On Bounding Saturated Models

Kenneth Harris *

June 22, 2006

Contents

*I would like to thank Pavel Semukhin and Rod Downey for their reading of an early version of this paper. This work has been undertaken with the assistance of my advisor Professor Robert Soare and Professor Denis Hirschfeldt.

Abstract

A degree is *saturated bounding* if it can compute a saturated model for any complete, decidable theory whose types are all computable. All high and PA degrees are saturated bounding (following from results of Jockusch and MacIntyre-Marker.) We show that for every n, no low$_n$ c.e. degree is saturated bounding, extending the previous known result that 0 is not saturated bounding, by Millar.

1 Introduction

Computable model theory introduces effective analogues of classical model-theoretic notions to investigate the effective content of constructions in model theory. For example, when does a complete decidable theory with a countable saturated model have a decidable presentation of the saturated model. A model is *decidable* if the elementary diagram of the theory is computable. The types of a theory with a decidable saturated model will all be computable, since each type will have to be realized in the model. The classical theorem is that a complete theory has a saturated model only when the theory has countably many types in n-variables for every $n \in \omega$. The proof is a construction of the model where the assumption is that all types of the theory are readily available and questions of the consistency of two types with overlapping variables can be immediately decided. In one of the earliest results in computable model theory, Morley and independently Millar (Theorem 2.2), showed that the classical construction can be made to work to produce a decidable saturated model, provided there was an effective listing of all types of the theory.

The ability to list the types of the theory is essential for constructing the saturated model. Millar (Theorem 2.3) also produced a counterexample of a complete, decidable theory whose types are all computable no decidable saturated model, although there does exist a saturated model. A natural follow-up question for a computability theorist is *which Turing degrees can compute a listing of the types of any complete, decidable theory whose types are all computable*? These will be degrees which can compute a presentation of the saturated model for any such theory, and we will call such degrees *saturated bounding degrees.* Another early result in computable model theory is that $0'$ is saturated bounding (due to Millar and independently Goncharov-

Nurtazin (Theorem 2.4).) MacIntyre-Marker later showed that the degrees of complete extensions of Peano Arithmetic are saturated bounding (Corollary 3.4)[1]. The high degrees are also saturated bounding (Corollary 3.2) following a result of Jockusch on the computational complexity of enumerating computable sets.

We will extend Millar's construction showing 0 is not saturated bounding to the low_n c.e. degrees (Theorem 5.9). The means to extend the construction was found through a finer analysis of Martin's classic characterization of the non-high degrees by the existence of *escape functions*. (The non-high degrees are those in which for every function computable in the degree there is a computable function which escapes domination from it.) The analysis of Martin's characterization introduced here is also behind a new characterization of the low_n degrees we provide in [Har].

2 Preliminaries

2.1 Model Theory

We consider presentations $\mathfrak{A}$ with universe $A \subseteq \omega$ over a language $\mathcal{L}_{\mathfrak{A}}$ and set of true sentences in this language, $\mathrm{Th}(\mathfrak{A})$. For $X \subset \omega$, let $\mathcal{L}_{\mathcal{X}}$ be the language $\mathcal{L} \cup \{\mathbf{a} : a \in X\}$ and $\mathfrak{A}_X$ be the expansion of the model $\mathfrak{A}$ by constants $\mathbf{a}$ for $a \in X$, where $\mathbf{a}$ is interpreted by the number a in the domain of $\mathfrak{A}$. The elementary diagram of $\mathfrak{A}$ is $\mathrm{Th}(\mathfrak{A}_A)$, the set of true sentences of $\mathfrak{A}$ in the language expanded by constants for the elements of $\mathfrak{A}$.

Let $\mathcal{L}$ be a computable language. Sentences and formulas of $\mathcal{L}$ will be identified with their Gödel numbers, so that sets of these syntactic entities are subsets of the natural numbers. A *theory* $\mathcal{T}$ is a set of sentences of $\mathcal{L}$, and is *complete* if for any sentence φ, either $\varphi \in \mathcal{T}$ or $\neg\varphi \in \mathcal{T}$ and *consistent* if there is some sentence which is not a consequence of $\mathcal{T}$. A set of formulas Γ whose free variables are among $\{x_0, \ldots, x_{m-1}\}$ is *maximal* if for any formula γ in these variables, either $\gamma \in \Gamma$ or $\neg\gamma \in \Gamma$. A set of formulas Γ is *consistent*

[1] We would like to thank David Marker for pointing out this consequence of [MM84].

over theory $\mathcal{T}$, if for every finite $\{\gamma_1, \ldots, \gamma_k\} \subseteq \Gamma$, the expanded theory

$$\mathcal{T} \cup \{(\exists x_0 \ldots x_{m-1})(\bigwedge_{i=1}^{k} \gamma_i)\}$$

is consistent . An *n-type* of $\mathcal{T}$ is a *maximal* and *consistent* set of formulas in the variables $\{v_0, \ldots, v_{n-1}\}$ (the first n variables in the language $\mathcal{L}$.) The set of all n-types over $\mathcal{T}$ is denoted by $S_n(\mathcal{T})$ and the set of all types over $\mathcal{T}$ by $S(\mathcal{T})$.

