Math 216
Matrix Algebra

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Course Data

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What is a differential equation?

1. A differential equation is an equation relating an unknown function and its derivatives.
2. A solution to a differential equation is a function which satisfies the equation

Example. $y = y(x)$ is the unknown function in the equation

$$\frac{dy}{dx} = 2x.$$

A solution is

$$y(x) = x^2 + 27,$$

which you can verify by substitution

$$\frac{d\left(x^2 + 27\right)}{dx} = 2x.$$

ConcepTest

Question. Which of the following are solutions to the equation

$$\frac{dy}{dx} = 2x?$$

(a) x^2
(b) $-x^2$
(c) $2x^2$
(d) $x^2 - 5$

Answer. (a) and (d).

Finding all solutions

Problem. Find all solutions to the following differential equation

$$\frac{dy}{dx} = 2x.$$

Solution. We want an unknown function whose derivative is $2x$.
Integrate: by integrating both sides of the equation

$$y(x) = \int \frac{dy}{dx} dx = \int 2x\, dx = x^2 + C.$$

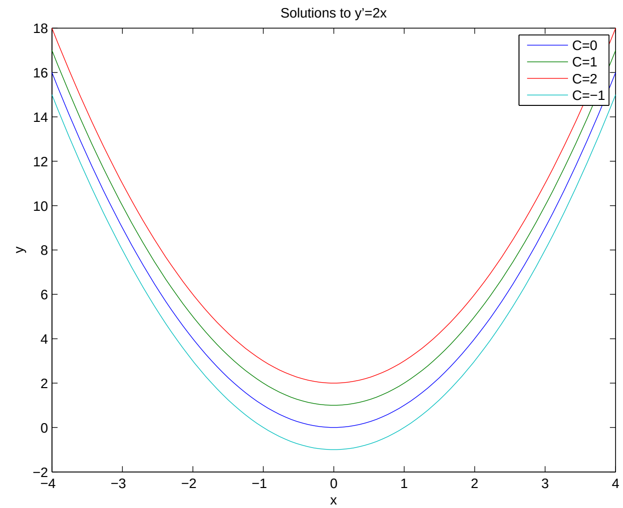
Verify. For each choice of the parameter C, $x^2 + C$ is a solution:

$$\frac{d(x^2 + C)}{dx} = 2x.$$

Important. These are all the solutions, since any two antiderivatives of $2x$ differ by a constant. .

Figure 1

Graph of some solutions, $x^2 + C$, to $\frac{dy}{dx} = 2x$. (See `fig1.m`.)



General Strategy: Section 1.2

Strategy. GIven any differential equation of the form

$$\frac{dy}{dx} = f(x),$$

we can solve the equation for $y = y(x)$ by integrating:

$$y_C(x) = \int \frac{dy}{dx} dx = \int f(x)\, dx + C.$$

The functions $y_C = y_C(x)$ are all the solutions to the original equation.
(Since any two antiderivatives of $f(x)$ differ by a constant.)

ConcepTest

Problem. Find all solutions $y = y(x)$ to the differential equation

$$y''(x) = a$$

where a is some constant real number.
Solution. Integrate both sides of the equation, twice.

$$\begin{aligned} y'(x) &= \int y''(x)\, dx = \int a\, dx = ax + B \\ y(x) &= \int y'(x)\, dx = \int (ax + B)\, dx = \frac{1}{2}ax^2 + Bx + C \end{aligned}$$

For each choice of value to the parameters B and C

$$y(x) = \frac{1}{2}ax^2 + Bx + C$$

is a solution to the equation $y''(x) = a$.

Ordinary Differential Equations

Equations. In Chapter 1 we will be interested in first-order ordinary differential equations (ODEs) of the form

$$\frac{dy}{dx} = F(x, y).$$

- first-order means that only the first derivative of the unknown function appears in the equation.
- ordinary means that the unknown function ($y = y(x)$ here) has a single independent variable (x here).

