Saturated Bounding Degrees

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Saturated Models

(*Vaught*) **countably saturated models**: Models which realize all types consistent over some finite subdomain of the model.

(*Millar, Morley*) A **complete**, **decidable** (CD) theory has a **decidable saturated model** iff there is a computable enumeration of the types of the theory.

(*Millar*) There is a complete, decidable (CD) theory with **types all computable** (TAC) which has no decidable saturated model.

(*Millar, Goncharov/Nurtazin*) Every CD theory with TAC has a $\mathbf{0}'$-decidable saturated model.

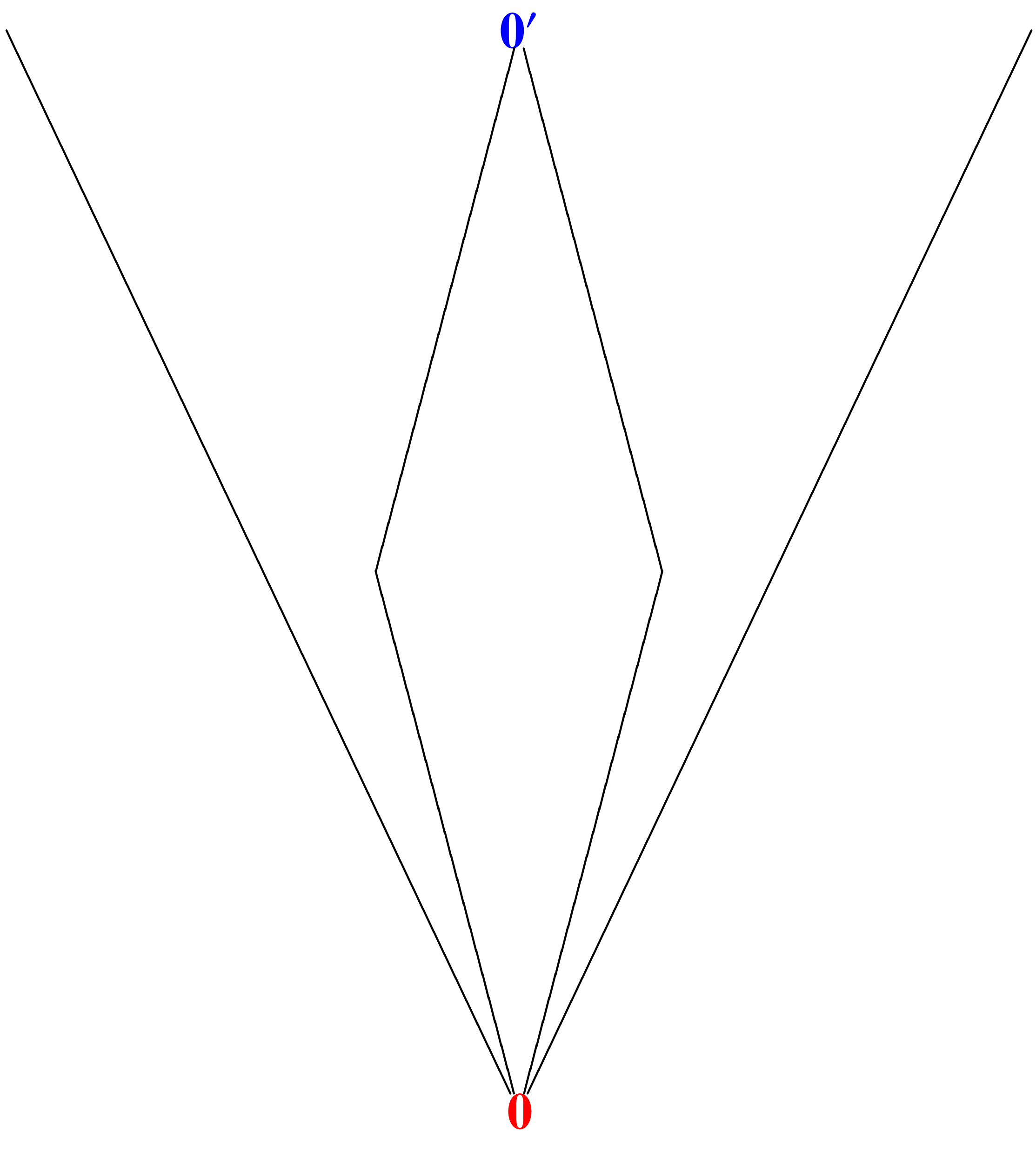
Saturated Bounding Degrees

Def. Degree $\mathbf{d}$ is *saturated bounding* if for any CD theory T with TAC, there is a saturated $\mathfrak{A} \models T$ with $D^e(\mathfrak{A}) \leq_T \mathbf{d}$.

