Math 216
Differential Equations

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Solutions to quadratic equations

The solutions to a quadratic equation

$$ax^2 + bx + c = 0$$

are given by the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

ConcepTest

Problem. Determine the solutions to the following equations

(a) $x^2 - 8x + 17 = 0$
(b) $x^2 - 6r - 2 = 0.$
(c) $4x^2 + 24x + 37 = 0$

Answer.

(a) $4 \pm i$
(b) $3 \pm \sqrt{11}.$
(c) $3 \pm \frac{1}{2}i$

Characteristic equation

For a homogeneous second-order equation with constant coefficients,

$$ay'' + by' + cy = 0$$

its associated characteristic equation is

$$ar^2 + br + c = 0.$$

Solutions

If r is a root of the characteristic equation

$$ar^2 + br + c = 0.$$

then e^{rx} is a solution to the differential equation

$$ay'' + by' + cy = 0$$

Since

$$a(e^{rx})'' + b(e^{rx})' + ce^{rx} = e^{rx}\left(ar^2 + br + c\right) = 0$$

This is true for any root r, real or complex.

Distinct real roots

Theorem

If r_1 and r_2 are distinct and real roots of the characteristic equation $ar^2 + br + c = 0$, then

$$c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is the general solution to the linear homogeneous equation

$$ay'' + by' + cy = 0$$

ConcepTest

Solve the initial value problem

$$y'' - 4y = 0, \qquad y(0) = 0, y'(0) = 3.$$

Answer. The characteristic equation is $r^2 - 4 = 0$, so the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

Substituting

$$\begin{aligned} 0 &= c_1 + c_2 \\ 3 &= 2c_1 - 2c_2 \end{aligned}$$

So, $c_1 = \frac{3}{4} = -c_2$.

$$y(x) = \frac{3}{4}\left(e^{2x} - e^{-2x}\right).$$

Example

Problem. The characteristic equation for

$$y'' - 6y' + 9y = 0.$$

is $r^2 - 6r + 9 = (r - 3)^2$, which has only one real (repeated) root, $r = 3$. So, e^{3x} is one solution. What is the other solution?

Verify that xe^{3x} is a second solution

$$\begin{aligned} 0 &= (xe^{3x})'' - 6(xe^{3x})' + 9xe^{3x} \\ &= \left(9xe^{3x} + 3e^3x + 3e^{3x}\right) - \left(6e^{3x} - 18xe^{3x}\right) + 9xe^{3x}. \end{aligned}$$

e^{3x} and e^{3x} are linearly independent. Thus, the general solution is

$$c_1 e^{3x} + c_2 xe^{3x}$$

Example

Problem. Find a general solution to the equation:

$$y'' - 2dy' + d^2 = 0.$$

where d is a real value.

Answer. The characteristic equation is

$$r^2 - 2dr + d^2 = (r - d)^2$$

has one real root $r = d$ with multiplicity two. The general solution is

$$c_1 e^{dx} + c_2 x e^{dx}.$$

Repeated real roots

Theorem

If the characteristic equation $ar^2 + br + c = 0$ has a single real (repeated) root r_1, then

$$(c_1 + c_2 x)e^{r_1 x}$$

is the general solution to the linear homogeneous equation

$$ay'' + by' + cy = 0$$

ConcepTest

Problem. Find the unique solution to the IVP

$$y'' + 2y' + y = 0, \quad y(0) = 1, y'(0) = b.$$

Answer. The characteristic equation is

$$r^2 + 2r + 1 = (r + 1)^2 = 0,$$

so, $r = -1$ is a root with multiplicity 2. The general solution is $c_1 e^{-x} + c_2 x e^{-x}$. Substituting for the initial conditions,

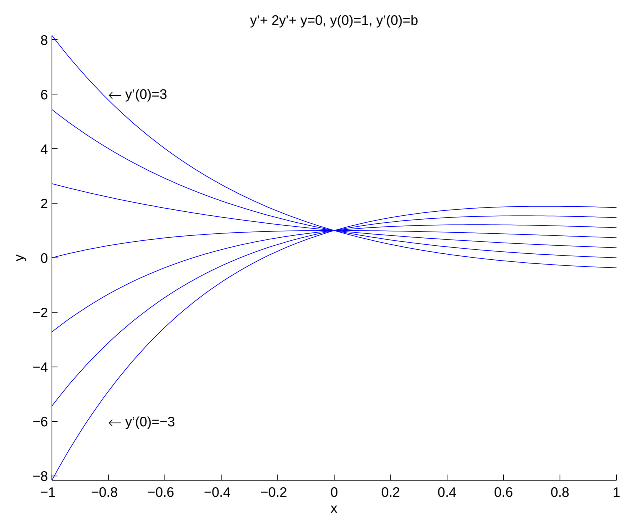
$$\begin{aligned} 1 &= c_1 \\ b &= -c_1 + c_2 \end{aligned}$$