ConcepTest

Question. Which of the following ODEs can be put into the form

$$\frac{dy}{dx} = F(x, y).$$

(a) $y' = 2x$
(b) $y'' = a$
(c) $xy' = y$
(d) $yy' = x^2$

Answer. (a), (c), (d).
(b) is an example of a second-order ODE, an equation involving a second-order derivative.

General solutions

A general solution to the ODE

$$\frac{dy}{dx} = F(x, y).$$

is a collection of functions $y_C = y_C(x)$. For each choice of the parameter C, y_C is a solution to the ODE:

$$\frac{d(y_C(x))}{dx} = F(x, y_C(x)).$$

ConcepTest

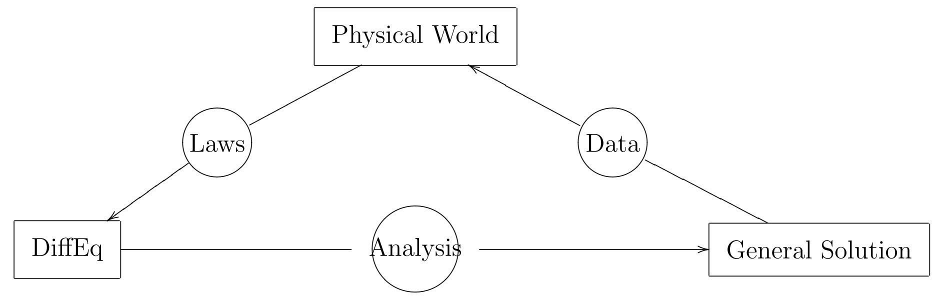
Problem. FInd a general solution for the ODE

$$\frac{dy}{dx} = \sin x$$

Solution. Integrate.

$$y(x) = \int \frac{dy}{dx} = \int \sin x + C = -\cos x + C.$$

- Many real-world problems involve the change over time of some physical property.
- Scientific laws postulate the rate of change (a derivative!) as a differential equation (which provides the mathematical model for the real-world problem).
- Mathematicians analyze differential equations, providing a general solution for the equation.
- The scientist must then provide additional data to determine which solution is the correct one.



Compound Interest

Problem. Bob and Alice place \$5000 in a savings account that pays 8% annual interest compounded continuously for their child's tuition to Michigan. If they allow the interest payments to accumulate, how much will Junior have for tuition at age 18?

- Let $A(t)$ be the amount in the savings account. We are given $A(0) = 5000$ and we want $A(18)$.
- Continuous compounding at interest rate r (here, $r = 0.08$) means

$$\Delta A = rA(t)\Delta t$$

  where Δt is a very small lapse in time.
- So, the instantaneous rate of change in the account is

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = rA.$$

Compound Interest

Analysis. We want a solution $A = A(t)$ to the ODE and initial value

$$\frac{dA}{dt} = 0.08A, \qquad A(0) = 5000.$$

Then $A(18)$ will be the savings Junior will have after 18 years.
Solution. The general solution is

$$A_C(t) = Ce^{0.08t}.$$

However, we want the solution which satisfies $A(0) = 5000$. Solve for C using the initial value:

$$5000 = A(0) = Ce^{0.08 \times 0} \qquad \text{so} \qquad C = 5000.$$

Thus, the particular solution is $A(t) = 5000e^{0.08t}$.
So, Junior has $A(18) \approx \$21,103.50$.

Natural growth equation

Many physical problems require solutions $y = y(x)$ to an ODE of the form

$$\frac{dy}{dt} = ky \quad \text{for some real-value } k.$$

The rate of growth of the quantity y is proportional to the value of y. (Examples of y are growth of savings due to interest, population growth when due to birth rates and death rates, radioactive decay.)