Question. Which Turing degrees are saturated bounding?

Saturated Bounding Degrees

Positive Degrees Negative Degrees



PAC Π^0_1 Classes

- An *extendable* tree is one without terminal nodes.
- A *PAC* tree is an extendable computable tree whose paths are all computable.
- A *PAC* Π^0_1 class is $[\mathcal{T}]$, for some PAC tree $\mathcal{T}$.

Lemma. The types of a CD theory with TAC form a PAC Π^0_1 class.

Enumerating Paths in a Π^0_1 Class

- A Turing degree **d** has an *enumeration* of a class $C \subset 2^\omega$ if there is a **d**-computable function of two arguments, $\lambda nx.F_n(x)$ such that

$$C = \{\lambda x.F_n(x)\}_{n\in\omega}$$

- A Turing degree **d** has a *subenumeration* of a class $C \subset 2^\omega$ if there is a **d**-computable function of two arguments, $\lambda nx.F_n(x)$ such that

$$C \subset \{\lambda x.F_n(x)\}_{n\in\omega}$$

Lemma. For any extendable computable tree, *subenumerating* $[\mathcal{T}]$ is equivalent to *enumerating* $[\mathcal{T}]$.

Saturated Bounding and Enumerations

Thm. (*Millar*, *Morley*) TFAE

(a) **d** is saturated bounding.

(b) There is a **d**-computable enumeration of the types for any CD theory with TAC.

(c) There is a **d**-computable enumeration of each PAC Π^0_1 classes.

Question. Which Turing degrees can enumerate each PAC Π^0_1 class?

Upper Bound: *high* degrees

A degree $\mathbf{d}$ is *high* if $\mathbf{d}' \geq \mathbf{0}''$.

Thm. (*Jockusch*) Any *high* degree can enumerate the computable sets.

Thm. Any *high* degree can enumerate every PAC Π^0_1 class.

