\theta \sin(4\theta)}{4}$$

Richardson Model

The English L.F. Richardson proposed a simple model of an arms race between two countries.

- $x(t)$, $y(t)$ is the expenditure per year of each country.
- Richardson assumed that the rate of change of expenditures was the sum of three factors:
  - a Expenditure for a country increases at a rate proportional to the other's expenditure.
  - b Expenditure for a country decreases at a rate proportional to its own expenditure.
  - c The rate of expenditure depends on a (constant) level of antagonism for the other country.

Richardson Model

The Richardson model for two countries with expenditures $x(t)$ and $y(t)$.

$$\begin{aligned} x' &= p_1 y - q_1 x + a_1, \quad x(0) = x_0 \\ y' &= p_2 x - q_2 y + a_2, \quad y(0) = y_0 \end{aligned}$$

where $p_1, p_2, q_1, q_2 > 0$ and $x_0, y_0 > 0$ are the initial expenditures.

The values a_1 and a_2 measure the antagonism of one country over another:

- $a > 0$: Country is antagonistic towards the other.
- $a = 0$: Country is neutral towards the other.
- $a < 0$: Country is trusting towards the other.

Richardson Model Outcomes

The Richardson model for countries with expenditures $x(t)$ and $y(t)$.

$$\begin{aligned} x' &= p_1 y - q_1 x + a_1, \quad x(0) = x_0 \\ y' &= p_2 x - q_2 y + a_2, \quad y(0) = y_0. \end{aligned}$$

There are three basic outcomes in model

- Disarmament : x and y go to zero as $t \to \infty$.
- Stable Arms Race : x and y approach a constant value as $t \to \infty$.
- Runaway Arms Race : x and y go to infinity as $t \to \infty$.

Example 1

Solve the Richardson model (both countries are neutral towards each other.)

$$\begin{aligned} x' &= 2y - x, \quad x(0) = 1 \\ y' &= 4x - 3y, \quad y(0) = 6 \end{aligned}$$

Eliminate y (using first equation)

$$\begin{aligned} y &= \frac{1}{2}x' + \frac{1}{2}x \\ y' &= \frac{1}{2}x'' + \frac{1}{2}x' \end{aligned}$$

Substitute for y (in second equation)

$$x'' + 4x' - 5x = 0$$

Solve for x

$$x = Ae^{-5t} + Be^{t}$$

Solve for y

$$y = -4Ae^{-5t} + Be^{t}$$

Example 1 continued

Solve the Richardson model

$$\begin{aligned} x' &= 2y - x, \quad x(0) = 1 \\ y' &= 4x - 3y, \quad y(0) = 6 \end{aligned}$$

where the expenditures are

$$\begin{aligned} x &= Ae^{-5t} + Be^{t} \\ y &= -4Ae^{-5t} + Be^{t} \end{aligned}$$

Solve for A and B.

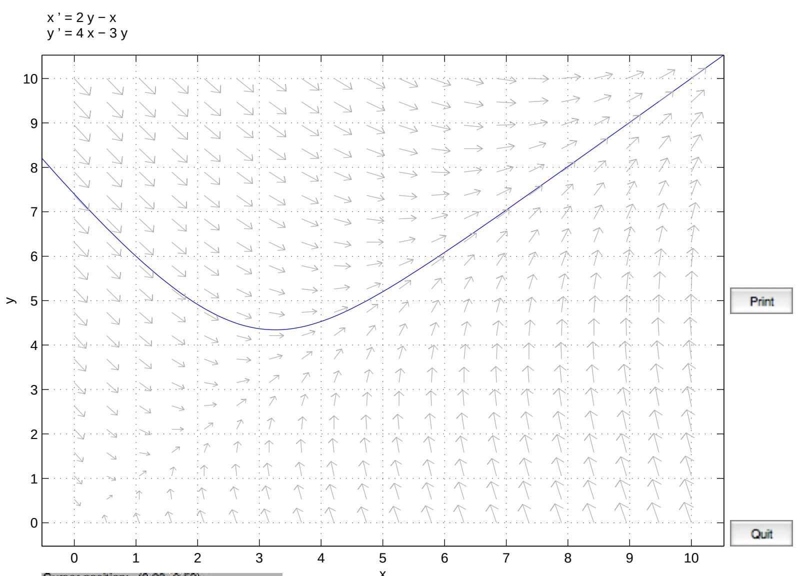
$$\begin{aligned} x(0) &= 1 = A + B \\ y(0) &= 6 = -4A + B \end{aligned}$$

So, $A = -1$ and $B = 2$ and

$$\begin{aligned} x &= 2e^{t} - e^{-5t} \\ y &= 2e^{t} + 4Ae^{-5t} \end{aligned}$$

A runaway arms race.

