Math 216 Differential Equations

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

September 29, 2008

Higher order linear equations

- The general *n-th order linear differential equation* is one of the form

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_{n-1}(x)y' + P_n(x)y = F(x).$$

- The *associated homogeneous equation* is the equation with the right-side set to 0:

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_{n-1}(x)y' + P_n(x)y = 0.$$

- We will assume that the coefficients $P_i(x)$ and $F(x)$ are continuous on some open interval I, and that $P_0(x)$ is never zero on I. Under these assumptions, we write the general form of the n-order linear equation as

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = f(x).$$

Principle of Superposition

Theorem

Let $y_1, y_2, \ldots, y_k$ be k solutions to the homogeneous linear equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = 0$$

on the interval I. Then for any constants $c_1, c_2, \ldots, c_k$, the linear combination

$$c_1y_1 + c_2y_2 + \ldots + c_ky_k$$

is also a solution to the homogeneous equation on I.

ConcepTest

Question. What interval can we expect to find a unique solution to the second order initial value problem

$$27y'' + 14y + 3 = 0, \quad y(0) = 1, y'(0) = -1?$$

(a) $(-\infty, \infty)$.
(b) A small interval around $x = 0$.
(c) Impossible to tell from the given data.

Answer. (a) – A linear equation always has a unique solution on the entire interval on which its coefficients are continuous. The constant coefficients are continuous on the entire real line.

Existence and Uniqueness Theorem

Theorem

Supose the functions $p_1, p_2, \ldots, p_n$ and f are continuous on the open interval I containing the point a. Then given any n numbers $b_0, b_1, \ldots, b_{n-1}$, the nth-order linear equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = f(x).$$

has a unique solution on the entire interval I that satisfies the n initial conditions

$$y(a) = b_0, y'(a) = b_1, \ldots, y^{(n-1)}(a) = b_{n-1}.$$

An nth-order equation with n initial conditions is called an nth-order initial value problem.

Example

Let $p_1, p_2, \ldots, p_n$ be continuous on some interval I containing a. What is the unique solution on I of the nth-order initial value problem

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = 0$$
$$y(a) = y'(a) = \ldots = y^{(n-1)}(a) = 0$$

Answer. The trivial solution $y \equiv 0$ is always a solution to a homogeneous equation, and satisfies the initial conditions. So, $y \equiv 0$ is the unique solution satisfying the nth-order initial value problem.

Example

The functions $y_1 = x^2$ and $y_2 = x^3$ and $y_3 \equiv 0$ are three different solutions to the second-order initial value problem

$$x^2y'' - 4xy + 6y = 0, \quad y(0) = y'(0) = 0$$

Why doesn't this contradict the Uniqueness Theorem?

Answer. The uniqueness theorem requires the leading coefficient of y'' to be nonzero in the interval containing 0.

ConcepTest

Solve the homogeneous third-order equation

$$y^{(3)} = 0$$

Answer. The general solution is

$$y(t) = c_1 + c_2x + c_3x^2$$

Goal. An n-order equation has "n parameters" and "n distinct solutions". By distinct solutions we mean linearly independent solutions.

Linear dependence

Definition

We say that n functions $f_1, f_2, \ldots, f_n$ are linearly dependent on an interval I when there are constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \qquad \text{for all } x \text{ in } I.$$

The functions are linearly independent if they are not linearly dependent.

Equivalent formulation

Lemma

The function $f_1, f_2, \ldots, f_n$ are linearly dependent if and only if one of the functions can be written as a linear combination of the other functions.

That is, there is some f_i and constants $c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n$ such that

$$f_i(x) = \sum_{j \neq i} c_j f_j(x) \qquad \text{for all } x \text{ in } I.$$

Example

Show the following functions are linearly dependent

$$\cosh x, \sinh x, e^x.$$

Recall the definition of the hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

Answer. Let $c_1 = c_2 = 1$ and $c_3 = -1$

$$0 = \cosh x + \sinh x - e^x$$

or equivalently,

$$e^x = \cosh x + \sinh x$$

ConcepTest

Show the following functions are linearly dependent on $(-\infty, \infty)$

$$0, \sin x, e^x.$$

Answer. Let $c_1 = 27$, $c_2 = c_3 = 0$

$$0 = (27)0 + 0 \sin x + 0 e^x.$$

In general, $0, f_1, f_2, \ldots, f_n$ are linearly dependent for any n functions.