A model $\mathfrak{A}$ in language $\mathcal{L}$ is *saturated* if for every finite $X \subset A$, $\mathfrak{A}_X$ realizes every type of $\mathrm{Th}(\mathfrak{A}_X)$ over $\mathcal{L}_X$. Saturated models for a theory $\mathcal{T}$ are internally the most complicated models for the theory, in the sense that any other (countable) model of $\mathcal{T}$ can be embedded in the saturated model. A saturated model for a theory $\mathcal{T}$ exists whenever there are only countably many types consistent with $\mathcal{T}$.

2.2 Computable Model Theory

We follow [Har98] for computable model theory.

A theory is *decidable* if it is a computable set of sentences. A type is decidable if it is a computable set of formulas. A model is *decidable* if its elementary diagram is a computable set of sentences. For a Turing degree $\mathbf{d}$, a model is $\mathbf{d}$-*decidable* if its elementary diagram is a $\mathbf{d}$-computable set of sentences.

I will use the abbreviation CD for a *complete* and *decidable* theory, and the abbreviation TAC for a CD theory whose types are all computable.

Definition 2.1. *Call a degree $\boldsymbol{d}$ saturated bounding if for any TAC theory, there is a $\boldsymbol{d}$-decidable saturated model for that theory.*

The following relativizes a result due independently to Morley and Millar:

Theorem 2.2 ([Mor76],[Mil78])**.** *For any degree $\boldsymbol{d}$, a TAC theory has a $\boldsymbol{d}$-decidable saturated model if and only if there is a $\boldsymbol{d}$-computable enumeration of the types of the theory.*

0 is not saturated bounding

Theorem 2.3 ([Mil78]). *There is a TAC theory which has no decidable saturated model.*

but $0'$ is saturated bounding

Theorem 2.4 ([Mil78], [GN73]). *Any saturated model of a TAC theory has a $0'$-computable presentation.*

2.3 Trees and Π^0_1 Classes

It is more direct to work with paths through trees, then traditional model-theoretic entities like theories, types and models. A *tree* $\mathscr{T}$ is a subset of $2^{<\omega}$ closed under the $\subseteq$ relation: if $\sigma \in \mathscr{T}$ and $\tau \subseteq \sigma$ then $\tau \in \mathscr{T}$. A node $\sigma \in \mathscr{T}$ is a *terminal node* if there is no $\tau \in \mathscr{T}$ with $\sigma \subset \tau$; An *extendable* tree is one with no terminal nodes. A tree is computable if it is computable as a subset of $2^{<\omega}$ (under some fixed computable representation of finite sequences as natural numbers.)

A *path* through a tree $\mathscr{T}$ is a function $f : \omega \mapsto \{0, 1\}$ such that for every n, $f \restriction n \in \mathscr{T}$. A path is also the characteristic function of a subset of the natural numbers. The collection of all paths through a tree $\mathscr{T}$ is denoted by $[\mathscr{T}]$. If a tree $\mathscr{T}$ is computable, then its class of paths $[\mathscr{T}]$ is definable by a Π^0_1 formula

$$f \in [\mathscr{T}] \iff (\forall n)\ f{\restriction}n \in \mathscr{T}$$

and is said to be a Π^0_1 class.

An *enumeration* of a class of sets is a function listing the members of the set. A *subenumeration* of a class of sets is an enumeration of a superclass. More formally, an *enumeration* of a class $\mathcal{C} \subseteq 2^\omega$ is a function $\lambda nx.F_n(x)$ such that

$$\mathcal{C} = \left\{\lambda x.F_n(x)\right\}_{n\in\omega},$$

and a *subenumeration* of the class $\mathcal{C}$ is a function $\lambda nx.F_n(x)$ such that

$$\mathcal{C} \subseteq \left\{\lambda x.F_n(x)\right\}_{n\in\omega}.$$

For a Turing degree $\mathbf{d}$, a class has a $\mathbf{d}$-computable enumeration if there is a $\mathbf{d}$-computable function which enumerates the class. A class has a $\mathbf{d}$-computable subenumeration if there is a **d**-computable function which subenumerates the class.

For Π^0_1 classes generated by extendable trees, the degrees of enumerations is the same as that of subenumerations:

Lemma 2.5. *For any extendable and computable tree $\mathscr{T}$ and Turing degree $\boldsymbol{d}$, if $[\mathscr{T}]$ has a $\boldsymbol{d}$-computable subenumeration then $[\mathscr{T}]$ has a $\boldsymbol{d}$-computable enumeration.*

Proof. Since $\mathscr{T}$ is computable and extendable, through any node $\sigma \in \mathscr{T}$ there is a computable path in $[\mathscr{T}]$: for example, the left-most path passing through σ, which takes the 0-branch whenever there is a choice of branches. Fix a **d**-computable subenumeration F of $[\mathscr{T}]$, and define an enumeration G as follows: for each function F_n in the subenumeration, follow F_n as a path through $\mathscr{T}$ until $F_n \restriction m \in \mathscr{T}$ but $F_n \restriction m + 1 \notin \mathscr{T}$ (if this ever occurs), then instead of continuing on F_n, follow the left-most path extending $F_n \restriction m$ through $\mathscr{T}$. Since F is a subenumeration of $[\mathscr{T}]$, G contains all paths in $[\mathscr{T}]$, but it also contains only paths in $[\mathscr{T}]$ by the construction. Finally, G is **d**-computable, since F is and the left-most path extending any node $\sigma \in \mathscr{T}$ is computable. □

2.4 Theories and Π^0_1 Classes

The purpose of this section is to show the connection between the types of a theory and Π^0_1 classes. Let $\mathcal{T}$ be a complete, decidable theory in language $\mathcal{L}$. Let $F_n(\mathcal{T})$ be the set of formulas in variables $\{v_0, \ldots, v_{n-1}\}$ (the first n variables of $\mathcal{L}$) and $S_n(\mathcal{T})$ the class of n-types of $\mathcal{T}$.

