Math 216
Differential Equations

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ConcepTest

Find the general form for the complementary and particular solution for

$$y'' + y = 4\cos x$$

Answer.

$$\begin{aligned} y_c &= C_1 \cos x + C_2 \sin x \\ y_p &= x\big(A\cos x + B\sin x\big). \end{aligned}$$

The form of a general solution is $y(x) = y_c(x) + y_p(x)$.

Particular solution

Find the coefficients of the particular solution for

$$y'' + y = 4\cos x$$

Guess. $y_p = Ax\cos x + Bx\sin x$.

Substitute.

$$y_p'' + y_p = 2B\cos x - 2A\sin x.$$

Match.

$$4\cos x = 2B\cos x - 2A\sin x.$$

So, $B = 1$, $A = 0$.

The particular solution is

$$y_p = x\sin x.$$

Initial Value Problem

Solve.

$$y'' + y = 4\cos x \quad y(0) = \frac{1}{2}, y'(0) = -\frac{1}{2}$$

The general solution is $y(x) = y_c(x) + y_p(x)$

$$y = C_1\cos x + C_2\sin x + x\sin x.$$

Substitute

$$\begin{aligned} y(0) &= \frac{1}{2} = C_1 \\ y'(0) &= -\frac{1}{2} = C_2 \end{aligned}$$

The solution is

$$y = \frac{1}{2}\cos x - \frac{1}{2}\sin x + x\sin x.$$

Method of variation of parameters

To determine a particular solution $y = y(x)$ to $ay'' + by' + cy = f(x)$:

1. Find two linearly independent solutions to the associated homogeneous equation.
2. Guess
$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$
3. Solve for $u_1'(x)$ and $u_2'(x)$ in the equations
$$\begin{aligned} 0 &= u_1'(x)y_1(x) + u_2'(x)y_2(x) \\ \frac{f(x)}{a} &= u_1'(x)y_1'(x) + u_2'(x)y_2'(x) \end{aligned}$$
4. Integrate $u_1 = \int u_1'\,dx$ and $u_2 = \int u_2'\,dx$

Example 1

Find a particular solution for
$$y'' - 2y' + y = x^{-1}e^x.$$

Step 1. Linearly independent complementary solutions $y_1 = xe^x$ and $y_2 = e^x$.

Step 2. Guess $y_p(x) = u_1(x)xe^x + u_2(x)e^x$.

Step 3. Solve the system of equations
$$\begin{aligned} 0 &= u_1'(xe^x) + u_2'(e^x) \\ x^{-1}e^x &= u_1'(xe^x + e^x) + u_2'(e^x) \end{aligned}$$

So, $u_1' = x^{-1}$ and $u_2' = -1$.

Step 4. Integrate
$$u_1 = \ln|x|, \quad u_2 = -x.$$

A particular solution is
$$y_p = u_1y_1 + u_2y_2 = x\ln|x|e^x - xe^x$$

Resonance

- Consider a mass-spring oscillator with no damping, but an external driving force syncronized to the natural frequency:
$$mx'' + kx = F\cos(\omega t) \quad x(0) = 0, x'(0) = 0$$
where $\omega = \sqrt{\frac{k}{m}}$.
- The complementary solution is
$$c_1\cos\omega t + c_2\sin\omega t.$$
- Guess the particular solution:
$$y_p(t) = t(A\cos\omega t + B\sin\omega t)$$

Resonance

Substitute. $y_p(t) = At\cos\omega t + Bt\sin\omega t$
$$my_p'' + ky_p = 2Bm\omega\cos\omega t - 2Am\omega\sin\omega t.$$