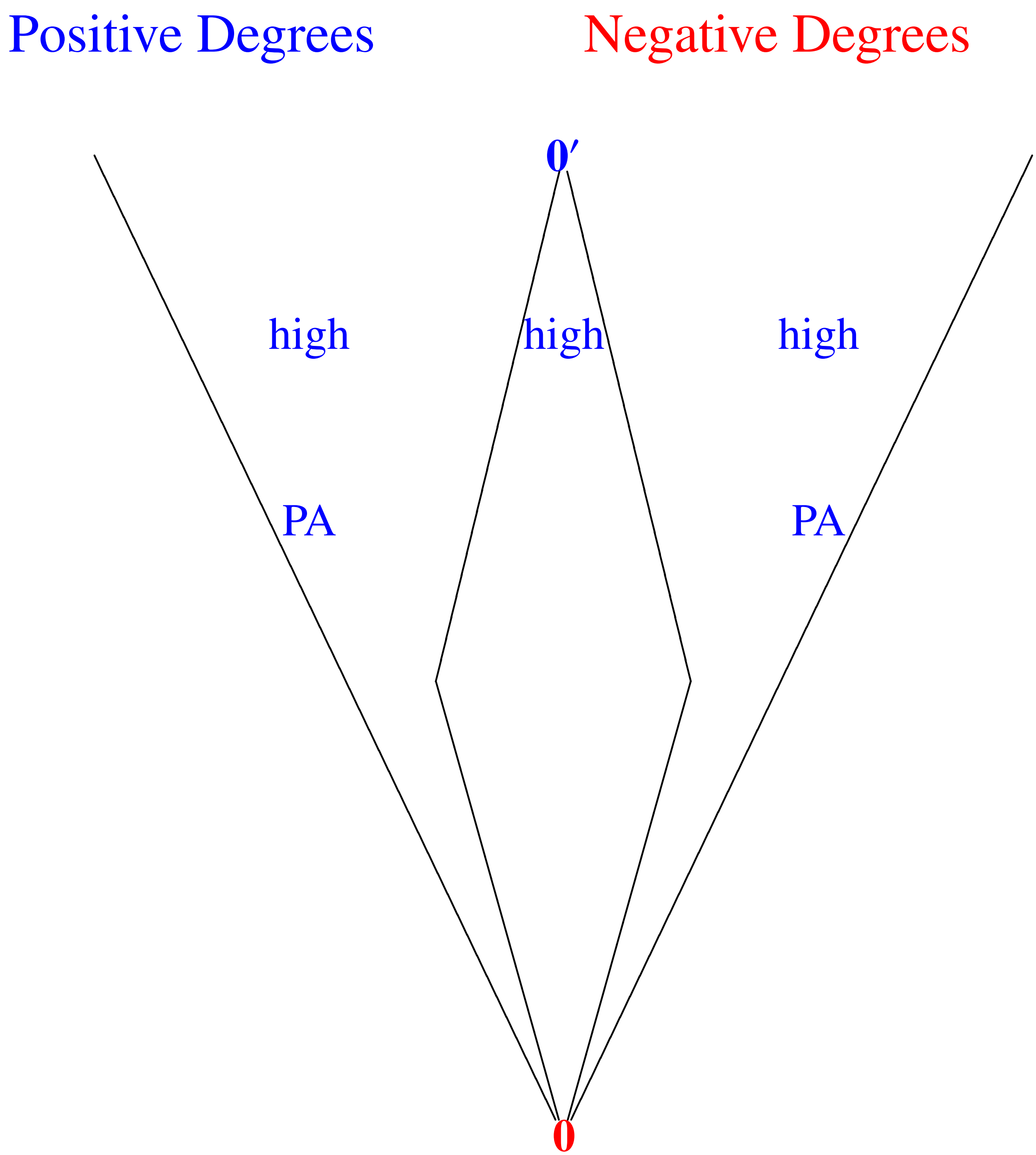
Upper Bound: *PA* degrees

A degree **d** is a *PA degree* if **d** computes a completion of Peano Arithmetic.

Thm. (*Jockusch*) Any *PA* degree can subenumerate the computable sets.

Thm. Any *PA* degree can enumerate every PAC Π^0_1 class.

