Math 216
Differential Equations

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Example

Problem. Find a particular solution to the problem

$$y'' - y = 8xe^x + 2e^x.$$

We could add particular solutions to the two nonhomogeneous equations:

$$y'' - y = 8xe^x \qquad y'' - y = 2e^x$$

Simplify. $r = \pm 1$ is a solution to the associated quadratic $r^2 - 1 = 0$, so we need to match the right-side as follows:

- $8xe^x$ is matched to the form $x(Ax + B)e^x$,
- $2e^x$ is matched to the form Cxe^x.

These two cases are redundant. We can simply match $8xe^x + 2e^x$ to the single form $(Ax^2 + Bx)e^x$.

Example

Problem. Find a particular solution to the problem

$$y'' - y = 8xe^x + 2e^x.$$

Guess. $y_p(x) = (Ax^2 + Bx)e^x$

Substitute.

$$\begin{aligned} y_p' &= (Ax^2 + Bx)e^x + (2Ax + B)e^x = (Ax^2 + (2A + B)x + B)e^x \\ y_p'' &= (2Ax + (2A + B))e^x + (Ax^2 + (2A + B)x + B)e^x \\ &= (Ax^2 + (4A + B)x + (2A + 2B))e^x \\ y_p'' - y_p &= (4Ax + (2A + 2B))e^x \end{aligned}$$

Match. $(4Ax + (2A + 2B))e^x = (8x + 2)e^x$.

$$\begin{aligned} 8 &= 4A \\ 2 &= 2A + 2B, \end{aligned}$$

Solve. $A = 2$ and $B = -1$.

$$y_p(x) = (2x^2 - x)e^x$$

Mechanical oscillator

Problem. A mass m kg is attached to a spring with elasticity k N/m. There is also an external driver acting on the mass which applies a force of $F\cos(\beta t)$ N. Find the displacement $x(t)$ from the equillibrium on the assumption that the mass starts at rest on the equillibrium point.

Equation. Use Newton's second law to set-up the IVP:

$$mx'' = -kx + F\cos(\beta t) \quad x(0) = 0, x'(0) = 0.$$

Equivalently,

$$mx'' + kx = F\cos(\beta t) \quad x(0) = 0, x'(0) = 0.$$

Complementary solution

The associated homogeneous equation is

$$mx'' + kx = 0,$$

and has characteristic equation $mr^2 + k = 0$ $(m, k > 0)$ with roots

$$r = \pm i\sqrt{\frac{k}{m}}$$

Let $\omega = \sqrt{\frac{k}{m}}$, so the complementary solution to the homogeneous equation is

$$c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

ω is called the natural frequency, and is the angular velocity of the shadow mass revolving around the circle as the mass oscillates between maximum stretch and maximum compression.

Particular solutions

We find a particular solution to the nonhomogeneous equation

$$mx'' + kx = F\cos(\beta t)$$

by guessing the solution has the form

$$y_p(t) = A\cos(\beta t) + B\sin(\beta t).$$

1. Notice that the right-side involves only $\cos(\beta t)$ and the left-side consists only of x'' and x, so we do not really need to consider $\sin(\beta t)$ in the particular solution. We might as well take only $y_p(t) = A\cos(\beta t)$.
2. If $\beta = \omega$ (the frequency of the driving force is in sync with the natural frequency), this guess will not be adequate.
3. We will assume $\beta \neq \omega$, for now.

Particular solutions

Guess a particular solution $y_p(t) = A\cos(\beta t)$ to the equation

$$mx'' + kx = F\cos(\beta t).$$

Substitute.

$$my_p'' + ky_p = -m\beta^2 A\cos(\beta t) + kA\cos(\beta t) = \big(-m\beta^2 A + kA\big)\cos(\beta t)$$

Match. $\big(-m\beta^2 A + kA\big)\cos(\beta t) = F\cos(\beta t)$.
Solve. $A = \frac{F}{k - m\beta^2}$. A particular solution is

$$\frac{F}{k - m\beta^2}\cos(\beta t)$$

We re-write this using the natural frequency $\omega^2 = \frac{k}{m}$,

$$\frac{F}{m(\omega^2 - \beta^2)}\cos(\beta t).$$

General solution

The general solution is

$$y_c + y_p = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{F}{m(\omega^2 - \beta^2)}\cos(\beta t).$$

Solve for the initial values $x(0) = 0, x'(0) = 0$,

$$\begin{aligned} 0 &= c_1 + \frac{F}{m(\omega^2 - \beta^2)} \\ 0 = &= -c_1\sin(0) - c_2\cos(0) - \frac{F}{m(\omega^2 - \beta^2)}\sin(0) = -c_2 \end{aligned}$$

so, $c_1 = -\frac{F}{m(\omega^2 - \beta^2)}$ and $c_2 = 0$.

