Math 216
Differential Equations

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Simple harmonic motion

Solve. $x(t)$ is the distance of an object O from a fixed point P. Find $x(t)$ in each case:

1. $4x'' + 24x' + 100x = 0.$
2. $4x'' + 40x' + 100x = 0.$
3. $4x'' + 48x' + 100x = 0$

What is the long term behavior of $x(t)$ in each case?

(a) O oscillates through P. (periodic motion)

(b) O slows to a halt without oscillating through P.

(c) O slows to a halt while oscillating through P. (pseudoperiodic motion)

(d) O drifts infinitely far from P.

Simple harmonic motion

Simple harmonic motion of a mass m on a spring of elasticity k:

$$mx'' + kx = 0, \quad x(0) = x_0, x'(0) = v_0$$

The solution to this problem is

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$$

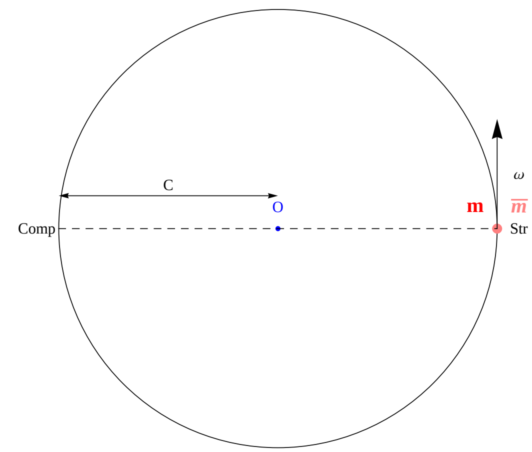
where $\omega = \sqrt{\frac{k}{m}}$.

There are four cases when one of $x_0 = 0$ or $v_0 = 0$.

Case 1

Case 1: $x_0 > 0$ and $v_0 = 0$

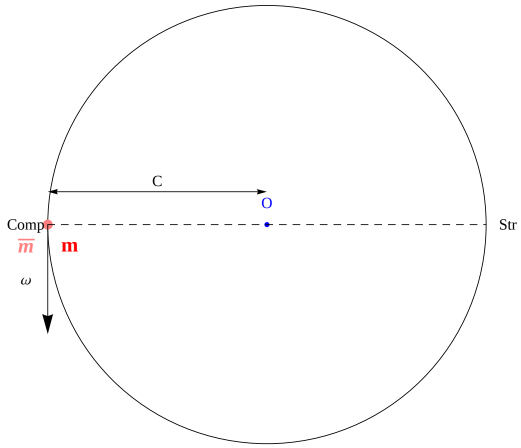
$$x(t) = x_0 \cos(\omega t)$$



Case 2

Case 2: $x_0 < 0$ and $v_0 = 0$

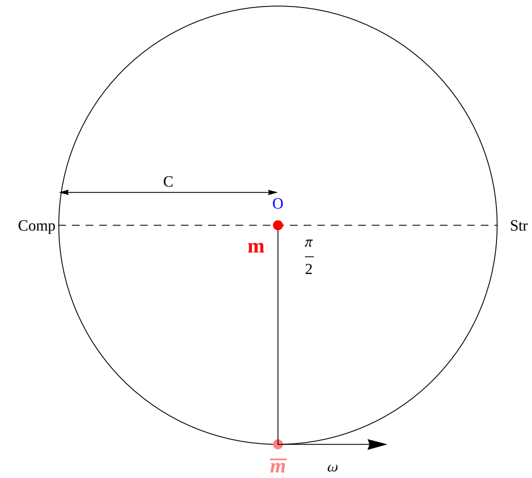
$$x(t) = x_0 \cos(\omega t) = -x_0 \cos(\omega t - \pi)$$



