A Characterization of the low$_n$ Degrees

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Dominate and Escape

Let $f, g : \mathbb{N} \to \mathbb{N}$.

- f *dominates* g if

$$(\forall^\infty x)\left[f(x) > g(x)\right]$$

  f is a *dominant function* if f dominates every computable function.

- g *escapes* (domination from) f if

$$(\exists^\infty x)\left[f(x) \leqslant g(x)\right]$$

  f has an *escape function* if there is a computable g which escapes domination by f.

Martin's Characterization

Theorem (*Martin*, 1966) Let $\mathbf{a}$ be a Turing degree.

- $\mathbf{a}$ is high $\left(\mathbf{a}' \geqslant \mathbf{0}''\right)$ *iff* there is an $\mathbf{a}$-computable dominant function:

$$(\exists f \leqslant \mathbf{a})(\forall g \leqslant \mathbf{0}) \left[f \text{ dominates } g\right]$$

- $\mathbf{a}$ is non-high $\left(\mathbf{a}' < \mathbf{0}''\right)$ *iff* every $\mathbf{a}$-computable function has an escape function:

$$(\forall f \leqslant \mathbf{a})(\exists g \leqslant \mathbf{0}) \left[g \text{ escapes } f\right]$$

Uniform Escape Property

Question: For what non-high degrees can escape functions be *effectively* produced?

Definition: A degree **a** has the *Uniform Escape Property* (UEP), or (1-UEP), when for any set $A \in \mathbf{a}$:

> There is a partial computable $\lambda ex.h_e(x)$ such that whenever Φ_e^A is total, then
>
> $$h_e \text{ total and escapes } \Phi_e^A$$

Recall, h_e *escapes* Φ_e^A if

$$(\exists^\infty x)\left[\Phi_e^A(x) \leqslant h_e(x)\right]$$

UEP Equivalent to low$_1$

Theorem: For all degrees $\mathbf{a}$ TFAE

(A) $\mathbf{a}$ is low$_1$ $\left(\mathbf{a}' \leqslant \mathbf{0}'\right)$.

(B) $\mathbf{a}$ has the Uniform Escape Property.

low$_n$ **Degrees and Escape Functions**

There is a hierarchy of properties characterized by progressively less effective procedures, *n-Uniform Escape Property* (n-UEP), starting with (1-UEP)=(UEP), such that

Theorem: For all degrees $\mathbf{a}$ and all $n \geqslant 1$ TFAE

(A) $\mathbf{a}$ is low$_n$ $\left(\mathbf{a}^{(n)} \leqslant \mathbf{0}^{(n)}\right)$.

(B) $\mathbf{a}$ has (n-UEP).

Quantifiers on Steroids

$(\forall^\infty x)$: *For almost every x.*
Reduces to $\exists\forall$ and behaves like $\forall$.

$(\exists^\infty x)$: *There exists infinitely many x.*
Reduces to $\forall\exists$ and behaves like $\exists$.

Fundamental Relations

$$\exists^\infty P \iff \neg\forall^\infty\neg P$$

$$\forall \implies \forall^\infty \implies \exists^\infty \implies \exists$$

Theorem (**Strong Normal Form**):
The arithmetic hierarchy is characterized by alternations of the two strongest quantifiers, $\forall$ and $\forall^\infty$.

low_1 Degrees and the 1-Uniform Escape Property

low$_1$ **implies Uniform Escape Property**

Theorem: All low$_1$ sets A have (1-UEP):

There is a partial computable function $\lambda ex.h_e(x)$ such that whenever Φ_e^A is total, then

$$h_e \text{ total and escapes } \Phi_e^A$$

The Key Idea

Let A be low$_1$, so $\Pi_2^A \subset \Pi_2$.

Want: Computable g such that for each total Φ_e^A, $W_{g(e)}$ satisfies

$$\textbf{(escape)} \qquad (\exists^\infty x)(\exists s)\left[\Phi_{e,s}^A(x)\downarrow \leqslant s \;\;\&\;\; x \notin W_{g(e),s}\right]$$

$$\textbf{(total)} \qquad W_{g(e)} = \omega$$

Define:

$$h_e(x) = (\mu s)\left[x \in W_{g(e),s}\right]$$

Problem: How to match (**escape**) with (**total**)?