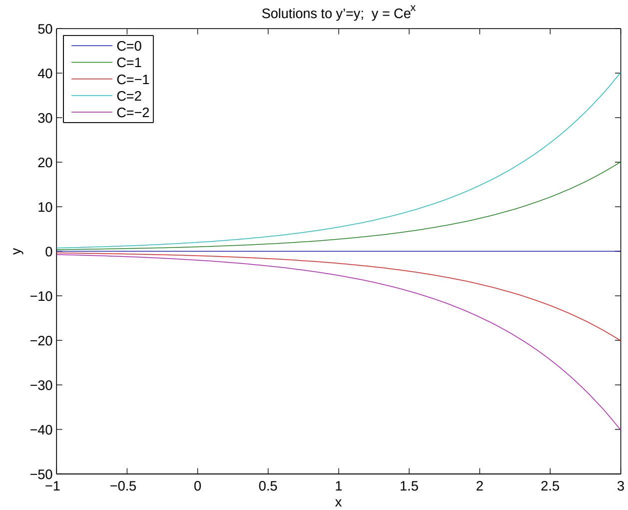
The general solution is

$$y_C(t) = Ce^{kt}.$$

We can solve this ODE by the method of separation of variables (Section 1.4). This is Friday's lecture.

Figure 2

Example. Some solutions $y(t) = Ce^t$ for $y' = y$. (See `fig2.m`.)



Initial value problems

- An Initial Value Problem (IVP) is an ODE together with an initial value the unknown function must satisfy

$$\frac{dy}{dx} = F(x, y), \qquad y(x_0) = y_0.$$

- To solve an IVP
  1. Find a general solution to the ODE, $y_C = y_C(x)$.
  2. Use the initial value of the unknown function to find the correct value for C.
- We need to know that
  - (i) there exists a solution $y = y(x)$, and
  - (ii) the solution is uniquely determined

  at least on some interval which include the initial value $x = x_0$.

ConcepTest

Question. What is the particular solution for the following IVP

$$\frac{dy}{dx} = 2x \qquad \text{and} \qquad y(10) = 113?$$

(a) $y(x) = 2x^2 - 87$

(b) $y(x) = x^2 + 13$

(c) $y(x) = x + 103$

Answer. (b).

Solving Chapter 1 ODEs

- **Problem.** Find IVP solutions for the following ODE:

$$y' = y^2 - x.$$

  where we pick initial values (x, y) in the rectangle $-2 \le x \le 10$ and $-4 \le y(x) \le 4$.
- We cannot apply integration since the right-side contains the unknown function $y = y(x)$. (In fact, none of the techniques in Sections 1.4 or 1.5 will work here.)
- Since we cannot produce an explicit solution, $y = y(x)$, we must resort to other strategies.
  1. Graphical solutions provide qualitative information about the shape of graphs of solutions to the ODE. (Section 1.3)
  2. Numerical solutions for computing the numeric value of a solution at a given point to as great an accuracy as desired. (Sections 2.4-2.6)

Graphical solutions for ODEs

Here is how we can get a graphical picture of the solution space.