Theorem 2.6 ([Sch60])**.** *Let $\mathcal{T}$ be a complete, decidable theory. Then there is a computable and extendable tree $\mathscr{T}$ such that $S_n(\mathcal{T}) = [\mathscr{T}]$.*

Proof. Let $\gamma_0, \gamma_1, \ldots$ be an effective enumeration of $F_n(\mathcal{T})$. Identify the formula γ_i with its index i. Let $\gamma^0 = \neg\gamma$ and $\gamma^1 = \gamma$. The idea is to define a tree $\mathscr{T}$ by associating the ith formula with the ith level of $\mathscr{T}$, the 0-branch with the negation of the formula and the 1-branch with the formula itself. Then a node $\sigma \in \{0,1\}^k$ on $\mathscr{T}$ is a finite subset which contains exactly one of a formula or its negation for each of the first k formulas, and which is consistent with $\mathcal{T}$. Given a finite sequence $\sigma \in \{0,1\}^{<\omega}$, let $\sigma \in \mathscr{T}$ if and only if

$$(\exists v_0, \ldots, v_{n-1})\big(\bigwedge_{i \in \mathrm{dom}(\sigma)} \gamma_i^{\sigma(i)} \big) \in \mathcal{T}.$$

Since $\mathcal{T}$ is decidable, $\mathscr{T}$ is computable. For any finite consistent set X of formulas and formula γ, one of $X \cup \{\gamma\}$ or $X \cup \{\neg\gamma\}$ is consistent, so $\mathscr{T}$ is also extendable. Each type $\Gamma \in S_n(\mathcal{T})$ corresponds to a path $f_\Gamma \in [\mathscr{T}]$ given by $\gamma_i \in \Gamma \implies f_\Gamma(i) = 1$ and else $f_\Gamma(i) = 0$. In addition, every path $g \in [\mathscr{T}]$ describes a maximal and consistent set of formulas, Γ_g, given by $f(i) = 1 \implies \gamma_i \in \Gamma_g$ and else $\neg\gamma_i \in \Gamma_g$. The consistency of Γ_g follows by compactness, as every initial segment of g describes a finite, consistent set. Thus, $S_n(\mathcal{T}) = [\mathscr{T}]$. □

Let $S(\mathcal{T})$ be the set of all types over theory $\mathcal{T}$. There is in fact, a single tree $\mathscr{T}$ whose paths include all n-types for each n, plus a single computable path, the back-bone. Let the back-bone be 1^ω, then for each n, let 1^n0 be the root of the tree giving the n-types from the previous theorem. The importance of this is that there is a computable transformation from $S(\mathcal{T})$ to $[\mathscr{T}]$.

A converse of this theorem also holds:

Theorem 2.7 ([Ehr61])**.** *Let $\mathscr{T}$ be a computable tree. Then there is a complete, decidable theory $\mathcal{T}$ with a 1-1 correspondence between $S_1(\mathcal{T})$ and $[\mathscr{T}]$ which preserves computational complexity. The n-types of $\mathcal{T}$ consists of combinations of n 1-types whose complexity is no more than the maximum complexity of the n 1-types.*

Proof. The language of $\mathcal{T}$ is equality together with $\{P_n(\cdot) : n \in \omega\}$, countably many unary predicate symbols. Let $P_k^0(x) = \neg P_k(x)$ and $P_k^1(x) = \neg P_k(x)$. For each $\sigma \in \{0,1\}^{<\omega}$

$$\gamma_\sigma(x) = \bigwedge_{i \in \mathrm{dom}(\sigma)} P_i^{\sigma(i)}(x)$$

Each predicate $P_k(x)$ corresponds to level k of $\mathscr{T}$, and the formula $\gamma_\sigma(x)$ will be used to say $\sigma \in \mathscr{T}$. The key to ensuring completeness is to specify the number of elements realizing each formula $\gamma_\sigma(x)$ for $\sigma \in \mathscr{T}$. The theory described will specify that there are infinitely many such x. The axioms of $\mathcal{T}$ consist of two groups:

Axiom I: $(\forall x)\neg\gamma_\sigma(x)$ for every $\sigma \in \{0,1\}^{<\omega}$ and $\sigma \notin \mathscr{T}$

Axiom II: $(\exists x_0,\ldots,x_{n-1})\big(\bigwedge_{i\neq j} x_i \neq x_j \ \wedge\ \bigwedge_i \gamma_\sigma(x_i)\big)$ for every $\sigma \in \mathscr{T}$ and $n \geq 1$