So, $c_1 = 1$ and $c_2 = b + 1$. Thus, the solution to the IVP is

$$e^{-x} + (b+1)xe^{-x}.$$

Graphing changing parameters

Plotting: $e^{-x} + (b+1)xe^{-x}$ for some values of b.



ConcepTest

Problem. Find the unique solution to the IVP

$$y'' + 2y' + y = 0, \quad y(0) = a, y'(0) = 1.$$

Answer. The general solution is $c_1 e^{-x} + c_2 x e^{-x}$. Substituting for the initial conditions, we have two equations

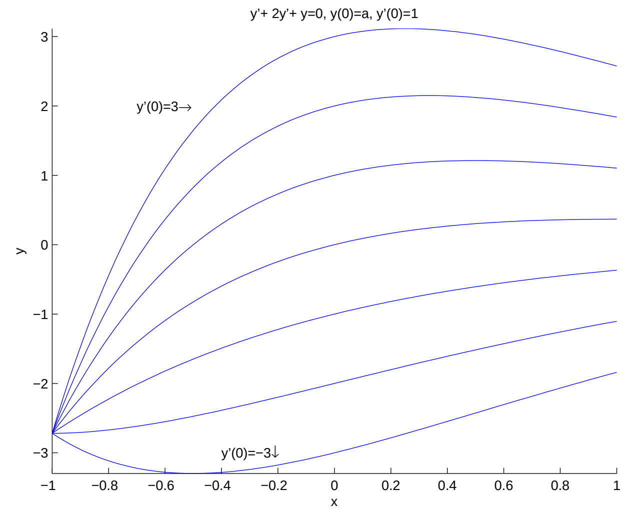
$$\begin{aligned} a &= c_1 \\ 1 &= -c_1 + c_2 \end{aligned}$$

So, $c_1 = a$ and $c_2 = a + 1$. Thus, the solution to the IVP is

$$ae^{-x} + (a+1)xe^{-x}.$$

Graphing changing parameters

Plotting: $ae^{-x} + (a+1)xe^{-x}$ for some values of a.



Complex roots

- Every quadratic equation has complex roots, although possibly no real roots. For example, the roots of $r^2 + 1$ are $\pm i$.
- If $a + bi$ is a root of a quadratic equation with real coefficients, then $\overline{a + bi} = a - bi$ is also a root. So, complex roots come in conjugate pairs.
- A polynomial with complex coefficients can have complex roots which are not conjugates:

$$x^2 + ix + 6 = (x - 2i)(x + 3i)$$

Linear independence

- We are now admitting complex-valued functions, so we need to extend the definition of linear independence to include complex numbers.
- Two complex-valued functions f_1 and f_2 are linearly dependent if there are complex constants c_1 and c_2, not both zero, such that

$$c_1 f_1 + c_2 f_2 = 0.$$

  The functions are linearly independent, if not linearly dependent.
- Two real-valued functions which are linearly independent over real linear combinations, are still linearly independent over complex linear combinations.

Complex solutions

- Suppose $a \pm bi$ are the roots of the characteristic equation for a second-order homogeneous differential equation.
- The functions
$$e^{(a+ib)x}, \qquad e^{(a-ib)x}$$
are linearly independent.
- The general solution to the differential equation is
$$y(x) = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$$
where c_1 and c_2 may take complex values.
- The problem is that we want real-valued solutions, and the general solution includes complex-valued solutions.