ConcepTest

Show the following functions are linearly dependent on $(-\infty, \infty)$

$$20x, 5x\sin^2 x, 10x\cos^2 x.$$

Answer. Let $c_1 = -1$, $c_2 = 4$ and $c_3 = 2$

$$-20x + 20x\sin^2 x + 20x\cos^2 x = -20x + 20x\big(\sin^2 x + \cos^2 x\big) = 0.$$

Linear independence

This is a rephrasing of our earlier definition in terms of linear dependence.

Definition

We say that n functions $f_1, f_2, \ldots, f_n$ are linearly independent on an interval I when there are no nontrivial choice of constants $c_1, c_2, \ldots, c_n$, such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \qquad \text{for all } x \text{ in } I.$$

(The trivial choice, $c_1 = c_2 = \ldots = c_n = 0$, is always possible.)

Question. How can we test for linear independence?

A test for linear independence

Lemma

Let f and g be any once differentiable functions on an interval I. If there is no x in I such that the following equation holds,

$$f(x)g'(x) - g(x)f'(x) = 0,$$

then f and g are linearly independent on the interval I.

Equivalently, if f and g are linearly dependent on an interval I, then there is an a in I satisfying the equation

$$f(a)g'(a) - g(a)f'(a) = 0.$$

(In fact, if f and g are linearly dependent on I then every a in I satisfies this equation.)

Proof of Lemma

Suppose f and g are linearly dependent on I. We will show that

$$f(x)g'(x) - g(x)f'(x) = 0.$$

for all x in I.

Let c_1 and c_2 not both zero such that

$$0 = c_1 f(x) + c_2 g(x) \qquad \text{for all } x \text{ in } I$$

Suppose $c_1 \neq 0$. Then $f(x) = \frac{-c_2}{c_1} g(x)$ for all x in I.
Let $k = \frac{-c_2}{c_1}$; so, for any x in I

$$\begin{aligned} f(x)g'(x) - g(x)f'(x) &= kg(x)g'(x) - g(x)kg'(x) \\ &= 0. \end{aligned}$$

If $c_1 = 0$ then $g \equiv 0$; but $f, 0$ are always linearly dependent.

A test for linear independence

We define the Wronskian of f and g to be the 2×2 determinant

$$W(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x)$$

Note that $W(f,g)$ is a function of x.

The following restates our condition for linear independence.

Lemma

Let f and g be once differentiable functions. If the Wronskian is nonzero for every x in I,

$$W(f,g)(x) = f(x)g'(x) - g(x)f'(x) \neq 0 \qquad \text{for every } x \text{ in } I,$$

then f and g are linearly independent on I.

ConcepTest

Show that $\sin(ax)$ and $\cos(ax)$ are linearly independent when $a \neq 0$.

Answer.

$$\begin{aligned} W(\cos(ax), \sin(ax))(x) &= \begin{vmatrix} \cos(ax) & \sin(ax) \\ -a\sin(ax) & a\cos(ax) \end{vmatrix} \\ &= a\cos^2(ax) + a\sin^2(ax) \\ &= a \neq 0 \end{aligned}$$

for all x. By the Lemma, $\cos(ax)$ and $\sin(ax)$ are linearly independent.

ConcepTest

Show that e^{ax} and e^{bx} are linearly independent when $a \neq b$.

Answer. Let x be any number.

$$\begin{aligned} W(e^{ax}, e^{bx})(x) &= \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix} \\ &= be^{(a+b)x} - ae^{(a+b)x} \\ &= (b-a)e^{(a+b)x} \\ &\neq 0 \end{aligned}$$

By the Lemma, e^{ax} and e^{-ax} are linearly independent.

ConcepTest

Compute $W(\cos^2 x, \sin^2 x)$.