Case 3

Case 3: $x_0 = 0$ and $v_0 > 0$

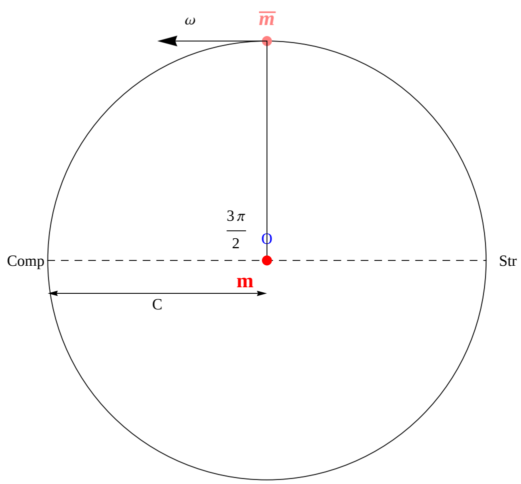
$$x(t) = \frac{v_0}{\omega} \sin(\omega t) = \frac{v_0}{\omega} \cos(\omega t - \frac{\pi}{2})$$



Case 4

Case 4: $x_0 = 0$ and $v_0 < 0$

$$x(t) = \frac{v_0}{\omega} \sin(\omega t) = -\frac{v_0}{\omega} \cos(\omega t - \frac{3\pi}{2})$$



Example

Problem. A 3 kg mass is attached to a spring of stiffness 48 N/m. When the mass is stretched $\frac{1}{2}$ m the velocity is 2 m/s away from the equillibrium point. Find the equation for displacement.

Answer. The initial value problem is given by

$$x'' + 16x = 0, \qquad x(0) = \frac{1}{2}, x'(0) = 2.$$

The general solution is

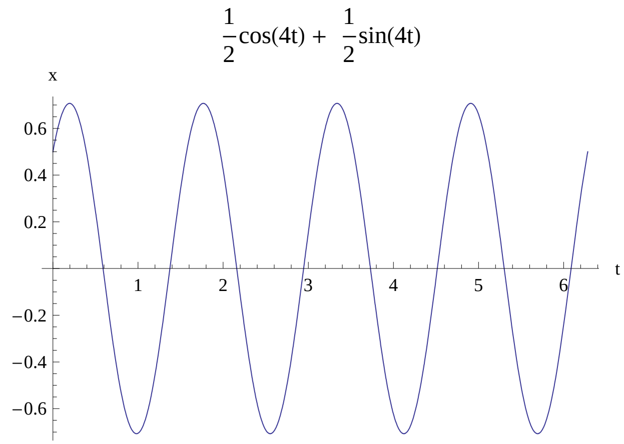
$$x(t) = A\cos(4t) + B\sin(4t).$$

In this case, $A = x_0 = \frac{1}{2}$ and $B = \frac{v_0}{\omega} = \frac{1}{2}$,

$$x(t) = \frac{1}{2}\cos(4t) + \frac{1}{2}\sin(4t).$$

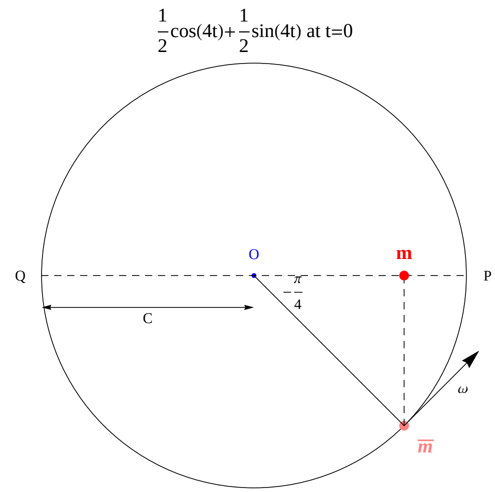
Graph of solution

The mass first reaches maximal stretch in $\frac{\pi}{16}$ seconds (time lag), then oscillates with period $\frac{\pi}{2}$ seconds and amplitude $\frac{1}{\sqrt{2}}$ meters.



Circular motion

At time $t = 0$ the shadow mass $\overline{m}$ is $\frac{\pi}{4}$ radians from P. The mass m will take $\frac{\pi}{16}$ seconds to reach its maximal displacement at P



Goal

Goal. Convert a periodic motion

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

to a circular motion

$$x(t) = C\cos(\omega t - \alpha)$$

where

- C is the amplitude
- ω is angular velocity or circular frequency,
- α is phase angle.