Complex solutions

Recall Euler's formula:
$$e^{x+iy} = e^x \left(\cos y + i \sin y \right)$$

Apply to our solution,
$$\begin{aligned} y(x) &= c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x} \\ &= e^{ax}\Big(c_1 \left(\cos(bx) + i \sin(bx) \right) + c_2 \left(\cos(-bx) + i \sin(-bx) \right) \Big) \\ &= e^{ax}\Big(c_1 \left(\cos(bx) + i \sin(bx) \right) + c_2 \left(\cos(bx) - i \sin(bx) \right) \Big) \\ &= e^{ax}\Big((c_1 + c_2) \cos(bx) + i(c_1 - c_2) \sin(bx) \Big) \end{aligned}$$
where c_1 and c_2 can be complex.

Complex solutions

The general solution over complex parameters c_1 and c_2 is
$$y(x) = e^{ax}\Big((c_1 + c_2) \cos(bx) + i(c_1 - c_2) \sin(bx) \Big)$$

- Let $A = c_1 + c_2$ and $B = i(c_1 - c_2)$, then the general solution is
$$\begin{aligned} y(x) &= e^{ax}\Big(A \cos(bx) + B \sin(bx) \Big) \\ &= Ae^{ax} \cos(bx) + Be^{ax} \sin(bx). \end{aligned}$$
- $Ae^{ax} \cos(bx) + Be^{ax} \sin(bx)$ are linearly independent, and real-valued when A and B are real-valued.
- Thus, every real-valued solution is given by
$$Ae^{ax} \cos(bx) + Be^{ax} \sin(bx),$$
where we restrict A and B to real-values.
- If we let A and B range over all complex values, we get all complex-valued solutions as well.

Complex solutions

Theorem (Theorem 3, Section 3.3)

If the characteristic equation (with real-coefficients)
$$ar^2 + br + c = 0$$
has complex conjugate roots $a \pm bi$ (where $b \neq 0$), then the homogeneous equation
$$ay'' + by' + cy = 0$$
has a real-valued general solution of the form
$$c_1 e^{ax} \cos(bx) + c_2 e^{ax} \sin(bx).$$

ConcepTest

Problem. FInd a general solution for the equation

$$y'' - 2y' + 5y = 0.$$

Answer. The characteristic equation is $r^2 - 2r + 5 = 0$, whose roots are $1 \pm 2i$. So, the general solution is

$$c_1 e^x \cos(2x) + c_2 e^x \sin(2x).$$

ConcepTest

Solve the IVP

$$y'' - 8y' + 17y = 0, \qquad y(0) = -4, y'(0) = 1$$

Answer. The characteristic equation = $r^2 - 8r + 17 = 0$, with roots $4 \pm i$. So, the general solution is

$$c_1 e^{4t} \cos x + c_2 e^{4t} \sin x.$$

Substituting

$$\begin{aligned} y(0) &= -4 = c_1 \\ y'(0) &= 1 = 4c_1 + c_2 \end{aligned}$$

So, $c_1 = -4$ and $c_2 = 17$. The solution is

$$-4e^{4t} \cos x + 17e^{4t} \sin x.$$

Higher-order equations

Consider a n-order homogeneous linear equation with constant coefficients $a_0, a_1, \ldots, a_n$

$$a_n y^{(n)} + \ldots a_2 y'' + a_1 y' + a_0 y = 0.$$

- As in the second-order case, if $y_1, \ldots, y_k$ is are solutions then so is any linear combination

$$c_1 y_1 + \ldots + c_k y_k.$$

- There will always be n linearly independent solutions to this equation. Furthermore, given n linearly independent solutions $y_1, \ldots, y_n$, any solution can be obtained by choice of constants $c_1, \ldots, c_n$ in

$$c_1 y_1 + \ldots + c_n y_n.$$

Solutions

Given an equation

$$a_n y^{(n)} + \ldots a_2 y'' + a_1 y' + a_0 y = 0,$$

we guess solutions should have the form e^{rx} and substitute:

$$\begin{aligned} 0 &= a_n (e^{rx})^{(n)} + \ldots a_2 (e^{rx})'' + a_1 (e^{rx})' + a_0 (e^{rx}) \\ &= a_n r^n e^{rx} + \ldots + a_2 r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} \\ &= e^{rx} \left(a_n r^n + \ldots + a_2 r^2 + a_1 r + a_0 \right) \end{aligned}$$

So, if r is a root of the characteristic equation

$$a_n r^n + \ldots a_2 r^2 + a_1 r + a_0 = 0,$$

then e^{rx} is a solution of the original equation.

Roots of polynomials

A degree n polynomial

$$a_n r^n + \dots a_2 r^2 + a_1 r + a_0 = 0,$$

has n roots (including multiplicities).

