Math 216
Differential Equations

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Judicious guessing

Goal. Use "judicious guessing" to derive a procedure for finding a solution to the nonhomogeneous linear equation with constant coefficient:

$$ay'' + by' + cy = f(x)$$

when $f(x)$ is a function of a special type.

Example of a judicious guess

Example. Find a particular solution to the equation

$$y'' + 3y' + 2y = 3x.$$

Guess that y is of the form $Ax + B$ and determine A and B.

Substitute.

$$\begin{aligned} y'' + 3y' + 2y &= (0) + 3(A) + 2(Ax + B) \\ &= 2Ax + (3A + 2B) \end{aligned}$$

Match. $2Ax + (3A + 2B) = 3x$.

$$\begin{aligned} 0 &= 3A + 2B \\ 3 &= 2A \end{aligned}$$

Solve. $A = \frac{3}{2}$ and $B = -\frac{9}{4}$. A particular solution is

$$y_p(x) = \frac{3}{2}x - \frac{9}{4}.$$

Polynomial terms

The previous example suggests a method for finding a particular solution to the equation

$$ay'' + by' + cy = Cx^m \qquad \text{where } m \geq 0$$

1. **Guess** the solution has the form

   $$y_p(x) = A_m x^m + \ldots + A_1 x + A_0$$

   with undetermined coefficients A_i. (We must retain all powers since they will occur as derivatives when we substitute into the equation.)
2. **Substitute** y_p into the differential equation to get a polynomial in x of degree m on the left and right-sides.
3. **Match** coefficients from each power x^i, to get $m + 1$ equations in $m + 1$ unknowns $A_0, \ldots, A_m$.
4. **Solve** for $A_0, \ldots, A_m$.

ConcepTest

Find a particular solution for

$$y'' + 2y' - y = 10x^2$$

Answer. Guess $y_p(x) = Ax^2 + Bx + C$.
Substitute.

$$y_p'' + 2y_p' - y_p = (2A) + (4Ax + 2B) - (Ax^2 + Bx + C)$$

Match. $10x^2 = -Ax^2 + (4A - B)x + (2A + 2B - C)$

$$\begin{aligned} 10 &= -A \\ 0 &= 4A - B \\ 0 &= 2A + 2B - C \end{aligned}$$

Solve. $A = -10$, $B = -40$, $C = -100$. A particular solution is

$$y_p(x) = -10x^2 - 40x - 100$$

Example

Example. Find a particular solution for

$$y'' + 3y' + 2y = 10e^{3x}$$

Guess $y_p = Ae^{3x}$ (differentiating e^{3x} does not change the exponential form).

Substitute.

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= 9Ae^{3x} + 3(3Ae^{3x}) + 2Ae^{3x} \\ &= 20Ae^{3x} \end{aligned}$$

Solve. $10e^{3x} = 20Ae^{3x}$, so $A = \frac{1}{2}$.

A particular solution is

$$y_p(x) = \frac{e^{3x}}{2}.$$

ConcepTest

Find a particular solution for

$$2x'' + x = 9e^{2t}$$

Answer.

Guess $x_p(t) = Ae^{2t}$.
Substitute.

$$\begin{aligned} 2x_p'' + x_p &= 2(4Ae^{2t}) + Ae^{2t} \\ &= 9Ae^{2t} \end{aligned}$$

Solve. $9Ae^{2t} = 9e^{2t}$. So, $A = 1$. A particular solution is

$$x_p(t) = e^{2t}$$

Example

Example. Find a particular solution for

$$y'' + 3y' + 2y = \sin x$$

Guess $y_p = A \sin x + B \cos x$ (differentiating sines and cosines yields sines and cosines.)

Substitute.