Strong Normal Form: Π_2, Σ_2

Normal Form: For $V \in \Pi_2$, there is some v (the Π_2 index for V) with

$$V(e) \iff (\forall y)\left[\langle e, y\rangle \in W_v\right]$$

Strong Normal Form (**SNF**): There is a computable g, such that for any $V \in \Pi_2$ with index e

$$\begin{aligned} V(e) &\implies \left[W_{g(v,e)} = \omega\right] \\ \neg V(e) &\implies \left[W_{g(v,e)} \text{ finite }\right] \end{aligned}$$

Implementation of Key Idea

Let A be low$_1$ (thus $\Pi_2^A \subset \Pi_2$).

Define Π_2^A predicate (**escape**)

$$(\exists^\infty x)(\exists s)\left[\Phi_{e,s}^A(x)\downarrow \leqslant s \;\;\&\;\; x \notin W_{g(v,e),s}\right]$$

where g is the computable function given by (**SNF**) from a Π_2 *index v for* (**escape**):

$$\begin{array}{rcl} (\textbf{escape}) & \Longrightarrow & \left[W_{g(v,e)} = \omega\right] \\ \neg(\textbf{escape}) & \Longrightarrow & \left[W_{g(v,e)} \text{ finite }\right] \end{array}$$

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then define

$$h_e(x) = (\mu s)\left[x \in W_{g(v,e),s}\right]$$

low$_2$ Degrees and the 2-Uniform Escape Property

2-Uniform Escape Property: First Change

Definition: A set A is low$_2$ if $A'' \leqslant \mathbf{0}''$.

With low$_2$ we add one jump class and one layer of quantifier complexity.

Our *first change* in defining (2-UEP):

> There are uniformly enumerable (u.e.) families of partial computable functions $\lambda e.\{h_{e,y}\}_{y\in\omega}$ such that whenever Φ_e^A is total, then
>
> $$(\forall^\infty y)\left[h_{e,y} \text{ total and escapes } \Phi_e^A\right]$$

2-Uniform Escape Property

Definition: A degree **a** has the *2-Uniform Escape Property* (2-UEP), when for any set $A \in \mathbf{a}$:

There are uniformly enumerable (u.e.) families of partial computable functions $\lambda e.\{h_{e,y}\}_{y\in\omega}$ such that for any u.e. family of functions $\{\Phi^A_{e,y}\}_{y\in\omega}$ satisfying

$$(\forall^\infty y)\ [\Phi^A_{e,y} \text{ total }]$$

then

$$(\forall^\infty y)\ \left[h_{e,y} \text{ total and escapes } \Phi^A_{e,y}\right]$$

low$_2$ **Equivalent to 2-UEP**

For all degrees $\mathbf{a}$ TFAE

(A) $\mathbf{a}$ is low$_2$ $\left(\mathbf{a}'' \leqslant \mathbf{0}''\right)$.

(B) $\mathbf{a}$ has the 2-Uniform Escape Property.

Strong Normal Form: Σ_3, Π_3

If A is low$_2$ then $\Sigma_3^A \subseteq \Sigma_3$.

Strategy of Proof: Pump (**escape**) property (Π_2^A) with strong quantifiers to Σ_3^A and exploit weakness of A.

$$(\forall^\infty y)(\exists^\infty x)(\exists s)\left[\Phi_{e,y,s}^A(x)\downarrow \leqslant s \ \ \& \ \ x \notin W_{g(v,e,y),s}\right]$$

Strong Normal Form (**SNF**): There is a computable g, such that for any $V \in \Sigma_3$ with index e

$$\begin{array}{rcl} (\textbf{escape}) & \Longrightarrow & (\forall^\infty y)\left[W_{g(v,e,y)} = \omega\right] \\ \neg(\textbf{escape}) & \Longrightarrow & (\forall y)\left[W_{g(v,e,y)} \text{ finite }\right] \end{array}$$

low$_3$ Degrees and Beyond

3-Uniform Escape Property

A degree **a** is low$_3$ if $\mathbf{a}''' = \mathbf{0}'''$.

Definition: A degree **a** has the *3-Uniform Escape Property* (3-UEP) when for any set $A \in \mathbf{a}$:

There are uniformly enumerable (u.e.) families of partial computable functions $\lambda e.\{h_{e,y_1,y_2}\}_{y_1,y_2\in\omega}$ such that for any u.e. family of functions $\{\Phi^A_{e,y_1,y_2}\}_{y_1,y_2\in\omega}$ satisfying

$$(\exists^\infty y_2)(\forall^\infty y_1)\ [\Phi^A_{e,y_1,y_2} \text{ total }]$$

then

$$(\exists^\infty y_2)(\forall^\infty y_1)\left[h_{e,y_1,y_2} \text{ total and escapes } \Phi^A_{e,y_1,y_2}\right]$$

low_3 Equivalent to 3-UEP

Theorem: For all degrees **a** TFAE

(A) **a** is low_3 $\left(\mathbf{a}''' \leqslant \mathbf{0}'''\right)$.