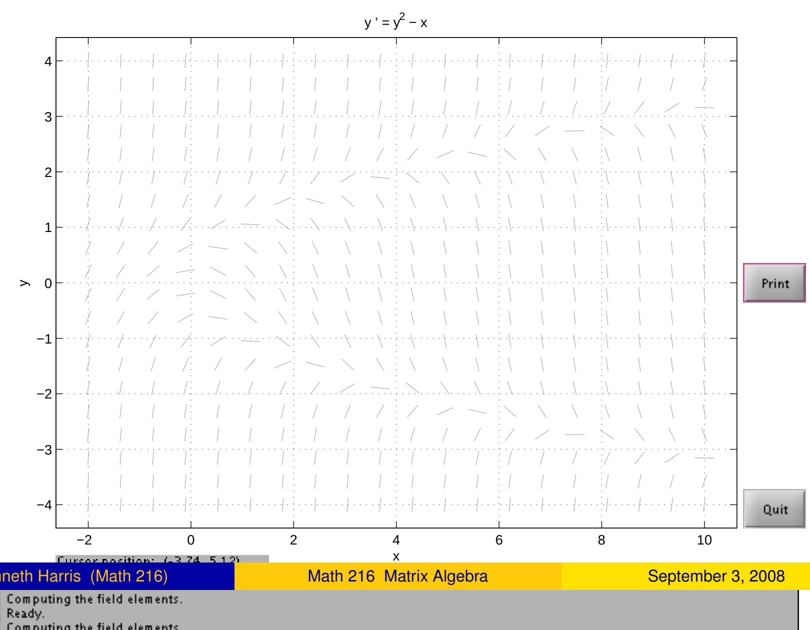
- Notice that

$$y' = y^2 - x.$$

  tells us that the slope, $\frac{dy}{dx}$ of the function $y = y(x)$ at the point (x, y) is $y^2 - x$.
- We can get a picture of the solution space initial values lying in a rectangle.
  1. Select representative points in a region where the function $y^2 - x$ is defined.
  2. Draw a short segment at each point (x, y) whose slope is $y^2 - x$.
- This picture is called a direction field (or slope field.)
- You can produce direction fields in *Matlab* by using `dfield7`.

Figure 3: Direction Field

There are 400 representative points in the region $-2 \le x \le 10$, $-4 \le y \le 4$.



Solution Curves

- We can sketch the graph of a solution (called the solution curve) for various points (x_0, y_0) of

$$\frac{dy}{dx} = y^2 - x, y(x_0) = y_0$$

  1. Draw a curve tangent to the line segment at (x_0, y_0) whose slope is $y_0^2 - x_0$.
  2. Extend the curve to be tangent to all representative lines whose midpoint it passes through in the direction field. (Recall, in a small neighborhood around $(x, y(x))$, the function y looks like the line with slope $y'(x) = y^2 - x$.)
- For each choice of intitial value (x_0, y_0), we obtain a different solution curve.

Figure 4: Solution Curves

Solution curves for the initial values $y(0) = 0$, $y(1) = 1$, and $y(2) = 2$.

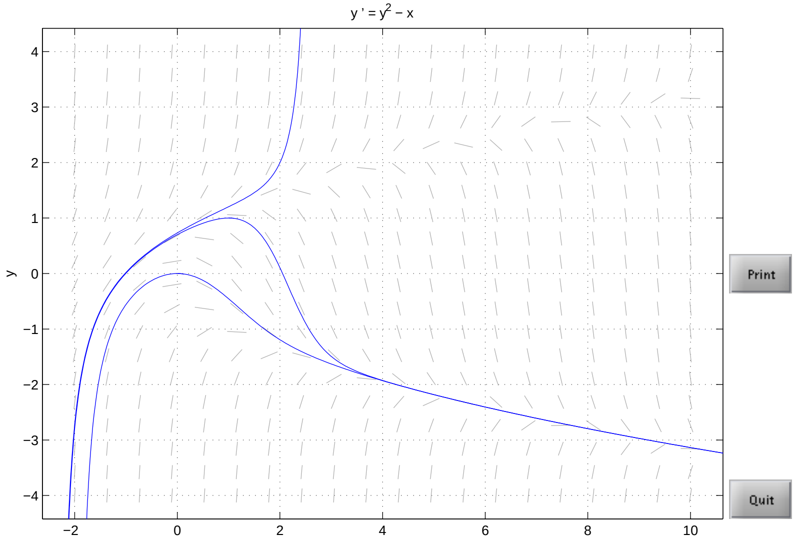
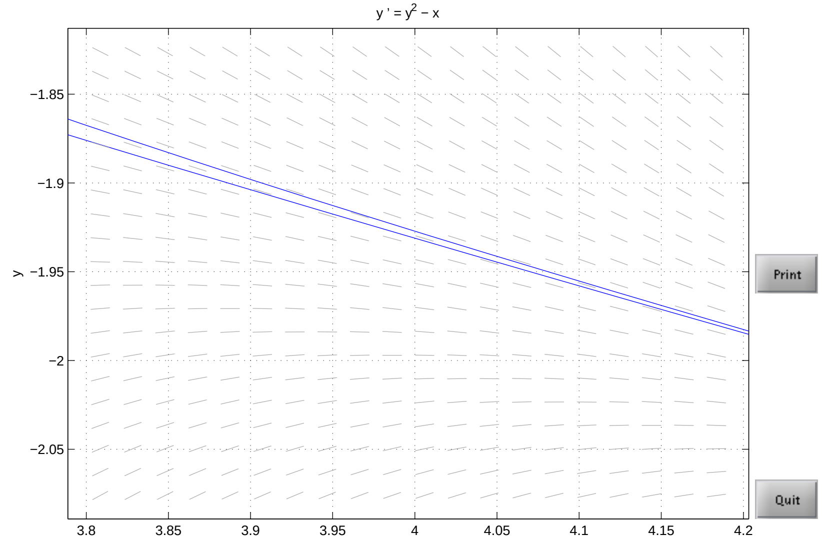