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= -(A \sin x + B \cos x) + 3(A \cos x - B \sin x) \\ &\quad + 2(A \sin x + B \cos x) \\ &= (-A - 3B + 2A) \sin x + (-B + 3A + 2B) \cos x. \end{aligned}$$

Match. $(A - 3B) \sin x + (3A + B) \cos x = \sin x$.

$$\begin{aligned} 1 &= A - 3B \\ 0 &= 3A + B, \end{aligned}$$

Solve. $A = \frac{1}{10}$ and $B = -\frac{3}{10}$. A particular solution is

$$y_p(x) = \frac{1}{10}\left(\sin x - 3 \cos x\right).$$

ConcepTest

Find a particular solution for

$$z'' + 4z = 3\cos(4t)$$

Answer.

Guess $z_p(t) = A\cos(4t) + B\sin(4t)$.
Substitute.

$$\begin{aligned} z_p'' + 4z_p &= \big(-16A\cos(4t) - 16B\sin(4t)\big) + 4\big(A\cos(4t) + B\sin(4t)\big) \\ &= -12A\cos(4t) - 12B\sin(4t) \end{aligned}$$

Match. $-12A\cos(4t) - 12B\sin(4t) = 3\cos(4t)$.
Solve. $A = -\frac{1}{4}, B = 0$.
A particular solution is

$$z_p(x) = -\frac{\cos(4t)}{4}.$$

Example

Example. Find a particular solution for

$$y'' + 4y = 5x^2e^x$$

Guess $y_p = (Ax^2 + Bx + C)e^x$ (includes all derivatives).
Substitute.

$$\begin{aligned} y_p' &= (2Ax + B)e^x + (Ax^2 + Bx + C)e^x \\ y_p'' &= 2Ae^x + 2(2Ax + B)e^x + (Ax^2 + Bx + C)e^x \\ &= Ax^2e^x + (4A + B)xe^x + (2A + 2B + C)e^x \\ y_p'' + 4y_p &= 5Ax^2e^x + (4A + 5B)xe^x(4A + 5B) + (2A + 2B + 5C)e^x. \end{aligned}$$

Match terms x^2e^x, xe^x, e^x:

$$5 = 5A, \quad 0 = 4A + 5B, \quad 0 = 2A + 2B + 5C.$$

Solve. $A = 1$, $B = -\frac{4}{5}$ and $C = -\frac{2}{25}$.

$$y_p(x) = \left(x^2 - \frac{4}{5}x - \frac{2}{25}\right)e^x$$

Problems for the method

Example 1. Find a particular solution for

$$y'' + 5y' = 5$$

Guess $y_p = A$.
Substitute. $y_p'' + y_p' = (A)'' + (A)' = 0$.

Example 2. Find a particular solution for

$$y'' - y = 3e^x$$

Guess $y_p = e^x$.
Substitute. $y_p'' - y_p' = (e^x)'' - (e^x)' = 0$.

The problem is that in both examples the solution we guessed happened to be a solution to the associated homogeneous equation.

Solution to problem

Example 1. Find a particular solution for

$$y'' + 5y' = 5$$

Use the same trick for we used for multiple roots: “boost" the guess by a polynomial factor.

Guess $y_p = Ax$.

Substitute. $y_p'' + 5y_p' = (Ax)'' + 5(Ax)' = 5A$.
Match. $5A = 5$.
Solve. $A = 1$

$$y_p(x) = x.$$

ConcepTest

Example 2. Find a particular solution for

$$y'' - y = 3e^x$$

by guessing $y_p = Axe^x$.

Answer.
Substitute.

$$\begin{aligned} y_p' &= Ae^x + Axe^x \\ y_p'' &= Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x \\ y_p'' - y_p &= 2Ae^x + Axe^x - Axe^x = 2Ae^x. \end{aligned}$$

Match. $2Ae^x = 3e^x$.
Solve. $A = \frac{3}{2}$. A solution is

$$y_p(x) = \frac{3}{2}xe^x.$$

Example

Example 3. Find a particular solution for

$$y'' - y = 4xe^x$$

Guess. $y_p = Axe^x$. (We don't need Be^x, since it is a solution to the homogeneous equation, so falls away.)
Substitute.

$$\begin{aligned} y_p' &= Ae^x + Axe^x \\ y_p'' &= Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x \\ y_p'' - y_p &= 2Ae^x + Axe^x - Axe^x = 2Ae^x. \end{aligned}$$

Match. $2Ae^x = 4xe^x$.
Problem. We lost the xe^x term in our solution because e^x is a solution.