Euler's formula

The ♡ is Euler's formula

$$e^{a+bi} = e^a\big(\cos b + i\sin b\big)$$

for converting polar coordinates $(re^{i\theta})$

$$r = e^a, \quad \theta = b$$

to rectangular coordinates $(x + iy)$

$$x = e^a\cos b, \quad y = e^a\sin b$$

Trig identities

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

Derivation of $\sin x$ using ♡ Euler's formula:

$$\begin{aligned}\frac{e^{ix}-e^{-ix}}{2i} &= \frac{1}{2i}\Big(\big(\cos x + i\sin x\big) - \big(\cos(-x) + i\sin(-x)\big)\Big)\\ &= \frac{1}{2i}\Big(\big(\cos x + i\sin x\big) - \big(\cos(x) - i\sin(x)\big)\Big)\\ &= \frac{1}{2i}\big(2i\sin x\big)\\ &= \sin x\end{aligned}$$

Converting to cos

$$\begin{aligned}A\cos(\omega t) + B\sin(\omega t) &= A\Big(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\Big) + B\Big(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\Big)\\ &= A\Big(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\Big) + -iB\Big(\frac{e^{i\omega t}-e^{-i\omega t}}{2}\Big)\\ &= (A-iB)\frac{e^{i\omega t}}{2} + (A+iB)\frac{e^{-i\omega t}}{2}\end{aligned}$$

Convert $A+iB$ and $A-iB$ to polar coordinates:

$$A+iB = Ce^{i\alpha} \quad A-iB = Ce^{-i\alpha}.$$

We will compute C and α shortly.

Converting to cos

Let $A+iB = Ce^{i\alpha}$, so $A-iB = Ce^{-i\alpha}$.

$$\begin{aligned}A\cos(\omega t) + B\sin(\omega t) &= (A-iB)\frac{e^{i\omega t}}{2} + (A+iB)\frac{e^{-i\omega t}}{2}\\ &= Ce^{-i\alpha}\frac{e^{i\omega t}}{2} + Ce^{i\alpha}\frac{e^{-i\omega t}}{2}\\ &= C\frac{e^{i(\omega t-\alpha)}}{2} + C\frac{e^{i(\alpha-\omega t)}}{2}\\ &= C\Big(\frac{e^{i(\omega t-\alpha)}+e^{-i(\omega t-\alpha)}}{2}\Big)\\ &= C\cos(\omega t-\alpha)\end{aligned}$$

Computing amplitude to Phase angle

Recap,

$$A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t-\alpha)$$

where $A+iB = Ce^{i\alpha}$. So,

$$C = \sqrt{A^2+B^2} \qquad \overline{\alpha} = \arctan\frac{|B|}{|A|}.$$

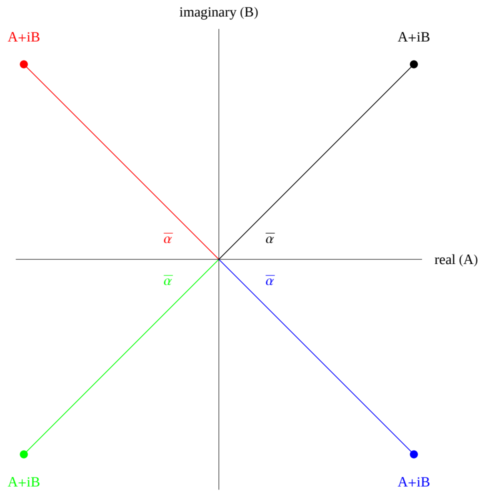
($\overline{\alpha}$ is the angle $A+iB$ makes with the x-axis.)

However, $0 \le \overline{\alpha} < \frac{\pi}{2}$, so we must determine the quadrant of $A+iB$:

$$\alpha = \begin{cases}\overline{\alpha} & \text{if } A, B > 0\\ \pi - \overline{\alpha} & \text{if } A < 0, B > 0\\ \pi + \overline{\alpha} & \text{if } A < 0, B < 0\\ 2\pi - \overline{\alpha} & \text{if } A > 0, B < 0\end{cases}$$

Determining quadrant

$$\alpha = \begin{cases} \overline{\alpha} & \text{if } A, B > 0 \\ \pi - \overline{\alpha} & \text{if } A < 0, B > 0 \\ \pi + \overline{\alpha} & \text{if } A < 0, B < 0 \\ 2\pi - \overline{\alpha} & \text{if } A > 0, B < 0 \end{cases}$$