Answer.

$$\begin{aligned} W(\cos^2 x, \sin^2 x)(x) &= \begin{vmatrix} \cos^2 x & \sin^2 x \\ -2\sin x \cos x & 2\sin x \cos x \end{vmatrix} \\ &= \cos^2 x(2\sin x\cos x)) - \sin^2 x(-2\cos x \sin x)) \\ &= 2\cos x \sin x(\cos^2 x + \sin^2 x) \\ &= 2\cos x \sin x. \end{aligned}$$

The Wronskian $W(\cos^2 x, \sin^2 x)(0) = 0$ but $\cos^2 x$ and $\sin^2 x$ are linearly independent. So, we do not have a test for linear dependence.

A test for linear dependence

Lemma

Let y_1 and y_2 are solutions on an interval I to the second-order homogeneous equation

$$y'' + p_0(x)y' + p_1(x)y = 0.$$

Then y_1 and y_2 are linearly dependent if and only if there is a point a in I such that the Wronskian is zero: $W(y_1, y_2)(a) = 0$.

Under these conditions, iff the Wronskian is zero on one point a in I, then it is zero on all x in I.

Recall,

$$W(y_1, y_2)(a) = y_1(a)y_2'(a) - y_2(a)y_1'(a)$$

A test for linear independence

Here is an equivalent formulation in terms of linear independence.

Lemma

Let y_1 and y_2 are solutions on an interval I to the second-order homogeneous equation

$$y'' + p_0(x)y' + p_1(x)y = 0.$$

Then y_1 and y_2 are linearly independent if and only if the Wronskian is never zero on I: $W(y_1, y_2)(x) \neq 0$ for any x in I.

Recall,

$$W(y_1, y_2)(a) = y_1(a)y_2'(a) - y_2(a)y_1'(a)$$

Proof of Lemma: $(\Rightarrow)$

Suppose y_1 and y_2 are solutions on an interval I to a second-order linear equation and for some a in I

$$y_1(a)y_2'(a) - y_2(a)y_1'(a) = 0.$$

Case 1. $y_1(a) \neq 0$. Let $k = \frac{y_2(a)}{y_1(a)}$ and consider the solution to the equation $y(x) = ky_1(x)$. Then

$$y(a) = \frac{y_2(a)}{y_1(a)}y_1(a) = y_2(a); \qquad y'(a) = \frac{y_2(a)}{y_1(a)}y_1'(a) = y_2'(a);$$

where the last equality is from our hypothesis about a.

Since y_2 is the unique solution determined by the values $y_2(a)$ and $y_2'(a)$, we have $y = y_2$ on I. So, $ky_1 = y_2$ on I, and y_1 and y_2 are linearly dependent on I.

Proof of Lemma: $(\Rightarrow)$

We are supposing y_1 and y_2 are solutions on an interval I to a second-order linear equation and for some a in I

$$y_1(a)y_2'(a) - y_2(a)y_1'(a) = 0.$$

Case 2. $y_1(a) = 0$ but $y_1'(a) \neq 0$. Note that our hypothesis implies $y_2(a) = 0$. Let $k = \frac{y_2'(a)}{y_1'(a)}$ and consider the solution to the equation $y(x) = ky_1(x)$. Then

$$y(a) = \frac{y_2'(a)}{y_1'(a)}y_1(a) = 0 = y_2(a); \qquad y'(a) = \frac{y_2'(a)}{y_1'(a)}y_1'(a) = y_2'(a);$$

where the first equality is from our hypothesis that $y_1(a) = 0$ and so, $y_2(a) = 0$.

Since y_2 is the unique solution determined by the values $y_2(a)$ and $y_2'(a)$, we have $y = y_2$ on I. So, $ky_1 = y_2$ on I, and y_1 and y_2 are linearly dependent on I.

Proof of Lemma: $(\Rightarrow)$

We are supposing y_1 and y_2 are solutions on an interval I to a second-order linear equation and for some a in I

$$y_1(a)y_2'(a) - y_2(a)y_1'(a) = 0.$$

Case 3. $y_1(a) = y_1'(a) = 0$. Then $y_1 \equiv 0$ on I, since the constant zero function is the unique solution y with $y(0) = y'(0) = 0$. So, y_1 and y_2 are linearly dependent (since $y_1 \equiv 0$ on I).