Match
$$F\cos\omega t = 2Bm\omega\cos\omega t - 2Am\omega\sin\omega t.$$

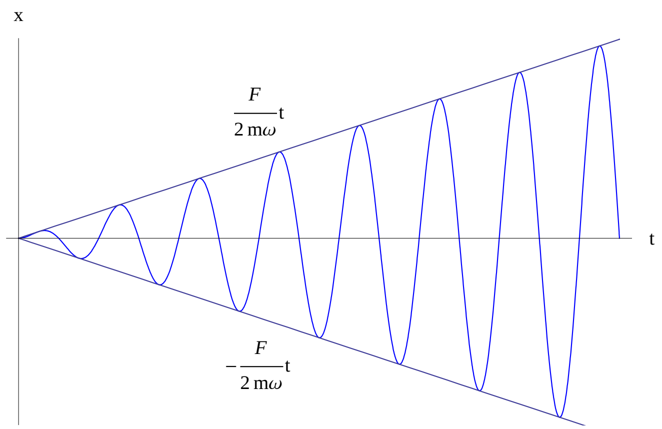
so, $A = 0$ and $B = \frac{F}{2\omega m}$.

The particular solution is
$$y_p(t) = \frac{F}{2\omega m}t\sin\omega t.$$

Note that $y_p \to \infty$ as $t \to \infty$!!

Example of resonance

$y_p = \frac{F}{2\omega m} t \sin(\omega t)$.



Dampening

The general model of a mechanical oscillator is of the form

$$mx'' + cx' + kx = F\cos\gamma t$$

where

- m is the mass of the object,
- c is the damping constant (often due to friction),
- k is the spring elasticity,
- $F\cos\gamma t$ is an external driving force,
- $x(t)$ is the displacement from a fixed point.

Solution

- The solution x_c to the associated homogeneous equation

  $$mx'' + cx' + kx = 0 \qquad m, c, k > 0$$

  is called the transient solution, because $x_c(t) \to 0$ as $t \to \infty$.
- The particular solution

  $$x_p = A\cos\gamma t + B\sin\gamma t$$

  is the steady periodic solution.
- x_p is a periodic motion which quickly becomes dominant.

Transient solution

The associated homogeneous equation is

$$mx'' + cx' + kx = 0.$$

whose characteristic equation is

$$mr^2 + cr + k = 0 \quad \text{where } m, c, k > 0,$$

and with roots

$$r = \frac{-c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$$

1. $c^2 - 4mk > 0$ is overdamping
2. $c^2 - 4mk = 0$ is critical damping
3. $c^2 - 4mk < 0$ is damping

Transient solution

1. $c^2 - 4mk > 0$. Two distinct negative roots:

$$r_1 = \frac{-c}{2m} + \frac{1}{2m}\sqrt{c^2 - 4mk} \quad r_2 = \frac{-c}{2m} - \frac{1}{2m}\sqrt{c^2 - 4mk}$$

and solution

$$x_c(t) = Ae^{r_1 t} + Be^{r_2 t}.$$

2. $c^2 - 4mk = 0$. One real root, $r = \frac{-c}{2m}$, and solution

$$x_c(t) = (A + Bt)e^{\frac{-c}{2m}t}.$$

3. $c^2 - 4mk < 0$. Complex conjugate roots: $\rho \pm \omega i$ where

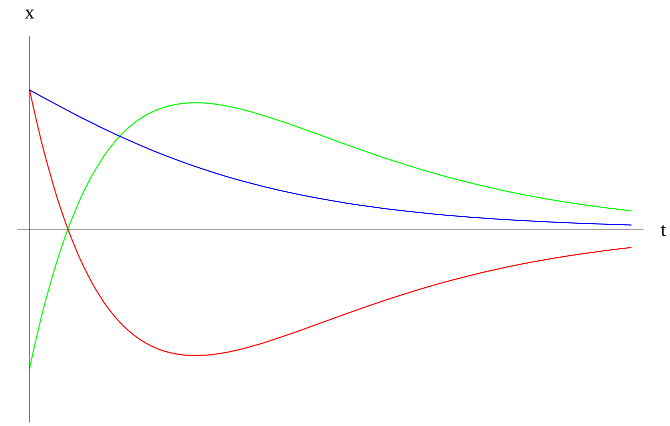
$$\rho = -\frac{c}{2m} \quad \omega = \frac{1}{2m}\sqrt{4mk - c^2}$$