Putting the pieces together

Simple harmonic motion with $x_0 = \frac{1}{2}m$ and $v_0 = 2m/s$,

$$x(t) = \frac{1}{2}\cos(4t) + \frac{1}{2}\sin(4t).$$

$C = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$ and $\alpha = \arctan(1) = \frac{\pi}{4}$ (since $x_0, v_0 > 0$, α is in the first quadrant). So,

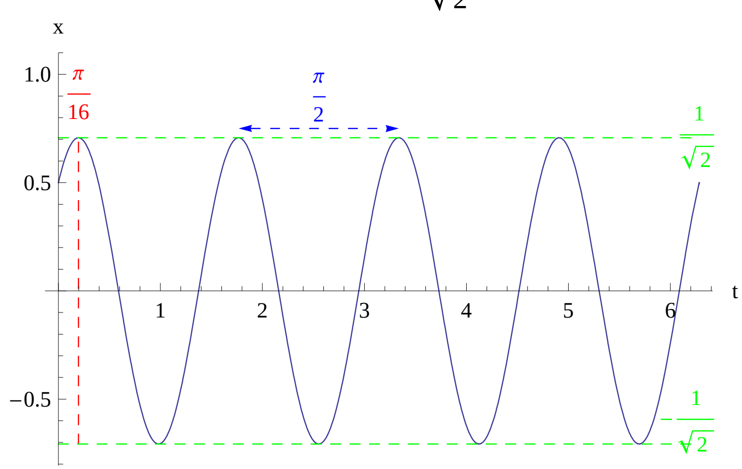
$$x(t) = \frac{1}{\sqrt{2}}\cos\left(4t - \frac{\pi}{4})\right) = \frac{1}{\sqrt{2}}\cos\left(4(t - \frac{\pi}{16})\right),$$

where

- Amplitude is $\frac{1}{\sqrt{2}} \approx 0.7$ meters,
- Period is $\frac{\pi}{2} \approx 1.6$ seconds per oscillation,
- Frequency is $\frac{2}{\pi} \approx 0.64$ oscillations per second,
- Time lag is $\frac{\pi}{16} \approx 0.2$ seconds (between start and maximum displacement).

Graph of solution

$$\frac{1}{2}\cos(4t) + \frac{1}{2}\sin(4t) = \frac{1}{\sqrt{2}}\cos(4t - \frac{\pi}{4})$$



Forces in Mass-Spring Oscillators

There are three forces in mass-spring oscillators

1. The spring elasticity, which by Hooke's law, is proportional to the displacement $x(t)$ in the opposite direction:

$$F_{spring} = -kx$$

2. Dampening forces, primarily friction, which is often proportional to the velocity (but opposing the direction of motion):

$$F_{dampening} = -cx'$$

3. External forces, such as gravity: F_{ext}.

By Newton's second law,

$$mx'' = -kx - cx' + F_{ext} \quad \text{equivalently} \quad mx'' + cx' + kx = F_{ext}.$$

Damped Ocillatory Motion

Section 3.4 considers damped oscillatory motion free of external forces

$$mx'' + cx' + kx = 0.$$

(Section 3.6 considers motions with external forces.)

The characteristic equation is $mr^2 + cr + k = 0$ ($m, c, k > 0$), and the roots are

$$r = \frac{-c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$$

There are three cases, depending on whether the discriminant $c^2 - 4mk$ is positive, zero, or negative.

Critical dampening

- Crically damped motion occurs when $c^2 - 4mk = 0$.
- There is only one real root:

$$r = \frac{-c}{2m}$$

- The general solution to $mx'' + cx' + kx = 0$ is

$$x(t) = \left(A + Bt\right)e^{\frac{-c}{2m}t}$$

- The mass eventually comes to a stop:

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{A + Bt}{e^{\frac{c}{2m}t}} = 0$$

(Use L'Hôpital's rule to verify this.)