Caveat

The test for linear dependence of f and g on an interval I given by

$$W(f,g)(a) = f(a)g'(a) - g(a)f'(a) = 0$$

for some a in I, is only appropriate when f and g are both solutions on I to a homogeneous linear equation

$$y'' + p_0(x)y' + p_1(x)y = 0.$$

Example. $\sin^2 x$ and $\cos^2 x$ are linearly independent on the interval $(-\infty, \infty)$, however

$$W(\cos^2 x, \sin^2 x)(x) = 2\cos x \sin x \left(\cos^2 x - \sin^2 x\right)$$

and $W(\cos^2 x, \sin^2 x)(0) = 0$.

It follows that $\sin^2 x$ and $\cos^2 x$ are not both solutions to the same homogeneous second-order linear equation.

The Wronskian for three functions

Definition

We define the Wronskian for three functions f, g and h which are twice differentiable using the 3×3 determinant.

$$\begin{aligned} W(f,g,h)(x) &= \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} \\ &= f(x)\begin{vmatrix} g'(x) & h'(x) \\ g''(x) & h''(x) \end{vmatrix} \\ &\quad -f'(x)\begin{vmatrix} g(x) & h(x) \\ g''(x) & h''(x) \end{vmatrix} \\ &\quad +f''(x)\begin{vmatrix} g(x) & h(x) \\ g'(x) & h'(x) \end{vmatrix} \end{aligned}$$

Example

Compute the Wronskian for $e^x, \cos x, \sin x$.

Answer.

$$\begin{aligned} W(e^x, \cos x, \sin x)(x) &= \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix} \\ &= e^x \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} \\ &\quad -e^x \begin{vmatrix} \cos x & \sin x \\ -\cos x & -\sin x \end{vmatrix} \\ &\quad +e^x \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} \\ &= 2e^x \end{aligned}$$

Note that $W(e^x, \cos x, \sin x)(x) \neq 0$ for all x.

A test for linear independence

Analogous to two functions, we have a test of linear independence using the Wronskian for three functions.

Lemma

Let f, g and h be twice differentiable functions. If the Wronskian is nonzero for every x in I,

$$W(f, g, h)(x) \neq 0 \qquad \text{for every } x \text{ in } I,$$

then f, g and h are linearly independent on I.

Example. $e^x, \cos x, \sin x$ are linearly independent.

A test for linear dependence

We get a test for linear dependence if the functions are solutions to a third-order homogeneous equation.

Lemma

Let y_1, y_2 and y_3 be solutions on an interval I to the third-order homogeneous equation

$$y^{(3)} + p_0(x)y'' + p_1(x)y' + p_2(x)y = 0.$$

Then y_1, y_2 and y_3 are linearly dependent if and only if there is a point a in I such that the Wronskian vanishes

$$W(y_1, y_2, y_3)(a) = 0.$$

In fact, if the Wronskian is zero on one point a in I then it is zero on all x in I.

A test for linear independence

Lemma

Let y_1, y_2 and y_3 be solutions on an interval I to the third-order homogeneous equation

$$y^{(3)} + p_0(x)y'' + p_1(x)y' + p_2(x)y = 0.$$

Then y_1, y_2 and y_3 are linearly independent if and only if the Wronskian $W(y_1, y_2, y_3)$ is never zero on I.

ConcepTest

Show that $e^x, \cos x, \sin x$ are linearly independent on $(-\infty, \infty)$.

Answer. We have seen that $W(e^x, \cos x, \sin x)(x) = 2e^x \neq 0$ for all x. It follows that $e^x, \cos x, \sin x$ are linearly independent.

General criterion for linear independence

- There is a generalization of the Wronskian to n functions which are n-times differentiable: it is an $n \times n$ determinant, $W(f_1, f_2, \ldots, f_n)$.
- Let $f_1, f_2, \ldots, f_n$ are n-times differentiable on an interval I. If $W(f_1, f_2, \ldots, f_n)(x) \neq 0$ for all x in I, then the functions are linearly independent.
- In addition, if $y_1, y_2, \ldots, y_n$ be n solutions on an interval I to the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = 0,$$

then, $y_1, y_2, \ldots, y_n$ are linearly independent on I if and only if $W(f_1, f_2, \ldots, f_n)(x) \neq 0$ for all x in I.