Example

Example 3. Find a particular solution for

$$y'' - y = 4xe^x$$

Guess. $y_p = (Ax^2 + Bx)e^x$. (We do need Bxe^x, since it is not a solution to the homogeneous equation, but does arise when we differentiate Ax^2e^x.)
Substitute.

$$\begin{aligned} y_p' &= Ax^2e^x + 2Axe^x + Be^x + Bxe^x \\ &= Ax^2e^x + (2Ax + Bx)e^x + Be^x \\ y_p'' &= Ax^2e^x + 2Axe^x + (2Ax + Bx)e^x + (2A + B)e^x + Be^x \\ &= Ax^2e^x + (4A + B)xe^x + (2A + 2B)e^x \\ y_p'' - y_p &= 4Axe^x + (2A + 2B)e^x. \end{aligned}$$

Match. $4Axe^x + (2A + 2B)e^x = 4xe^x$.
Solve. $4A = 4$, $2A + 2B = 0$, so $A = 1$ and $B = -1$. A particular solution is

$$y_p(x) = x^2e^x - xe^x$$

Example

Solve. Find a particular solution for

$$y'' - 2y' + y = 4e^t.$$

Note: $r^2 - 2r + 1 = (r-1)^2$.
Guess. $y_p(t) = At^2e^t$. Both e^t and te^t are solutions to the associated homogeneous equation. (We don't need $Bte^t + Ce^t$, which vanish on substitution into the equation.)
Substitute.

$$\begin{aligned} y_p' &= 2Ate^t + At^2e^t \\ y_p'' &= 2Ae^t + 2Ate^t + 2Ate^t + At^2e^t \\ &= 2Ae^t + 4Ate^t + At^2e^t \\ y_p'' - 2y_p' + y_p &= (2Ae^t + 4Ate^t + At^2e^t) - 2(2Ate^t + At^2e^t) + At^2e^t \\ &= 2Ae^t \end{aligned}$$

Match. $2Ae^t = 4e^t$.
Solve. $A = 2$. A particular solution is

$$y_p(x) = 2t^2e^t.$$

Method of Undetermined Coefficients: Case 1

Method. To find a particular solution to the equation $ay'' + by' + cy = Cx^m e^{rx}$, use the guess

$$y_p(x) = t^s(A_m x^m + \ldots + A_1 x + A_0)e^{rt}$$

where

(i) $s = 0$ if r is not a root of the associated homogeneous equation,

(ii) $s = 1$ if r is a simple root of the associated homogeneous equation,

(iii) $s = 2$ if r is a double root of associated homogeneous equation.

Remark. The case of $ay'' + by' + cy = Cx^m$ corresponds to $r = 0$.

Method of Undetermined Coefficients: Case 2

Method. To find a particular solution to the equation $ay'' + by' + cy = R$ where

$$R = P_m(x)e^{\alpha x}\cos(\beta t) \quad \text{or} \quad R = P_m(x)e^{\alpha x}\cos(\beta t),$$

guess

$$\begin{aligned} y_p(x) \quad &= \quad t^s(A_m x^m + \ldots + A_1 x + A_0)e^{\alpha x}\cos(\beta x)) \\ &\qquad + t^s(B_m x^m + \ldots + B_1 x + B_0)e^{\alpha x}\sin(\beta x)) \end{aligned}$$

where

(iv) $s = 0$ if $\alpha + i\beta$ is not a root of the associated homogeneous equation,

(v) $s = 1$ if $\alpha + i\beta$ is a root of the associated homogeneous equation.