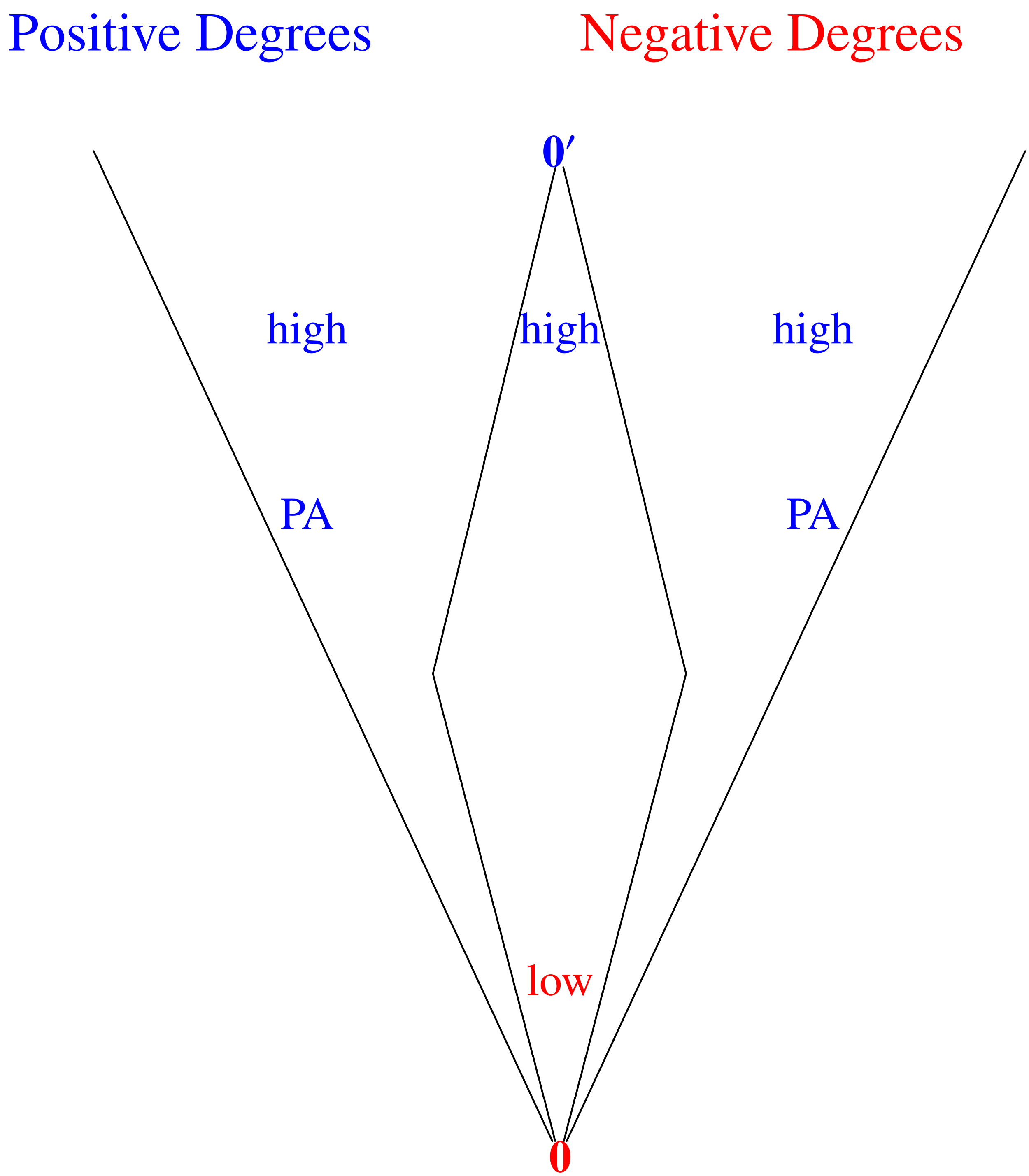
Saturated Bounding Degrees



New Negative Result

Thm 1. No *low c.e.* degree is saturated bounding.