Figure 5: Zoom in at $(4, -2)$

Zooming in at $(4, -2)$. The two solution curves do not make contact.



Existence and Uniqueness Theorem

$$\frac{dy}{dx} = F(x, y) \qquad \text{and} \qquad y(a) = b$$

When does this IVP have a unique solution?

Theorem (Theorem 1, Section 1.3)

In any rectangle R in which the function $F(x, y)$ is nicely behaved, the IVP

$$\frac{dy}{dx} = F(x, y), \qquad y(a) = b$$

has exactly one solution on some open interval I containing a.

F is nicely behaved on a rectangle R if

(a) *F is continuous on R,*

(b) *$\frac{\partial F}{\partial y}$ is continuous on R.*

ConcepTest

Question. Which of the following functions are nicely behaved in the rectangle $-2 \le x \le 2$ and $-2 \le y \le 2$.

(a) $\frac{dy}{dx} = 2\sqrt{y}$ at $(0, 0)$.

(b) $xy' - y = 0$ at $(0, 0)$.

(c) $\frac{dy}{dx} = x \ln y$ at $(1, 1)$.

Answer.

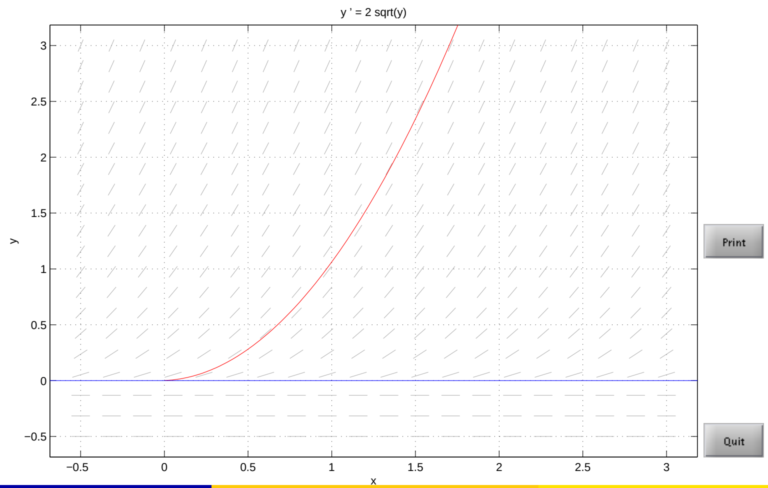
(a) Not nice. $\frac{\partial\left(2\sqrt{y}\right)}{\partial y} = \frac{1}{\sqrt{y}}$ is not continuous at $(0, 0)$.

(b) Not nice. $\frac{y}{x}$ is not continuous at $(0, 0)$.

(c) Nice.

ConcepTest (a)

Both $y_1(x) = 0$ and $y_2(x) = x^2$ are solutions of $\frac{dy}{dx} = 2\sqrt{y}$ through $(0, 0)$. These are the only two solutions through $(0, 0)$. The general solutions is $y(x) = (x + C)^2$ (solved by separation of variables, Section 1.4.)



ConcepTest (b)

The general solution for $xy' - y = 0$ is $y_C(x) = Cx$. There are infinitely many solutions which pass through $(0, 0)$, and none which pass through $(0, b)$ when $b \neq 0$.