(B) **a** has the 3-Uniform Escape Property.

Strong Normal Form: Π_4, Σ_4

If A is low$_3$ then $\Pi_4^A \subseteq \Pi_4$.

Strategy of Proof: Pump (**escape**) property (Π_2^A) with strong quantifiers to Π_4^A and exploit weakness of A.

$$(\exists^\infty y_2)(\forall^\infty y_1)(\exists^\infty x)(\exists s)\left[\Phi^A_{e,y_1,y_2,s}(x)\downarrow\leqslant s \right.$$
$$\left. \& \quad x \notin W_{g(v,e,y_1,y_2),s}\right]$$

Strong Normal Form (**SNF**): There is a computable g, such that for any $V \in \Pi_4$ with index e

$$(\textbf{escape}) \implies (\forall y_2)(\forall^\infty y_1)\left[W_{g(v,e,y_1,y_2)} = \omega\right]$$
$$\neg(\textbf{escape}) \implies (\forall^\infty y_2)(\forall y_1)\left[W_{g(v,e,y_1,y_2)} \text{ finite }\right]$$

n-Uniform Escape Property

A degree $\mathbf{a}$ is low$_n$ if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$.

Definition: A degree $\mathbf{a}$ has the *n-Uniform Escape Property* (n-UEP) when for any set $A \in \mathbf{a}$:

> There are uniformly enumerable (u.e.) families of partial computable functions $\lambda e.\{h_{e,\bar{y}}\}_{\bar{y}\in\omega}$ such that for any u.e. family of functions $\{\Phi^A_{e,\bar{y}}\}_{\bar{y}\in\omega}$ satisfying
>
> $$(Q_1 y_{n-1})(Q_2 y_{n-2})\ldots\ [\Phi^A_{e,\bar{y}} \text{ total}\ ]$$
>
> then
>
> $$(Q_1 y_{n-1})(Q_2 y_{n-2})\ldots\ \left[h_{e,\bar{y}} \text{ total and escapes } \Phi^A_{e,\bar{y}}\right]$$

where $Q_1, Q_2 \in \{\exists^\infty, \forall^\infty\}$ by

- For *odd* n: alternate $\exists^\infty\forall^\infty$
- For *even* n: alternate $\forall^\infty\exists^\infty$

low$_n$ **Equivalent to** n**-UEP**

Theorem: For all degrees **a** TFAE

(A) **a** is low$_n$ $\left(\mathbf{a}^{(n)} \leqslant \mathbf{0}^{(n)}\right)$.

(B) **a** has the n-Uniform Escape Property.

Strong Normal Form Theorem

Strong Normal Form Theorem (**SNF**) (with $n \geqslant 1$)
All arithmetic formulas equivalent to formulas using only the *beefiest* quantifiers $\{\forall, \forall^\infty\}$:

For any $V \in \Sigma_{2n+1}$ with index v there is a computable g, such that

$$\begin{aligned} V(e) &\implies (\forall^\infty y_{2n-1})(\forall y_{2n-2})\ldots \left[W_{g(v,e,\bar{y})} = \omega\right] \\ \neg V(e) &\implies (\forall y_{2n-1})(\forall^\infty y_{y2n-2})\ldots \left[W_{g(v,e,\bar{y})} \text{ finite }\right] \end{aligned}$$

For any $U \in \Pi_{2n}$ with index u there is a computable g, such that

$$\begin{aligned} U(e) &\implies (\forall y_{2n-2})(\forall^\infty y_{2n-3})\ldots \left[W_{g(u,e,\bar{y})} = \omega\right] \\ \neg U(e) &\implies (\forall^\infty y_{2n-3})(\forall y_{2n-3})\ldots \left[W_{g(u,e,\bar{y})} \text{ finite }\right] \end{aligned}$$

Application of Escape Functions

Bounding Saturated Models

Theorem: There is a complete decidable theory $\mathcal{T}$ whose types are all computable, which has *no* saturated model of low$_n$ c.e. degree for any n.

Bibliography

My work:

Kenneth Harris, ”A Characterization of the low$_n$ Degrees using Escape Functions, preprint at **people.cs.uchicago.edu/∼kaharris/papers/lown.pdf**

Kenneth Harris, ”On Bounding Saturated Models”, preprint at **people.cs.uchicago.edu/∼kaharris/papers/sat.pdf**