The solution is

$$\frac{F}{m(\omega^2 - \beta^2)}\big(\cos(\omega t) - \cos(\beta t)\big).$$

Summary

The nonhomogeneous IVP (where $\omega = \sqrt{\frac{k}{m}} \neq \beta$)

$$mx'' + kx = F\cos(\beta t) \quad x(0) = 0, x'(0) = 0$$

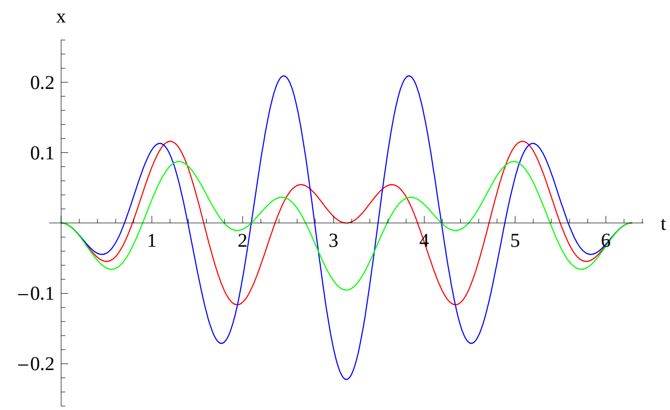
has the solution

$$\frac{F}{m(\omega^2 - \beta^2)}\big(\cos(\omega t) - \cos(\beta t)\big).$$

- The smaller the difference $|\omega - \beta|$, the greater the amplitude of displacement.
- The mass is oscillating with a period which is the least common multiple of 2π of the frequencies $\frac{2\pi}{\omega}$ and $\frac{2\pi}{\beta}$.

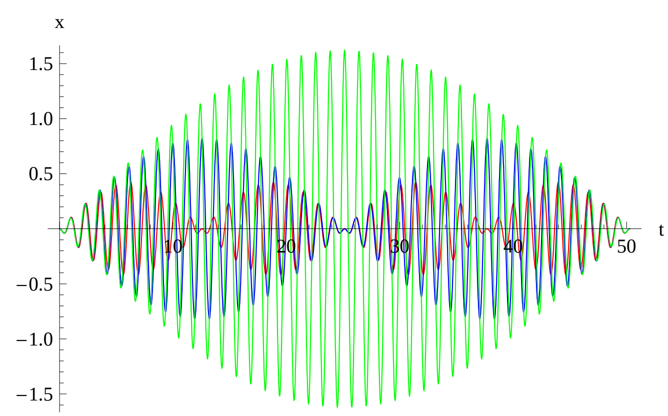
Simple examples

$F = 1, m = 1, \omega = 5$. Driving frequency: $\beta = 2$, $\beta = 3$, $\beta = 4$.
The period is 2π in each case.



Simple examples

$F = 1, m = 1, \omega = 5$. Driving frequency: $\beta = 4.5$, $\beta = 4.75$, $\beta = 4.875$.
The periods are $\beta = 8\pi$, $\beta = 16\pi$, $\beta = 32\pi$.



Variation of parameters

- The method of variation of parameters allows you to find a particular solution to any nonhomogeneous equation

$$ay'' + by' + cy = f(x)$$

  provided you have two linearly independent solutions $y_1(x), y_2(x)$ for the associated homogeneous equation

$$ay'' + by' + cy = 0$$

- The complementary function is

$$y_c(x) = c_1 y_2(x) + c_2 y_2(x)$$

The method

- We will seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

  where we replaced the coefficients c_1 and c_2 of the complementary function with functions $u_1(x)$ and $u_2(x)$.
- We will find $u_1(x)$ and $u_2(x)$ by imposing two requirements on these functions, in the form of equations. (Just as we would in solving two unknowns in two equations.)
- We will choose the equations
  1. to simplify finding u_1 and u_2, and
  2. to guarantee that the u_1 and u_2 we find really give us a particular solution to the nonhomogeneous equation.

Computing derivatives: first equation

- Start with what we need: y_p is a solution to the nonhomogeneous equation:

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ y_p' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \\ &= (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2') \end{aligned}$$

- Make solving the equation easier and eliminate having to compute second derivatives of u_1 and u_2:

$$u_1'y_1 + u_2'y_2 = 0.$$

- Continue computing:

$$\begin{aligned} y_p' &= u_1y_1' + u_2y_2' \\ y_p'' &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \end{aligned}$$