$\mathcal{T}$ is decidable from its characterization. It is straightforward to verify that $\mathcal{T}$ is finitely satisfiable using the fact that $\mathscr{T}$ is a tree (namely, if $\sigma \in \mathscr{T}$ then for each $\tau \subseteq \sigma$, $\tau \in \mathscr{T}$.) $\mathcal{T}$ is also complete (and in fact admits elimination of quantifiers). See [Har98, Lemma 7.8, Lemma 7.9] for a proof of this. The 1-types of $\mathcal{T}$ corresponds to paths through $\mathscr{T}$: If $f \in [\mathscr{T}]$ then the corresponding 1-type is the completion of $\{x = x\} \cup \{P_k^{f(k)}(x)\}_{k\in\omega}$. Since $\mathcal{T}$ is decidable, this 1-type has the same computational complexity as f. Similarly, if Γ is a 1-type, the function g defined by $g(k) = 1$ if $P_k(x) \in \Gamma$ and $g(k) = 0$ if $\neg P_k(x) \in \Gamma$ defines a path through $\mathscr{T}$. The n-types consist of n 1-types $\Gamma_i(x_i)$ and $x_i = x_j$ iff $\Gamma_i(x_i) = \Gamma_j(x_j)$. □

These two theorems justify our study of enumerations in Π^0_1 classes to investigating saturated-bounding degrees. A PAC Π^0_1 class is a Π^0_1 class generated from a computable and extendible tree whose paths are all computable.

Corollary 2.8. *The saturated-bounding degrees are precisely those degrees which can compute an enumeration of the paths in any PAC Π^0_1 class.*

2.5 Computability

We follow [Soa87] as a reference for computability theory.

We will use two additional quantifiers besides $\{\forall, \exists\}$

- $(\forall^\infty y)\ P$ to assert *for almost every y, P*
- $(\exists^\infty y)\ P$ to assert *for infinitely many y, P*

with the following relations to the standard quantifiers $\forall$ and $\exists$

$$\begin{aligned}(\forall^\infty y)\ P &\iff (\exists x)(\forall y)\ \big[y \geq x \implies P\big]\\ (\exists^\infty y)\ P &\iff (\forall x)(\exists y)\ \big[y \geq x\ \ \&\ \ P\big].\end{aligned}$$

The quantifier $\forall^\infty$ behaves similarly to $\forall$ with respect to the propositional connectives, and $\exists^\infty$ behaves similarly to $\exists$. The logical relations between the quantifiers is given by

$$\forall \implies \forall^\infty \implies \exists^\infty \implies \exists.$$

The following summarizes some key logical laws for these quantifiers to which we will appeal

Lemma 2.9. *In the following P, Q are any relations, and R is a relation in which the variable x does not occur free.*

$$(\exists^\infty x)P \iff \neg(\forall^\infty x)\,\neg P \tag{a}$$
$$(\forall^\infty x)P \wedge (\forall^\infty x)Q \iff (\forall^\infty x)\,\big[P \wedge Q\big] \tag{b}$$
$$R \rightarrow (\forall^\infty x)P \iff (\forall^\infty x)\,\big[R \rightarrow P\big] \tag{c}$$
$$(\forall^\infty x)P \rightarrow R \iff (\exists^\infty x)\,\big[P \rightarrow R\big] \tag{d}$$
$$(\forall^\infty x)\,\big[P \rightarrow Q\big] \implies (\forall^\infty x)P \rightarrow (\forall^\infty x)Q \tag{e}$$
$$(\forall^\infty x)P \wedge (\exists^\infty x)Q \implies (\exists^\infty x)\,\big[P \wedge Q\big] \tag{f}$$
$$(\forall^\infty x)P \wedge (\forall^\infty x)Q \implies (\forall^\infty x)\,\big[P \wedge Q\big] \tag{g}$$
$$(\forall^\infty x)P \wedge (\forall x)Q \implies (\forall^\infty x)\,\big[P \wedge Q\big] \tag{h}$$
$$(\exists^\infty x)P \wedge (\forall x)Q \implies (\exists^\infty x)\,\big[P \wedge Q\big] \tag{i}$$

When $\mathcal{Q}_1, \mathcal{Q}_2 \in \big\{\forall, \exists, \forall^\infty, \exists^\infty\big\}$ we will write

$$(\mathcal{Q}_1 y_n)(\mathcal{Q}_2 y_{n-1}) \ldots\ P$$

for the assertion obtained by alternating the quantifiers $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in turn, starting with $\mathcal{Q}_1$.

Let $\omega = \big\{0, 1, 2, \ldots\big\}$. We write $\big\{\varphi_e\big\}_{e\in\omega}$ for a standard enumeration of the partial computable functions, and define

$$W_e = \big\{x \in \omega \mid \varphi_e(x)\downarrow\big\}$$

where $\varphi_e(x)\downarrow$ means $x \in \operatorname{dom}\varphi_e$. Fix some computable pairing function $\langle\cdot\rangle$ so that for every k-tuple of natural numbers $y_1, \ldots, y_k$, $\langle y_1, \ldots, y_k\rangle \in \omega$. We will write $\overline{y}$ for $\langle y_1, \ldots, y_k\rangle$ where the natural numbers $y_1, \ldots, y_k$ and arity k

are determined by the context. Any c.e. k-ary relation R can be represented by some c.e. set W_e using the pairing function:

$$R(y_1, \ldots, y_k) \iff \langle y_1, \ldots, y_k \rangle \in W_e$$

We will say in this case that the c.e. relation R has Σ_1-*index* e.