Saturated Bounding Degrees

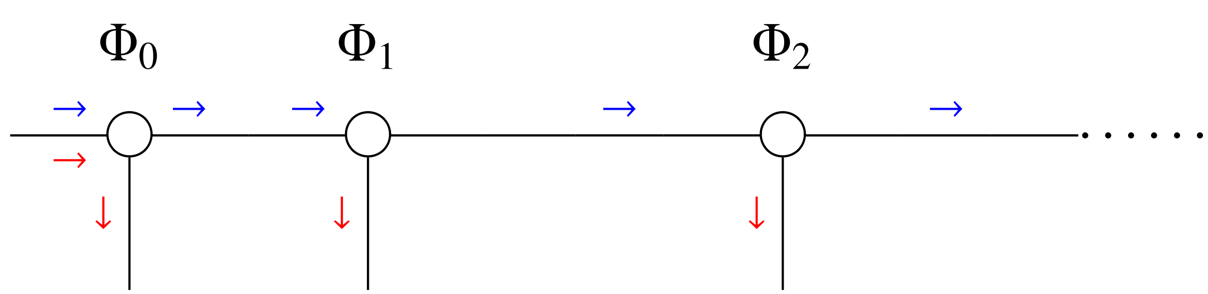


Millar's Construction

Against Computable Opponent

Opponent: $\{\Phi_n\}$ $\longrightarrow$

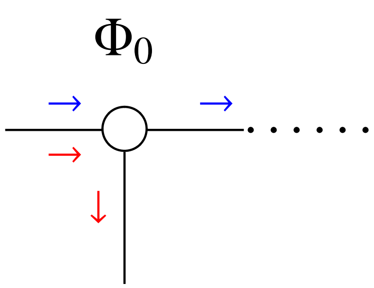
Our Avoiding Path: $\longrightarrow$



Difficulties with Construction

Against Noncomputable Opponent

Opponent: $\{\Phi_n\}$ $\longrightarrow$
Our Avoiding Path: $\longrightarrow$



Difficulties with Construction

Against Noncomputable Opponent

Opponent: $\{\Phi_n\}$ $\longrightarrow$
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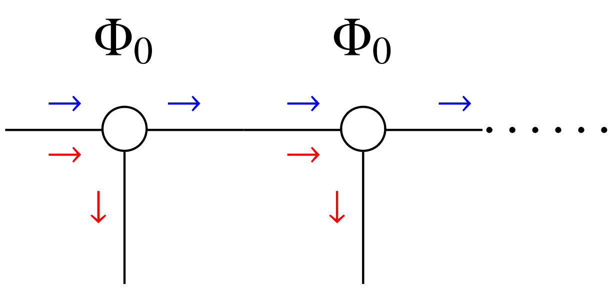
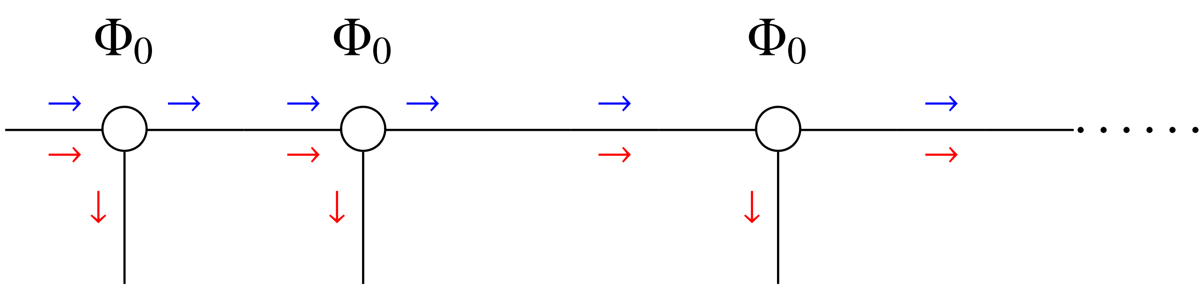


Difficulties with Construction

Against Noncomputable Opponent

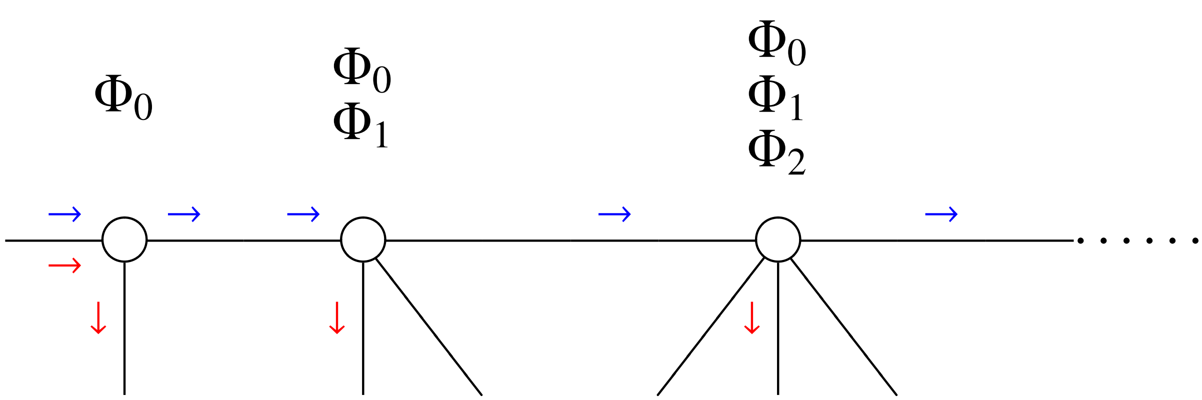
Opponent: $\{\Phi_n\}$ $\longrightarrow$
Our Avoiding Path: $\longrightarrow$