Computing solution: second equation

- Substitute

$$\begin{aligned} f &= ay_p'' + by_p' + cy_p \\ &= a(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') \\ &\quad + b(u_1y_1' + u_2y_2') + c(u_1y_1 + u_2y_2) \\ &= a(u_1'y_1' + u_2'y_2') + (ay_1'' + by_1' + cy_1)u_1 \\ &\quad + (ay_2'' + by_2' + cy_2)u_2 \\ &= a(u_1'y_1' + u_2'y_2') + 0 + 0 \end{aligned}$$

- Our second requirement guarantees that the functions u_1 and u_2 really help us find a particular solution:

$$\frac{f}{a} = u_1'y_1' + u_2'y_2'$$

Solving for $u_1'(x)$

- We have two equations and two unknowns ($u_1 = u_1(x), u_2 = u_2(x)$)

$$\begin{aligned} 0 &= u_1'y_1 + u_2'y_2 \\ \frac{f}{a} &= u_1'y_1' + u_2'y_2' \end{aligned}$$

- Solve for $u_1'(x)$. Multiply first equation by y_2' and second equation by y_2:

$$\begin{aligned} 0 &= u_1'(x)y_1(x)y_2'(x) + u_2'(x)y_2(x)y_2'(x) \\ y_2(x)\frac{f(x)}{a} &= u_1'(x)y_1'(x)y_2(x) + u_2'(x)y_2'(x)y_2(x) \end{aligned}$$

  Subtract the second from the first to eliminate $u_2'(x)$

$$-y_2(x)\frac{f(x)}{a} = u_1'(x)y_1(x)y_2'(x) - u_1'(x)y_1'(x)y_2(x)$$

- Solve for $u_1'(x)$

$$u_1'(x) = \frac{-y_2(x)f(x)}{a(y_1(x)y_2'(x) - y_1'(x)y_2(x))}$$

Solving for $u_2'(x)$

- We have two equations and two unknowns $(u_1(x), u_2(x))$.

$$\begin{aligned} 0 &= u_1'(x)y_1(x) + u_2'(x)y_2(x) \\ \frac{f(x)}{a} &= u_1'(x)y_1'(x) + u_2'(x)y_2'(x) \end{aligned}$$

- Solve for $u_2'(x)$. Multiply first equation by y_1' and second equation by y_1:

$$\begin{aligned} 0 &= u_1'(x)y_1(x)y_1'(x) + u_2'(x)y_2(x)y_1'(x) \\ y_1(x)\frac{f(x)}{a} &= u_1'(x)y_1'(x)y_1(x) + u_2'(x)y_2'(x)y_1(x) \end{aligned}$$

  Subtract the first from the second to eliminate $u_1'(x)$

$$y_1(x)\frac{f(x)}{a} = u_2'(x)y_2'(x)y_1(x) - u_2'(x)y_2(x)y_1'(x)$$

- Solve for $u_2'(x)$

$$u_2'(x) = \frac{y_1(x)f(x)}{a(y_2'(x)y_1(x) - y_2(x)y_1'(x))}$$

The Wronskian

- Remember the Wronskian

$$W(y_1, y_2)(x) = y_2'(x)y_1(x) - y_2(x)y_1'(x)$$

- The $W(y_1, y_2)(x) \neq 0$ for any x on an interval where y_1 and y_2 are linearly independent.
- The solutions for u_1', u_2':

$$u_1'(x) = \frac{-y_2(x)f(x)}{aW(y_1, y_2)(x))} \qquad u_2'(x) = \frac{y_1(x)f(x)}{aW(y_1, y_2)(x))}$$

Solve for u_1, u_2

- $$u_1'(x) = \frac{-y_2(x)f(x)}{aW(y_1, y_2)(x))} \qquad u_2'(x) = \frac{y_1(x)f(x)}{aW(y_1, y_2)(x))}$$
- Solve for u_1, u_2 by integrating:

$$u_1(x) = \int \frac{-y_2(x)f(x)}{aW(y_1, y_2)(x))}\,dx \qquad u_2(x) = \int \frac{y_1(x)f(x)}{aW(y_1, y_2)(x))}\,dx$$

- The method is guaranteed to produce a particular solution

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Method of variation of parameters

To determine a particular solution to $ay'' + by' + cy = f$:

1. Find two linearly independent solutions to the associated homogeneous equation, and take

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

2. Determine $u_1(x)$ and $u_2(x)$ by solving the equations

$$\begin{aligned} 0 &= u_1'(x)y_1(x) + u_2'(x)y_2(x) \\ \frac{f(x)}{a} &= u_1'(x)y_1'(x) + u_2'(x)y_2'(x) \end{aligned}$$

3. Substitute $u_1(x)$ and $u_2(x)$ into $y_p(x)$ to obtain a particular solution.

Remark. You could memorize the formula for u_1 and u_2, but I strongly recommend that you derive these from the equation in step 2.