We take Σ_n and Π_n (standardly written Σ^0_n and Π^0_n) to be the usual arithmetic classes of sets defined by quantifier complexity (see [Soa87, Section IV.1].) If A is any Σ_{2n+1} relation then there is a c.e. set W_e such that

$$A(x) \iff (\exists y_{2n-1})(\forall y_{2n-2}) \ldots (\exists y_1)(\forall z) \left[\langle x, z, y_1, \ldots, y_{2n-1} \rangle \in W_e \right]$$

and we will take e to be a Σ_{2n+1}-*index* of A. (The important point is that there are $2n$ alternating quantifiers starting with $\exists$ and ending with $\forall$.) If $B \in \Pi_{2n+1}$ then a Π_{2n+1}-*index* for B will be a a Σ_{2n+1}-index for $\neg B$. Similarly, if A is any k-ary Π_{2n} relation then there is a c.e. set W_e such that

$$A(x) \iff (\forall y_{2n-2})(\exists y_{2n-3}) \ldots (\exists y_1)(\forall z) \left[\langle x, z, y_1, \ldots, y_{2n-2} \rangle \in W_e \right]$$

and we will take e to be a Π_{2n}-*index* for A. (What is important is that there are $2n - 1$ alternating quantifiers beginning with $\forall$ and ending with $\forall$.) If $B \in \Sigma_{2n}$ then a Σ_{2n}-*index* for B will be a a Π_{2n}-index for $\neg B$. (See [Rog87, §14.2] for the *index* notation.)

We write $\left\{\Phi^X_e\right\}_{e\in\omega}$ for a standard enumeration of the partial computable functionals For any set A and any A-computable function h, there is an index e such that Φ^A_e is f. (See [Soa87, Section III.1].)

Let $A \leq_{\mathrm{T}} 0'$. There is a computable sequence of computable sets $\left\{A_s\right\}_{s\in\omega}$ such that

$$A = \lim_s A_s.$$

If Φ^A_e is total, then for every $x \in \omega$

$$\Phi^A_e(x) = \lim_s \Phi^{A_s}_{e,s}(x)$$

(see [Soa87, Limit Lemma, III.3.3].) A *modulus* for an A-computable function Φ^A_e, is a function m which has the property that for every $x \in \omega$

$$\Phi^A_e(x) = \Phi^{A_s}_{e,s}(x) \qquad \text{whenever } s \geq m(x).$$

(A modulus function for Φ_e^A is always with respect to some computable approximation of A.)

If W is a c.e. set, then there is a *standard enumeration* of W, a computable enumeration of computable sets $\{W_s\}_{s\in\omega}$, such that $W_s \subseteq W_t$ whenever $s \le t$ and $W = \lim_s W_s$. Relative to this standard enumeration, for each W-computable function Φ_e^W, we will take the function $\hat{\Phi}_e$ to be the computable function given by

$$\hat{\Phi}_e(x,s) = \Phi_{e,s}^{W_s}(x).$$

The function m_e^W will be a modulus for $\hat{\Phi}_e$ defined by

$$m_e^W(x) = (\mu s)(\forall t \ge s)\big[\hat{\Phi}_e(x,s) = \hat{\Phi}_e(x,t)\big].$$

The essential property about this modulus for a function computable from a c.e. set is the following:

Lemma 2.10 (Modulus Lemma)**.** *Let W be a c.e. set. Then for each $e \in \omega$ such that Φ_e^W is total, the modulus function m_e^W is W-computable.*

(See [Soa87, Modulus Lemma III.3.2].)

For a set A, $A^{(n)}$ is the nth Turing jump of A. We always have $0^{(n)} \le_{\mathrm{T}} A^{(n)}$.

Definition 2.11 (low$_n$ Degrees)**.** *Let $n \ge 1$. A set A is* low$_n$ *if $A^{(n)} \equiv_{\mathrm{T}} 0^{(n)}$. A Turing degree is* low$_n$ *if some set in that degree is* low$_n$*.*

The following is the main fact about low$_n$ sets we will use

Theorem 2.12. *Let A be* low$_n$ *(for $n \ge 1$). Then $\Delta_{n+1}^A = \Delta_{n+1}$, so that*

(A) $\Sigma_{n+1}^A = \Sigma_{n+1}$

(B) $\Pi_{n+1}^A = \Pi_{n+1}$

(see [Soa87, Post's Theorem, IV.2.2].)

We review the notion of function *domination* and *escaping domination*:

Definition 2.13. *Let $f, g : \mathbb{N} \to \mathbb{N}$ be total functions.*

- f dominates g *if*

$$(\forall^\infty x)\ \big[f(x) > g(x)\big]$$

f is said to be a dominant function if it dominates all computable functions.

- g escapes *(domination by) f if*

$$(\exists^\infty x)\ \big[f(x) \leq g(x)\big]$$

g has an escaping function if some computable function escapes domination by g.