Phase Plane Portrait



Example 2

Solve the Richardson model

$$\begin{aligned} x' &= 3y - 4x + 6, \quad x(0) = 0 \\ y' &= x - 2y + 1, \quad y(0) = 0 \end{aligned}$$

Eliminate x (using second equation)

$$\begin{aligned} x &= y' + 2y - 1 \\ x' &= y'' + 2y' \end{aligned}$$

Substitute for x (in first equation)

$$y'' + 6y' + 5y = 10$$

Solve for y (note that $y_p = 2$)

$$y = Ae^{-5t} + Be^{-t} + 2$$

Solve for x

$$x = -3Ae^{-5t} + Be^{t} + 3$$

Example 2 continued

Solve the Richardson model

$$\begin{aligned} x' &= 3y - 4x + 6, \quad x(0) = 0 \\ y' &= x - 2y + 1, \quad y(0) = 0 \end{aligned}$$

where the expenditures are

$$\begin{aligned} x &= -3Ae^{-5t} + Be^{t} + 3 \\ y &= Ae^{-5t} + Be^{-t} + 2 \end{aligned}$$

Solve for A and B.

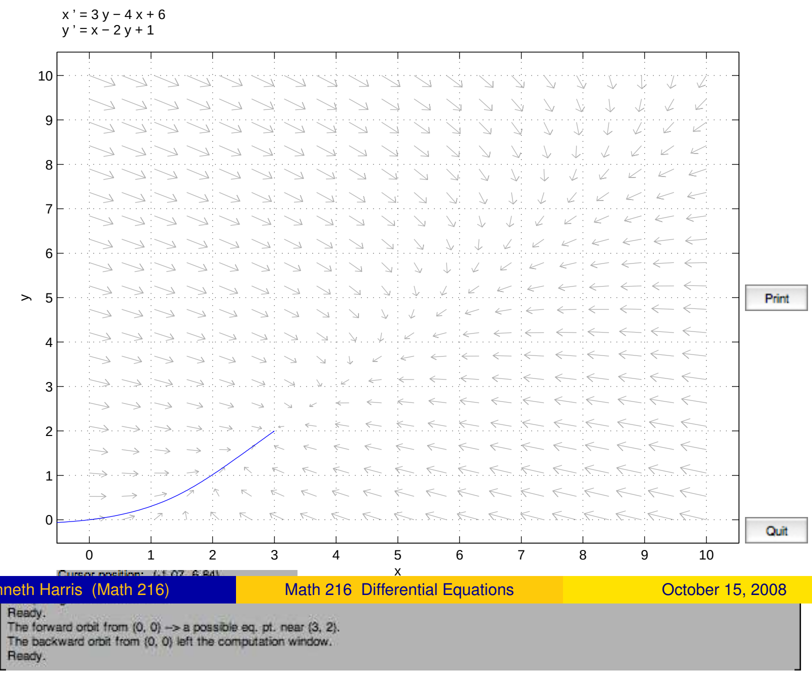
$$\begin{aligned} x(0) &= 0 = -3A + B + 3 \\ y(0) &= 0 = A + B + 2 \end{aligned}$$

So, $A = \frac{1}{4}$ and $B = -\frac{9}{4}$ and

$$\begin{aligned} x &= -\frac{3}{4}e^{-5t} - \frac{9}{4}e^{-t} + 3 \\ y &= \frac{1}{4}e^{-5t} - \frac{9}{4}e^{-t} + 2 \end{aligned}$$

A stable arms race with $x(t) \to 3$ and $y(t) \to 2$.