Example 1

Find a particular solution for

$$y'' - 2y' + y = x^{-1}e^x.$$

Step 1. Two linearly independent solutions to the homogeneous equation are $y_1 = xe^x$ and $y_2 = e^x$.

Step 2. Solve the system of equations

$$\begin{aligned} 0 &= u_1'(xe^x) + u_2'(e^x) \\ x^{-1}e^x &= u_1'(xe^x + e^x) + u_2'(e^x) \end{aligned}$$

So, $u_2' = -u_1'x$. Solve for u_1':

$$\begin{aligned} 0 &= u_1'(xe^x) + u_2'(e^x) \\ x^{-1}e^x &= u_1'xe^x + u_1'e^x - u_1'xe^x \\ &= u_1'e^x \end{aligned}$$

So, $u_1' = x^{-1}$ and $u_2' = -1$. Integrate

$$u_1 = \ln|x|, \quad u_2 = -x.$$

Example 1 continued

Step 3. A particular solution is

$$y_p = u_1y_1 + u_2y_2 = x\ln|x|e^x - xe^x$$

Since $-xe^x$ is a solution to the homogeneous equation, we simplify

$$y_p = x\ln|x|e^x$$

A general solution is

$$y(x) = c_1xe^x + c_2e^x + x\ln|x|e^x.$$

Example 2

Find a particular solution for

$$y'' + 2y' + y = e^{-x}.$$

Step 1. Two linearly independent solutions to the homogeneous equation are $y_1 = xe^{-x}$ and $y_2 = e^{-x}$.

Step 2. Solve the system of equations

$$\begin{aligned} 0 &= u_1'(xe^x) + u_2'(e^x) \\ e^{-x} &= u_1'(e^{-x} - xe^{-x}) - u_2'(e^{-x}) \end{aligned}$$

So, $u_2' = -u_1'x$. Solve for u_1':

$$\begin{aligned} e^{-x} &= u_1'e^{-x} - u_1'xe^{-x} + u_1'xe^{-x} \\ &= u_1'e^{-x} \end{aligned}$$

So, $u_1' = 1$ and $u_2' = -x$. Integrate

$$u_1 = x, \quad u_2 = -\frac{x^2}{x}.$$

Example 2 continued

Step 3. A particular solution is

$$y_p = u_1y_1 + u_2y_2 = x^2e^{-x} - \frac{x^2}{2}e^{-x} = \frac{x^2}{2}e^{-x}$$

A general solution is

$$y(x) = c_1xe^{-x} + c_2e^{-x} + \frac{x^2}{2}e^{-x}.$$

Example 3

Find a particular solution $y = y(\theta)$ for

$$y'' + 16y = \sec(4\theta).$$

Step 1. Two linearly independent solutions to the homogeneous equation are $y_1 = \cos(4\theta)$ and $y_2 = \sin(4\theta)$.

Step 2. Solve the system of equations

$$\begin{aligned} 0 &= u_1' \cos(4\theta) + u_2' \sin(4\theta) \\ \sec(4\theta) &= -4u_1' \sin(4\theta) + 4u_2' \cos(4\theta) \end{aligned}$$

So, $u_1' = -u_2' \tan(4\theta)$. Solve for u_2':

$$\begin{aligned} \sec(4\theta) &= 4u_2' \tan(4\theta) \sin(4\theta) + 4u_2' \cos(4\theta) \\ &= 4u_2' \left(\frac{\sin^2(4\theta) + \cos^2(4\theta)}{\cos(4\theta)}\right) \\ &= 4u_2' \frac{1}{\cos(4\theta)} = 4u_2' \sec(4\theta). \end{aligned}$$

So, $u_2' = \frac{1}{4}$ and $u_1' = -\frac{\tan(4\theta)}{4}$.

Example 3 continued

Step 2, continued. Integrate $u_2' = \frac{1}{4}$ and $u_1' = -\frac{\tan(4\theta)}{4}$

$$u_1 = \frac{\ln|\cos(4\theta)|}{16}, \quad u_2 = \frac{\theta}{4}.$$

Note that (use integration by substitution, $u = \cos(4\theta)$)

$$\int \tan(4\theta)\, d\theta = \int \frac{\sin(4\theta)}{\cos(4\theta)}\, d\theta = -\frac{1}{4} \ln|\cos(4\theta)|$$

Step 3. A particular solution is

$$y_p = u_1 y_1 + u_2 y_2 = \frac{\cos(4\theta) \ln|\cos(4\theta)|}{16} + \frac{\theta \sin(4\theta)}{4}.$$

A general solution is

$$y(x) = c_1 \cos(4\theta) + c_2 \sin(4\theta) + \frac{\cos(4\theta) \ln|\cos(4\theta)|}{16} + \frac{\theta \sin(4\theta)}{4}$$