Martin characterized the high degrees as those which can compute a function which dominates all computable functions :

Theorem 2.14 (Martin's Characterization)**.** *For any degree $\boldsymbol{a}$,*

- *$\boldsymbol{a}$ is high $\big(\boldsymbol{a}' \geq \boldsymbol{0}''\big)$ iff some $f \leq \boldsymbol{a}$ is dominant.*
- *$\boldsymbol{a}$ is non-high $\big(\boldsymbol{a}' < \boldsymbol{0}''\big)$ iff every $f \leq \boldsymbol{a}$ has an escaping function.*

(see [Soa87, Theorem XI.1.3].)

3 Enumerating Paths in Π^0_1 Classes

Jockusch ([Joc72]) characterized the Turing degrees which have an enumeration or subenumeration of the computable sets. This will be exploited here to show that the high degrees and PA degrees are saturated bounding.

Jockusch used Martin's characterization of the high degrees (Theorem 2.14) to prove the following:

Theorem 3.1 ([Joc72])**.** *Any high Turing degree can enumerate the computable sets.*

Proof. By Theorem 2.14 choose a dominating function g over the computable functions. If φ_e is computable, then by Lemma 2.10, the modulus function m_e is also computable and so dominated by g. Define

$$f_{e,s}(n) = \begin{cases} \varphi_{e,s+g(n)}(n) & \text{if } \varphi_{e,s+g(m)}(m)\downarrow \forall m \leq n \\ 0 & \text{otherwise} \end{cases}$$

so that either $f_{e,s}$ is almost everywhere 0 or $f_{e,s} = \varphi_e$ where φ_e is total, and thus $f_{e,s}$ is computable. If φ_e is total then there is an s with $f_{e,s} = \varphi_e$ by choosing s large enough so that $s + g(n) > m_e(n)$ for all n. □

An enumeration of the computable sets is a subenumeration for every PAC Π^0_1 class (a Π^0_1 class whose paths are all computable), so we have

Corollary 3.2. *Every high degree is saturated bounding.*

A Turing degree is a *PA degree* if it is the degree of a complete theory extending Peano Arithmetic. The key fact about these degrees is that they can compute a path in any Π^0_1 class (whether all nodes are extendible or not) and can do it uniformly: For any PA degree $\mathbf{d}$ and any computable g such that for each e, $g(e)$ codes a computable infinite tree $\mathscr{T}_e$, there is a $\mathbf{d}$-computable function $\lambda ex.G_e(x)$ such that

$$\lambda x.G_e(x) \in \left[\mathscr{T}_e\right].$$

Using this Jockusch also showed

Theorem 3.3 ([Joc72])**.** *Any PA Turing degree can subenumerate the computable sets.*

Proof. Given a partial computable 0,1-valued function φ_e, construct a computable tree $\mathscr{T}_e$ by $\sigma \in \mathscr{T}$ if for every $n < |\sigma| = s$, if $\varphi_{e,s}(n) \downarrow$ then $\varphi_e(n) = \sigma(n)$. This is a computable condition, although not all nodes will be extendable, and there is a uniform computable enumeration of these trees $\mathscr{T}_e$. If φ_e is total, then $[\mathscr{T}_e] = \{\varphi_e\}$, and otherwise, the paths in $[\mathscr{T}_e]$ are total extensions of φ_e, so may not be computable. There is $\mathbf{d}$-computable function G, such that for each e, $G(e) \in \mathscr{T}_e$. Thus, G provides a $\mathbf{d}$-computable subenumeration of the computable sets. □

It then follows that[2],

Corollary 3.4 ([MM84])**.** *Every PA degree is saturated bounding.*

4 The Main Construction

In this section we show that Millar's construction in the proof of Theorem 2.3 can be extended to the class of degrees satisfying the *aligned escape property.*

[2]We first came across this result through Theorem 3.3 in [Joc72], but it was earlier noticed by [MM84].

4.1 Aligned Escape Property

Martin's Theorem (Theorem 2.14) characterizes the non-high degrees by a bald existence of escape functions. There are two directions for refining this result. First, under what conditions can the escape functions from the Theorem be effectively found? An effective procedure for producing escape function for a set A is a partial computable function h such that for any $e \in \omega$ in which the function Φ_e^A is total, then the function $\varphi_{h(e)}$ is also total and escapes domination from Φ_e^A. We investigate this question in [Har], and show that the low_1 degrees are exactly those degrees for which escape functions can be found effectively. We also produce conditions which characterize each of the low_n degrees.

A second direction for modifying Martin's Theorem is by finding conditions on which we can *align* the escape functions on a chosen subset of the domain. Given a c.e. set C and an A-computable function h, no extension of Martin's characterization is needed to show that there exists an escape function φ_e, which escapes h on C

$$(\exists^\infty x \in C)\ \big[h(x) \leq \varphi_e\big]$$

But suppose that the choice of where to escape depends upon the escape function itself:

$$(\exists^\infty x \in C_e)\ \big[h(x) \leq \varphi_e\big]$$

where C_e effectively depends upon φ_e. This is the problem we will investigate in the remainder of this paper; its solution has some bearing on the problem of the saturated bounding degrees.