Modified Construction

Against Noncomputable Opponent

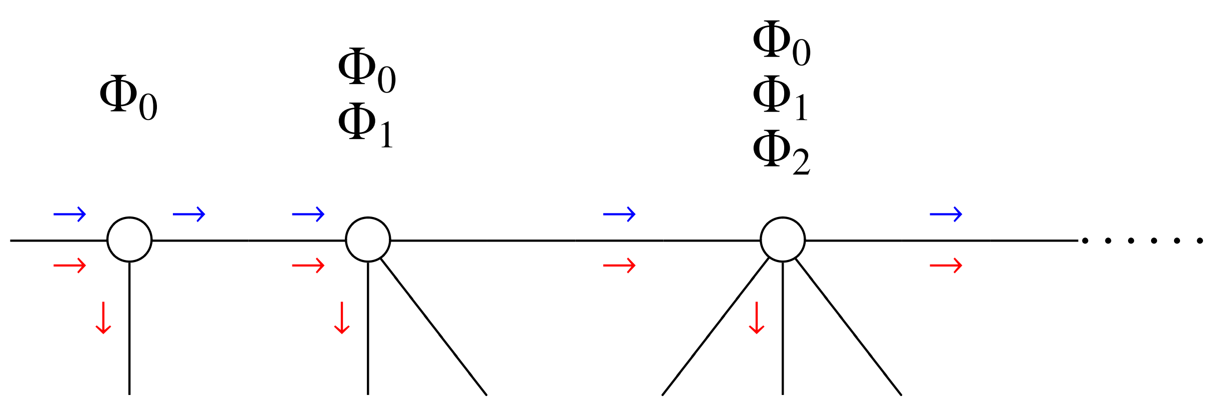
Opponent: $\{\Phi_n\}$ $\longrightarrow$
Our Avoiding Path: $\longrightarrow$



Modified Construction

Against Noncomputable Opponent

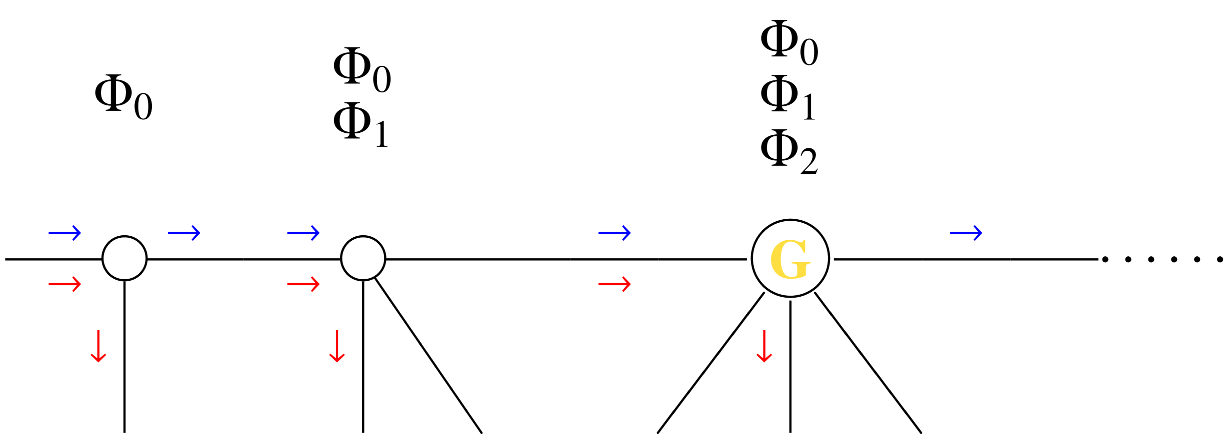
Opponent: $\{\Phi_n\}$ $\longrightarrow$
Our Avoiding Path: $\longrightarrow$



Modified Construction

Against Noncomputable Opponent

Opponent: $\{\Phi_n\}$ $\longrightarrow$
Our Avoiding Path: $\longrightarrow$



Escape Property

Def. A degree $\mathbf{d}$ has the *Escape Property* if

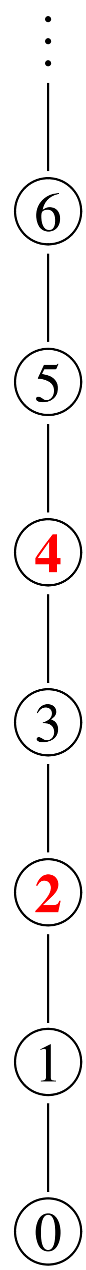
$$(\forall \Phi \leq_T \mathbf{d})(\exists f \leq_T \mathbf{0})(\exists^\infty x)[\Phi(x) \leq f(x)]$$

The computable functions *escape* domination from any $\mathbf{d}$-computable function.

Thm. (*Martin*) All *nonhigh* degrees have the Escape Property.