Application of linear independence

The following sets of functions are linearly independent.

- $1, x, x^2, x^3, \ldots, x^n$.
- $1, \sin x, \cos x, \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx)$.
- $e^{a_1 x}, e^{a_2 x}, \ldots, e^{a_n x}$ where the a_i are distinct constants.

General solutions

Theorem

Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions to the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = 0$$

on an open interval I in which each of $p_1, p_2, \ldots, p_{n-1}$ are continuous. Then, for any solution to the equation, there are constants $c_1, c_2, \ldots, c_n$ such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

for all x in I.

General solutions

Definition

Given n linearly independent solutions $y_1, y_2, \ldots, y_n$ on an interval I, and parameters $c_1, c_2, \ldots, c_n$, the general solution to the homogeneous equation is

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

Proof of Theorem

We prove the theorem in the case of $n = 2$. The general case is essentially the same.

Let y_1 and y_2 be linearly independent solutions to the homogeneous equation

$$y'' + p_1(x)y' + p_2(x)y = 0$$

on an open interval I in which each of p_1 and p_2 are continuous.

We know that the Wronskian is never zero on I:

$$W(y_1, y_2)(x) \neq 0 \qquad \text{for all } x \text{ in } I$$

Proof of Theorem

Let y be a solution to the same equation with $y(a) = Y_0$ and $y'(a) = Y_1$. We want solutions c_1 and c_2 for the equations:

$$\begin{aligned} Y_0 &= c_1 y_1(a) + c_2 y_2(a) \\ Y_1 &= c_1 y_1'(a) + c_2 y_2'(a) \end{aligned}$$

Multiply the first equation by $y_2'(a)$ and the second by $y_2(a)$:

$$\begin{aligned} Y_0 y_2'(a) &= c_1 y_2'(a) y_1(a) + c_2 y_2'(a) y_2(a) \\ Y_1 y_2(a) &= c_1 y_2(a) y_1'(a) + c_2 y_2(a) y_2'(a) \end{aligned}$$

Subtract the second equation from the first and solve for c_1:

$$c_1 = \frac{Y_0 y_2'(a) - Y_1 y_2(a)}{W(y_1, y_2)(a)}$$

This is OK since $W(y_1, y_2)(a) \neq 0$.

Proof of Theorem

$$\begin{aligned} Y_0 &= c_1 y_1(a) + c_2 y_2(a) \\ Y_1 &= c_1 y_1'(a) + c_2 y_2'(a) \end{aligned}$$

Multiply the second equation by $y_1'(a)$ and the second by $y_1(a)$:

$$\begin{aligned} Y_0 y_1'(a) &= c_1 y_1'(a) y_1(a) + c_2 y_1'(a) y_2(a) \\ Y_1 y_1(a) &= c_1 y_1(a) y_1'(a) + c_2 y_1(a) y_2'(a) \end{aligned}$$

Subtract the second equation from the first and solve for c_2:

$$c_2 = \frac{Y_0 y_1'(a) - Y_1(a) y_1(a)}{W(y_1, y_2)(a)}$$

This is OK since $W(y_1, y_2)(a) \neq 0$.

Proof of Theorem

So, the choice of constants

$$c_1 = \frac{Y_0 y_2'(a) - Y_1 y_2(a)}{W(y_1, y_2)(a)}; \qquad c_2 = \frac{Y_0 y_1'(a) - Y_1(a) y_1(a)}{W(y_1, y_2)(a)}$$

for

$$c_1 y_1 + c_2 y_2$$

guarantee

$$\begin{aligned} y(a) &= c_1 y_1(a) + c_2 y_2(a) \\ y'(a) &= c_1 y_1'(a) + c_2 y_2'(a) \end{aligned}$$

Then, by the uniqueness of the solution on I satisfying $y(a)$ and $y'(a)$,

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$