Definition 4.1. *Let $A \subseteq \omega$.*

1. *Let f and g be functions. We will say that the domain of f* depends upon *the domain of g, if for every n, $f(n)$ is defined whenever for all $m < n$, $g(m)$ is defined.*

2. *Let $A \subseteq \omega$. A has the* aligned escape property *if for every A-computable function h, and every partial computable function k satisfying the condition that the domain of the function $\varphi_{k(i)}$ depends upon the domain*

of the function φ_i, there is an $e \in \omega$ such that φ_e is total and escapes domination from h on the range of $\varphi_{k(e)}$

$$(\exists^\infty x)\big[h(\varphi_{k(e)}(x)) \leq \varphi_e(x)\big].$$

3. *A degree will be said to have the* aligned escape property *if some set of that degree has the property.*

Note that the aligned escape property is downward closed in the Turing degrees:

Lemma 4.2. *If $\boldsymbol{a}$ has the aligned escape property and $\boldsymbol{b} \leq_{\mathrm{T}} \boldsymbol{a}$, then $\boldsymbol{b}$ has the aligned escape property.*

There is a significant class of degrees which satisfy the aligned escape property, the low$_n$ c.e. degrees (Theorem 5.7), and any c.e. degree with this property will not be saturated-bounding (Theorem 4.4).

4.2 Main Result

It follows from the next theorem, that no c.e. degree with the aligned escape property can be saturated bounding (Corollary 4.4.)

Theorem 4.3. *For every c.e. set satisfying the aligned escape property, there is a PAC Π^0_1 class whose paths it cannot enumerate.*

Proof. Let A be a c.e. set satisfying the aligned escape property. We need to construct a computable tree $\mathscr{T}$ with no terminal nodes and whose paths are all computable, but for which there is no A-computable enumeration. An A-computable enumeration is a function of two arguments, $\lambda nx.\Phi^A_e(n,x)$. We will write the class which the function Φ^A_e enumerates as $\left\{\Phi^A_{e,n}\right\}_{n\in\omega}$. Then, we will need the tree $\mathscr{T}$ to meet the following requirements (for each $e \in \omega$):

$$\mathcal{R}_e : \ [\mathscr{T}] \neq \left\{\Phi^A_{e,n}\right\}_{n\in\omega}$$

If $\mathcal{R}_e$ is satisfied, then Φ^A_e is not an enumeration of the paths in $[\mathscr{T}]$.

Let the standard enumeration of the c.e. set A be $\{A_s\}_{s\in\omega}$. For each e, $\hat{\Phi}_e$ is the computable approximation to Φ_e^A relative to this standard enumeration, and m_e^A is the least modulus for $\hat{\Phi}_e$. By Lemma 2.10, $m_e^A \leq_{\mathrm{T}} A$. Define

$$M_e^A(z) = \max\{m_e^A(n,x) : n, x < z\}$$

so that the value $M_e^A(z)$ provides a modulus for the computable approximation of the partial enumerations

$$\{\Phi_{e,n}^A \restriction z\}_{n<z}.$$

Note also that $M_e^A \leq_{\mathrm{T}} A$.

Let $\alpha_{e,i} = 1^{\langle e,i\rangle}{}^\frown 0$, and begin the construction of $\mathscr{T}$ by including

$$\{\sigma : (\exists e, i)\ \sigma \subseteq \alpha_{e,i}\} \subseteq \mathscr{T}.$$

The construction will ensure that $\mathscr{T}$ has no terminal nodes by putting $\sigma^\frown 0$ into $\mathscr{T}$ whenever $\sigma \in \mathscr{T}$. Let $\mathscr{T}_{e,i}$ be the subtree of $\mathscr{T}$ above $\alpha_{e,i}$ together with all subsequences of $\alpha_{e,i}$. The construction will ensure that for each $e \in \omega$ there is some $i \in \omega$ such that

$$[\mathscr{T}_{e,i}] \neq \{\Phi_{e,n}^A\}_{n\in\omega}$$

This is sufficient to meet requirement $\mathcal{R}_e$. The rest of the proof will concentrate on building a single $\mathscr{T}_{e,i}$.

The index e refers to the potential A-computable enumeration Φ_e^A, and the index i refers to the partial computable function φ_i, which will act as a stand-in for the modulus Ψ_e^A in the construction. The construction will determine partial computable functions:

- $F_{e,i}$: if $F_{e,i}$ is total then it is the only path through $\mathscr{T}_{e,i}$ which is not isolated. If $F_{e,i}$ is not total then all paths through $\mathscr{T}_{e,i}$ are isolated
- $Q_{e,i}$: the *queries* to the stand-in function φ_i for the modulus function Ψ_e^A. If $\varphi_i \restriction n$ is defined, then $Q_{e,i}(n)$ will be defined.

Additionally, the construction will ensure that if Φ_e^A is total and enumerates only paths through $\mathscr{T}$, and if φ_i is the escape function given by the aligned escape property with

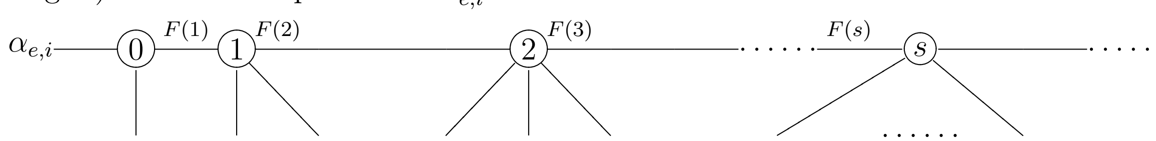
$$(\exists^\infty x)\left[M_e^A(Q_{e,i}(x)) \leq \varphi_i(x)\right],$$

then $F_{e,i}$ will be a path in $\mathscr{T}$ which Φ_e^A fails to enumerate.