- If $r_1, \dots, r_k$ are the distinct real roots of this equation, then

$$e^{r_1 x}, e^{r_2 x}, \dots, e^{r_k x}$$

will be linearly independent solutions to the associated equation.
- If $a \pm bi$ is a conjugate pair of solutions (so, $b \neq 0$) then

$$e^{ax}\cos(bx), \quad e^{ax}\sin(bx)$$

will be linearly independent solutions to the associated equation.

Multiple roots

Suppose ρ is a root (real or complex) of the polynomial

$$a_n r^n + \dots a_2 r^2 + a_1 r + a_0 = 0,$$

occurring with multiplicity k. Then the following k functions are also solutions to the associated linear equation

$$e^{\rho x}, xe^{\rho x}, \dots, x^{k-1}e^{\rho x}$$

If $\rho = a + bi$ ($b \neq 0$) is a root with multiplicity k, then there are $2k$ linearly independent solutions (with conjugate pairs). These can be written using the real-valued functions

$$e^{ax}\cos(bx), xe^{ax}\cos(bx), \dots, x^{k-1}e^{ax}\cos(bx)$$
$$e^{ax}\sin(bx), xe^{ax}\sin(bx), \dots, x^{k-1}e^{ax}\sin(bx)$$

These $2k$ solutions are linearly independent.

Example

Solve the IVP

$$y^{(3)} - 5y'' - 22y' + 56y = 0, \quad y(0) = 1, y'(0) = -2, y''(0) = -4$$

The characteristic equation is

$$r^3 - 5r^2 - 22r + 56 = (r+4)(r-2)(r-7) = 0.$$

So, we have three distinct roots, and a general solution

$$y(t) = c_1 e^{-4x} + c_2 e^{2x} + c_3 e^{7t}.$$

Differentiate twice, and apply the initial conditions

$$\begin{aligned} 1 &= c_1 + c_2 + c_3 \\ -2 &= -4c_1 + 2c_2 + 7c_3 \\ -4 &= 16c_1 + 4c_2 + 49c_3 \end{aligned}$$

So, $c_1 = \frac{14}{33}$, $c_2 = \frac{13}{15}$, and $c_3 = -\frac{16}{35}$. The actual solution is

$$y(t) = \frac{14}{33}e^{-4x} + \frac{13}{15}e^{2x} - \frac{16}{35}e^{7t}.$$

Example

Find a general solution to the following equation

$$2y^{(4)} + 11y^{(3)} + 18y'' + 4y' - 8y = 0$$

The characteristic equation is

$$2r^4 + 11r^3 + 18r^2 + 4r - 8 = (2r-1)(r+2)^3 = 0.$$

So, we have roots $r = \frac{1}{2}$ and $r = -2$, which has multiplicity three. The general solution is

$$y(t) = c_1 e^{\frac{1}{2}x} + c_2 e^{-2x} + c_3 xe^{-2x} + c_4 x^2 e^{-2x}.$$

Example

Find a general solution to the following equation

$$y^{(5)} + 12y^{(4)} + 104y^{(3)} + 408y'' + 1156y' = 0$$

The characteristic equation is

$$r^5 + 12r^4 + 104r^3 + 408r^2 + 1156r = r(r^2 + 6r + 34)^2 = 0.$$

So, we have one real root $r = 0$ and a pair of complex conjugate roots $r = 3 \pm 5i$ of multiplicity two. The general solution is

$$y(t) = c_1 + c_2 e^{3x} \cos(5x) + c_3 e^{3x} \sin(5x) + c_4 x e^{3x} \cos(5x) + c_5 x e^{3x} \sin(5x).$$

Example

Find a general solution to the following equation

$$y^{(5)} - 15y^{(4)} + 84y^{(3)} - 220y'' + 275y' - 125y = 0$$

The characteristic equation is

$$r^5 - 15r^4 + 84r^3 - 220r^2 + 275r - 125 = (r-1)(r-5)^2(r^2 - 4r + 5) = 0.$$

So, we have roots $r = 1$, a double root $r = 5$and complex conjugates $r = 2 \pm i$. The general solution is

$$y(t) = c_1 e^x + c_2 e^{5x} + c_3 x e^{5x} + c_4 e^{2x} \cos x + c_5 e^{2x} \sin x.$$