Remark. The cases of $x^m \sin(\beta x)$ and $x^m \cos(\beta x)$ correspond to $\alpha = 0$.

ConcepTest

Question. What should you guess in each instance?

$$y'' + 2y' - 3y = f(t)$$

where $f(t)$ is

(a) $\cos(3t)$

(b) e^{-3t}

(c) $te^t \sin t$

(d) $t^2 e^t$

Answer. $r^2 + 2r - 3 = (r + 3)(r - 1) = 0$.

(a) $A\cos(3t) + B\sin(3t)$, case (iv)

(b) Ate^{-3t}, case (ii)

(c) $(A_1 t + A_0)e^t \cos t + (B_1 t + B_0)e^t \sin t$, case (iv)

(d) $t(A_2 t^2 + A_1 t + A_0)e^t$, case (ii)

ConcepTest

Question. What should you guess in each instance?

$$y'' + y = f(t)$$

where $f(t)$ is

(a) $\cos t$

(b) e^{-3t}

(c) $te^t \sin t$

(d) $t^2 e^t$

Answer. $r^2 + 1 = 0$, $r = \pm i$.

(a) $tA\cos t + tB\sin t$, case (v)

(b) Ae^{-3t}, case (i)

(c) $(A_1 t + A_0)e^t \cos t + (B_1 t + B_0)e^t \sin t$, case (iv)

(d) $(A_2 t^2 + A_1 t + A_0)e^t$, case (i)

Superposition Theorem

The following theorem unables us to extend the method of undetermined coefficients to solve initial value problems.

Theorem

Let y_1 be a solution to the differential equation

$$ay'' + by' + cy = f_1(x),$$

and let y_2 be a solution to the differential equation

$$ay'' + by' + cy = f_2(x).$$

Then for any constants A and B, the function $Ay_1 + By_2$ is a solution to the equation

$$ay'' + by' + cy = Af_1(x) + Bf_2(x).$$

Example

Find a particular solution to

$$y'' + 3y' + 2y = 9t + 20e^{3x}$$

We have seen that $y_1(x) = \frac{3}{2}x - \frac{9}{4}$ is a solution to

$$y'' + 3y' + 2y = 3t$$

and $y_2(x) = \frac{e^{3x}}{2}$ is a solution to

$$y'' + 3y' + 2y = 10e^{3x}$$

So, by the Theorem, the following is a solution to the above equation

$$3y_1 + 2y_2 = \frac{9}{2}x - \frac{27}{4} + e^{3x}.$$

Corollary

Corollary

Let y_p be a solution to the differential equation

$$ay'' + by' + cy = f(x),$$

and let $Ay_1 + By_2$ be a general solution to the associated homogeneous equation

$$ay'' + by' + cy = 0.$$

Then, $y_p + Ay_1 + By_2$ is a solution as well to the equation

$$ay'' + by' + cy = f(x) + 0 = f(x).$$

Remark. The general solution $Ay_1 + By_2$ to the associated homogeneous equation is called the complementary solution.

Initial value problems for nonhomogeneous equations

Theorem

For any real numbers a, b, c, t_0, Y_0, Y_1, suppose that $y_p(x)$ is a particular solution on an interval I containing t_0 to

$$ay'' + by' + cy = f(x),$$

and y_1 and y_2 are linearly independent solutions on I to the associated homogeneous equation.

Then there exists a unique solution $y = y(x)$ in I to the initial value problem

$$ay'' + by' + cy = f(x), \quad y(t_0) = Y_0, y'(t_0) = Y_1$$

and it is given by appropriate choices of A and B for

$$y = y_p + Ay_1 + By_2.$$

Example

Problem. Find a particular solution to the initial value problem

$$y'' - y = 2 - x^2, \quad y(0) = 1, y'(0) = 0$$

given that $y_p(x) = x^2$ is a particular solution to the equation.