Escape Property

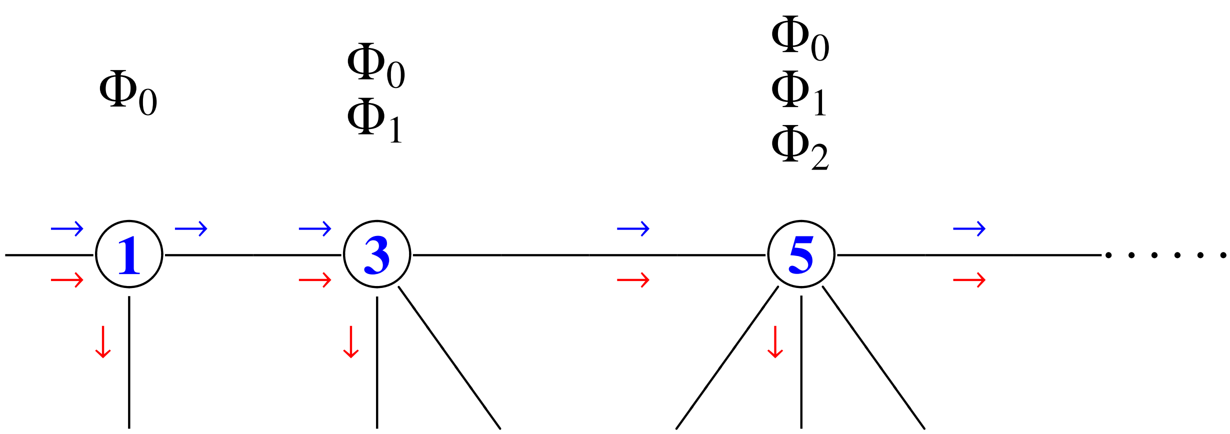
Escaping Values



Review: Modified Construction

Needed Nodes on Path

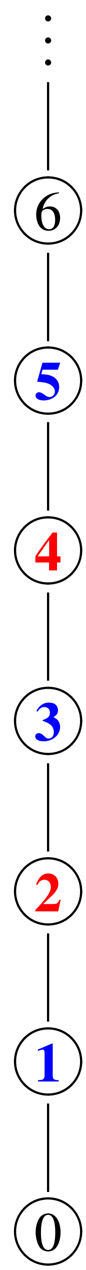
Opponent: $\{\Phi_n\}$ $\longrightarrow$
Our Avoiding Path: $\longrightarrow$



Escape Property

Escaping Values

Values Needed by construction



Uniform Escape Property

Def. A degree **d** has the *Uniform Escape Property* if there is an $h \leq_{\mathrm{T}} \mathbf{0}$ such that

$$(\forall e)(\exists^{\infty} x)[\Phi_e^{\mathbf{d}}(x) \leq \Phi_{h(e)}(x)]$$

Escape functions for **d**-computable functions can be found effectively.

Thm 2. For *c.e.* degrees **d**, the following are equivalent

(a) **d** is *low*.

(b) **d** has the Uniform Escape Property

Aligned Escape Property

Def. A degree $\mathbf{d}$ has the *Aligned Escape Property* if for any $g \leq_T \mathbf{0}$ where $\varphi_{g(e)}$ is total provided φ_e is total,

$$(\forall \Phi \leq_T \mathbf{d})(\exists e)(\exists^\infty x)[\Phi(\varphi_{g(e)}(x)) \leq \varphi_e(x)]$$

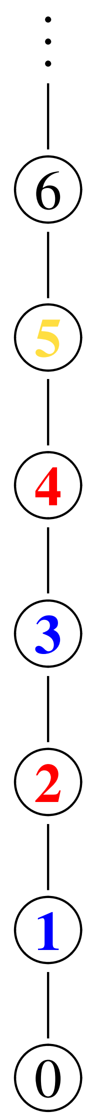
Lemma. No degree with the Aligned Escape Property is saturated bounding.

Aligned Escape Property

Escaping Values

Values Needed by construction

Aligned Values



Extending Negative Results

A degree **d** is *low*$_n$ if $\mathbf{d}^{(n)} = \mathbf{0}^{(n)}$, for some $n \in \omega$.

Thm 3. All *low*$_n$ *c.e.* degrees have the Aligned Escape Property.

Cor. No *low*$_n$ *c.e.* degree is saturated bounding.

Saturated Bounding Degrees

Positive Degrees

Negative Degrees