so, we can express the solution as

$$x_c(t) = Ae^{\rho t}\cos(\omega t - \alpha).$$

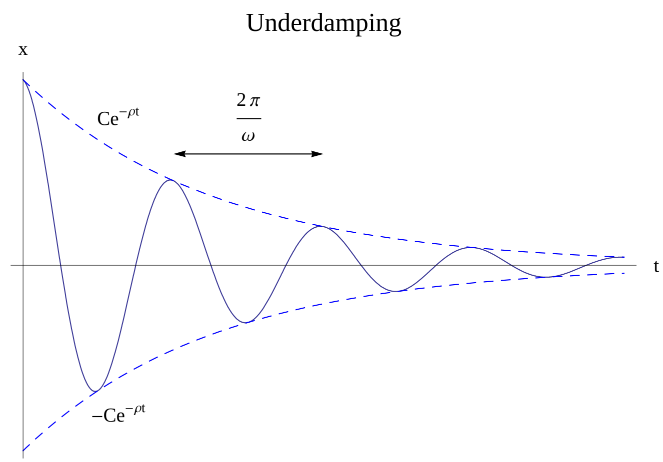
Examples of critically damped motion

The possible shapes of overdamped and critically damped motion.



Example of underdamped motion

The possible shape of underdamped motion.



Particular equation: coefficients

Compute the coefficients of the particular solution:

$$x_p = A\cos\gamma t + B\sin\gamma t$$

Substitute.

$$mx_p'' + cx_p' + kx_p = \left(Ak + Bc\gamma - Am\gamma^2\right)\cos\gamma t + \left(Bk - Ac - Bm\gamma^2\right)\sin\gamma t$$

Match to $F\cos\gamma t$.

$$\begin{aligned} F &= A(k - m\gamma^2) + Bc\gamma \\ 0 &= -Ac\gamma + (k - m\gamma^2)B \end{aligned}$$

Solution.

$$A = \frac{F(k - m\gamma^2)}{(k - m\gamma^2)^2 + c^2\gamma^2} \quad B = \frac{Fc\gamma}{(k - m\gamma^2)^2 + c^2\gamma^2}$$

Convert: amplitude

Convert. $A\cos\gamma t + B\sin\gamma t = C\cos(\gamma t - \alpha)$ where

$$A = \frac{F(k - m\gamma^2)}{(k - m\gamma^2)^2 + c^2\gamma^2} \quad B = \frac{Fc\gamma}{(k - m\gamma^2)^2 + c^2\gamma^2}$$

The amplitude is

$$C = \sqrt{A^2 + B^2} = \frac{F}{\sqrt{(k - \gamma^2)^2 + c^2\gamma^2}}$$

Convert: angle

Convert. $A\cos\gamma t + B\sin\gamma t = C\cos(\gamma t - \alpha)$ where

$$A = \frac{F(k - m\gamma^2)}{(k - m\gamma^2)^2 + c^2\gamma^2} \quad B = \frac{Fc\gamma}{(k - m\gamma^2)^2 + c^2\gamma^2}$$

Note that $B > 0$ and $k - m\gamma^2 > 0$ determines the sign of A.

$$\frac{B}{A} = \frac{c\gamma}{k - m\gamma^2}$$

so,

$$\alpha = \begin{cases} \arctan\frac{c\gamma}{k - m\gamma^2} & \text{if } k - m\gamma^2 > 0 \\ \pi + \arctan\frac{c\gamma}{k - m\gamma^2} & \text{if } k - m\gamma^2 < 0 \\ \frac{\pi}{2} & \text{if } k - m\gamma^2 = 0 \end{cases}$$

Steady state solution

We can express the steady periodic solution in an alternative form

$$x_p(t) = \frac{F}{\sqrt{(k - \gamma^2)^2 + c^2\gamma^2}}\cos(\gamma t - \alpha)$$

where $\tan\alpha = \frac{c\gamma}{k - m\gamma^2}$.

Compare to $F\cos\gamma t$.

- y_p has the same angular frequency as driving force.
- y_p is out of phase by α.
- y_p has different magnitude by a factor of

$$\frac{1}{\sqrt{(k - \gamma^2)^2 + c^2\gamma^2}}$$

called the gain factor.

Graph

Solution to $mx'' + cx' + kx = F\cos\gamma t$ with steady periodic solution and combined solution. (Transient solution is underdamped.)