Phase Plane Portrait



Method

To solve a system of linear equations in unknowns $x(t), y(t)$

$$\begin{aligned} x' &= a_{11}x + a_{12}y + f_1(t) \\ y' &= a_{21}x + a_{22}y + f_2(t) \end{aligned}$$

where the a_{ij} are constants.

Step 1. Solve for y in the first equation

$$y = \frac{1}{a_{12}}x' - \frac{a_{11}}{a_{12}}x + \frac{f_1(t)}{a_{12}}.$$

Step 2. Eliminate y in second equation by substiuting for y and y'.

Step 3. Solve the resulting second-order ODE for x, a solution with two parameters.

Step 4 Substitute the solution for x into the equation for y in Step 1.

Example 3

Solve the linear system of equations for unknowns $x(t), y(t)$.

$$\begin{aligned} x' &= 4x + y + 2t \\ y' &= -2x + y \end{aligned}$$

Step 1. Solve for y (using first equation)

$$\begin{aligned} y &= x' - 4x - 2t \\ y' &= x'' - 4x' - 2 \end{aligned}$$

Step 2. Eliminate y (in second equation)

$$x'' - 5x' + 6x = 2 - 2t$$

Step 3. Solve for x

$$x_c = Ae^{3t} + Be^{2t} \qquad x_p = \frac{1}{18} - \frac{1}{3}t$$

Example 3 continued

Step 4. Solve for y given x:

$$y = x' - 4x - 2t.$$

Compute

$$\begin{aligned} x &= Ae^{3t} + Be^{2t} + \frac{1}{18} - \frac{1}{3}t \\ x' &= 3Ae^{3t} + 2Be^{2t} - \frac{1}{3} \end{aligned}$$

and substitute:

$$y = -Ae^{3t} - 2Be^{2t} - \frac{2}{3}t - \frac{5}{9}$$

The solution to the linear system is

$$\begin{aligned} x &= Ae^{3t} + Be^{2t} + \frac{1}{18} - \frac{1}{3}t \\ y &= -Ae^{3t} - 2Be^{2t} - \frac{2}{3}t - \frac{5}{9} \end{aligned}$$

Generalized Arms Race

- The expenditures $x(t), y(t)$ for our two countries are coupled in the system of equations

$$\begin{aligned} x' &= p_1 y - q_1 x + a_1, \quad x(0) = x_0 \\ y' &= p_2 x - q_2 y + a_2, \quad y(0) = y_0. \end{aligned}$$

- There is no reason to stop with two countries. We could generalize to n countries, whose rate of expenditure on arms is proportional to the rate of other countries.
- RIchardson modeled the expenditure of several European nations leading up to World War I and concluded that a war had to occur because the expenditures were leading to a runaway arms race.

Linear systems of equations

A linear first-order system of equations with constant coefficients in unknowns $x_1(t), x_2(t), \ldots, x_n(t)$ is n equations of the form

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + f_1(t) \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + f_2(t) \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + f_n(t) \end{aligned}$$

where each a_{ij} is a constant and each f_i is continuous on some common interval I.

- The system is homogeneous if $f_1, \ldots, f_n$ are all identically zero; otherwise it is nonhomogeneous.
- A solution of the system are n functions $x_1(t), \ldots, x_n(t)$ that identically satisfy each of the equations on I.

Examples

Example 1 was a homogeneous linear system of equations.

$$\begin{aligned} x' &= -x + 2y \\ y' &= 4x - 3y \end{aligned}$$

Example 2 was a nonhomogeneous linear system of equations.

$$\begin{aligned} x' &= -4x + 3y + 6 \\ y' &= x - 2y + 1 \end{aligned}$$

Example 3 is a nonhomogeneous linear system of equations

$$\begin{aligned} x' &= 4x + y + 2t \\ y' &= -2x + y \end{aligned}$$

Existence and Uniqueness Theorem

Theorem

Given a linear system of equations

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + f_1(t) \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + f_2(t) \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + f_n(t) \end{aligned}$$

where the functions $f_1, \ldots, f_n$ are continuous on an open interval I containing the point a.

Given n constants $b_1, b_2, \ldots, b_n$, the system of equations has a unique solution on the entire interval I satisfying the n initial conditions

$$x_1(a) = b_1 \quad x_2(a) = b_2 \quad \ldots \quad x_n(a) = b_n.$$