The associated homogeneous equation is $y'' - y = 0$ has solutions

$$Ae^x + Be^{-x}$$

The nonhomogeneous general solution is $x^2 + Ae^x + Be^{-x}$.

Substitute.

$$\begin{aligned} y(0) &= 1 = 0^2 + Ae^0 + Be^0 = A + B \\ y'(0) &= 0 = 2(0) + Ae^0 - Be^0 = A - B \end{aligned}$$

Solve. $A = B = \frac{1}{2}$. The solution is

$$y(x) = x^2 + \frac{1}{2}(e^x + e^{-x}).$$

Example

Problem. Find a particular solution to the problem

$$y'' - y = 8xe^x + 2e^x.$$

Solution. $r = \pm 1$ is a solution to the associated quadratic $r^2 - 1 = 0$, so we need to match the right-side as follows:

- $8xe^x$ is matched to the form $x(Ax + B)e^x$,
- $2e^x$ is matched to the form Cxe^x.

These two cases are redundant. We can simply match $8xe^x + 2e^x$ to the single form $(Ax^2 + Bx)e^x$.

Example

Problem. Find a particular solution to the problem

$$y'' - y = 8xe^x + 2e^x.$$

Guess. $y_p(x) = (Ax^2 + Bx)e^x$

Substitute.

$$\begin{aligned} y_p' &= (Ax^2 + Bx)e^x + (2Ax + B)e^x = (Ax^2 + (2A + B)x + B)e^x \\ y_p'' &= (2Ax + (2A + B))e^x + (Ax^2 + (2A + B)x + B)e^x \\ &= (Ax^2 + (4A + B)x + (2A + 2B))e^x \\ y_p'' - y_p &= (4Ax + (2A + 2B))e^x \end{aligned}$$

Match. $(4Ax + (2A + 2B))e^x = (8x + 2)e^x$.

$$\begin{aligned} 8 &= 4A \\ 2 &= 2A + 2B, \end{aligned}$$

Solve. $A = 2$ and $B = -1$.

$$y_p(x) = (2x^2 - x)e^x$$

General Method of Undetermined Coefficients: Case 1

Let $P_m(x)$ be a polynomial of degree m.

Method. To find a particular solution to the equation $ay'' + by' + cy = P_m(x)e^{rx}$, use the guess

$$y_p(x) = t^s(A_m x^m + \ldots + A_1 x + A_0)e^{rt}$$

where

(i) $s = 0$ if r is not a root of the associated homogeneous equation,

(ii) $s = 1$ if r is a simple root of the associated homogeneous equation,

(iii) $s = 2$ if r is a double root of associated homogeneous equation.

Remark. The case of $ay'' + by' + cy = P_m(x)$ corresponds to $r = 0$.

General Method of Undetermined Coefficients: Case 2

Let $P_m(x)$ be a polynomial of degree m.

Method. To find a particular solution to the equation $ay'' + by' + cy = R$ where

$$R = P_m(x)e^{\alpha x}\cos(\beta t) \quad \text{or} \quad R = P_m(x)e^{\alpha x}\cos(\beta t),$$

guess

$$\begin{aligned} y_p(x) &= t^s\left(A_m x^m + \ldots + A_1 x + A_0\right)e^{\alpha x}\cos(\beta x)) \\ &\quad + t^s\left(B_m x^m + \ldots + B_1 x + B_0\right)e^{\alpha x}\sin(\beta x)) \end{aligned}$$

where

(iv) $s = 0$ if $\alpha + i\beta$ is not a root of the associated homogeneous equation,

(v) $s = 1$ if $\alpha + i\beta$ is a root of the associated homogeneous equation.

Remark. The cases of $P_m(x)\sin(\beta x)$ and $P_m(x)\cos(\beta x)$ correspond to $\alpha = 0$.