Critical dampening: non-oscillatory

- The general solution to $mx'' + cx' + kx = 0$ in the critical damping case is

$$x(t) = \left(A + Bt\right)e^{\frac{-c}{2m}t}$$

- The $x'(t)$ vanishes at most one value of t (provided A or B is nonzero)

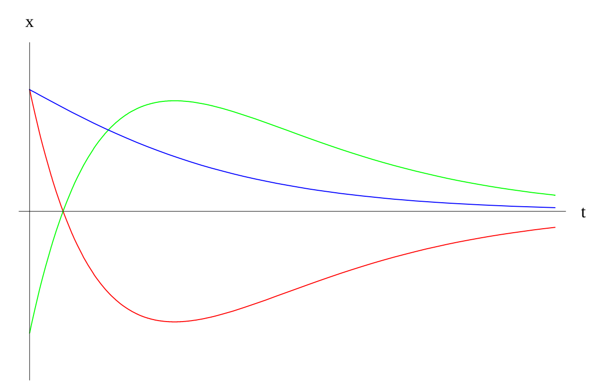
$$x'(t) = \left(B - \frac{c}{2m}A - \frac{c}{2m}Bt\right)e^{\frac{-c}{2m}t}$$

- Therefore, the mass m does not oscillate.

Examples of critically damped motion

Equation: $x'' + 2x + 1 = 0$, Solutions:

$x_0 = .3, v_0 = -1.3$	$(0.3 - t)e^{-t}$
$x_0 = .3, v_0 = 1.3$	$(t - 0.3)e^{-t}$
$x_0 = .3, v_0 = -0.15$	$(0.3 + 0.15)e^{-t}$



Overdamped motion

- Overdamped motion occurs when $c^2 - 4mk > 0$, so the effect of dampening is large compared to the mass and spring elasticity.
- There are two distinct real roots:

$$r_1 = \frac{-c}{2m} + \frac{1}{2m}\sqrt{c^2 - 4mk} \quad r_2 = \frac{-c}{2m} - \frac{1}{2m}\sqrt{c^2 - 4mk}$$

  Since $c^2 > c^2 - 4mk$, both r_1 and r_2 are negative.
- The general solution to $mx'' + cx' + kx = 0$ is

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

  where $x(t) \to 0$ as $t \to \infty$

Overdampening: non-oscillatory

- The general solution to $mx'' + cx' + kx = 0$ in the overdamping case is

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

  where $r_1, r_2 < 0$.
- The $x'(t)$ vanishes at most one value of t (provided A or B is nonzero)

$$x'(t) = Ar_1 e^{r_1 t} + Br_2 e^{r_2 t} = e^{r_1 t}\left(Ar_1 + Br_2 e^{(r_1 - r_2)t}\right)$$

- Therefore, the mass m does not oscillate.

Underdampening

- Underdamped motion occurs when $c^2 - 4mk < 0$. The dampening is too weak to prevent oscillatory behavior.
- There are complex conjugate roots $\rho \pm i\omega$ where

$$\rho = -\frac{c}{2m} \quad \omega = \frac{1}{2m}\sqrt{4mk - c^2}$$

- The general solution to $mx'' + cx' + kx = 0$ is

$$x(t) = e^{\rho t}\left(A\cos(\omega t) + B\sin(\omega t)\right).$$

- We can express $x(t)$ in the alternative form

$$Ce^{\rho t}\cos\left(\omega t - \alpha\right).$$

  where $C = \sqrt{A^2 + B^2}$ and $\tan\alpha = \frac{B}{A}$.

Underdampening

- The general solution to $mx'' + cx' + kx = 0$ in the underdamping case is

$$Ce^{\rho t}\cos\left(\omega t - \alpha\right).$$

  where $\rho = -\frac{c}{2m} < 0$.
- $\cos\left(\omega t - \alpha\right)$ oscillates between $-Ce^{\rho t}$ and $Ce^{\rho t}$, so that $x(t) \to 0$ as $t \to \infty$.
- The motion is not actually periodic, since the amplitude diminishes over time. However, motion periodically reverses:
  - Time-varying amplitude $Ce^{\rho t} = Ce^{-\frac{c}{2m}t}$,
  - pseudoperiod $\frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{4mk - c^2}}$,
  - pseudofrequency $\frac{\omega}{2\pi}$.

Example of underdamped motion