Begin Construction.
The construction is local for each $e, i \in \omega$. Fix e, i. The construction builds, in stages $s \in \omega$, a tree $\mathscr{T}_{e,i}^s$ and two functions $F_{e,i}$ and $Q_{e,i}$. Reference to e and i will be dropped during the construction from the two functions, F and Q.

Stage 0:
Let $\mathscr{T}_{e,i}^s = \{\sigma : \sigma \subseteq \alpha_{e,i}\}$, $F(0) = \alpha_{e,i}$ and $Q = \emptyset$.

Stage s+1:
We are given $\mathscr{T}_{e,i}^s$, F defined for values $t \leq s$ and Q defined for all values $t < s$. Additionally, $F(s) \in \mathscr{T}_{e,i}^s$ and $F(t) \subseteq F(t+1)$ for all $t \leq s$. Extend $\mathscr{T}_{e,i}^s$ by introducing $s+2$ new branches to the end of F^s and let $Q(s)$ be the height of the tree $\mathscr{T}_{e,i}^{s+1}$ after the branches are added (or s, whichever is larger.) The current picture of $\mathscr{T}_{e,i}^{s+1}$ looks like



The construction at stage s+1 waits for $\varphi_i(s)\downarrow$, so while t is such that $\varphi_{i,t}(s)\uparrow$ extend $\mathscr{T}_{e,i}^{s+1}$ by incrementing each end node σ by $\sigma\hat{\ }0$.

Let t be least such that $\varphi_{i,t}(s)\downarrow$ (if such a t exists), and let $u = \varphi_{i,t}(s)$. If some of the $s+2$ branches extending $F(s)$ are not enumerated by any of $\Phi_{e,n,u}^{A_u} \upharpoonright Q(s)$ for $n \leq s$ (for each $n \leq s$ this is the stage u approximation to $\Phi_{e,n}^A$ of length $Q(s)$) then choose the first such branch, and let $F(s+1)$ be the end of the current path along this branch (there is only one); and otherwise, let $F(s+1)$ be the current path along the first branch. This concludes the construction in stage $s+1$.

End of Construction.

It is clear from the construction that for each $e, i \in \omega$:

- $F_{e,i}$ is total and computable if and only if φ_i is total.

- If $F_{e,i}$ is total, then $\cup_{s<\omega} F_{e,i}(s)$ is the only non-isolated branch in $\mathscr{T}_{e,i}$. If $F_{e,i}$ is not total, then all paths in $\mathscr{T}_{e,i}$ are isolated.
- If $\varphi_i \upharpoonright n$ is defined then so is $Q_{e,i}(n)$, and if φ_i is total then so is $Q_{e,i}$. Thus, the domain of $Q_{e,i}$ depends upon the domain of φ_i.

Sublemma 1. *$\mathcal{R}_e$ is satisfied for all e.*

Proof of Sublemma 1. If Φ_e^A is either partial or fails to enumerate paths on $\mathscr{T}$, $\mathcal{R}_e$ is satisfied. Suppose Φ_e^A is total and enumerates only paths in $\mathscr{T}$. It will be shown that for some i, $\cup_{s<\omega} F_{e,i}(s)$ is a path in $\mathscr{T}$ such that

$$\cup_{s<\omega} F_{e,i}(s) \neq \Phi_{e,m}^A \qquad \text{for all } m$$

The set A satisfies the aligned escape property, the function M_e^A is A-computable (since Φ_e^A is total), and the domain of $Q_{e,i}$ depends upon the domain of φ_i, thus for some i, φ_i is total and

$$(\exists^\infty x)\big[M_e^A(Q_{e,i}(x)) \leq \varphi_i(x)\big].$$

For any $s \in \omega$ such that $M_e^A(Q_{e,i}(s)) \leq \varphi_i(s) = u$, and for every $m < n$

$$\Phi_{e,m}^A \upharpoonright Q_{e,i}(s) = \Phi_{e,m,u}^{A_u} \upharpoonright Q_{e,i}(s).$$

Note also that $F_{e,i}$ is total, since φ_i is.

If $\Phi_{e,m}^A$ enumerates a path in $\mathscr{T}_{e,i}$, then $\cup_{s<\omega} F_{e,i}(s) \neq \Phi_{e,m}^A$: At the completion of stage s,

$$\begin{aligned} F_{e,i}(s) &\neq \Phi_{e,m,u}^{A_u} \upharpoonright Q_{e,i}(s) \qquad && \text{for all } m < s \\ &= \Phi_{e,m}^A \upharpoonright Q_{e,i}(s) && \text{for all } m < s. \end{aligned}$$

But

$$\begin{aligned} F_{e,i}(s) &\subset \cup_{s<\omega} F_{e,i}(s) \\ \Phi_{e,m}^A \upharpoonright Q_{e,i}(s) &\subset \Phi_{e,m}^A \qquad && \text{for all } m < s \end{aligned}$$

so that

$$\cup_{s<\omega} F_{e,i}(s) \neq \Phi_{e,m}^A \qquad \text{for all } m < s.$$

This is true for inifinitely many stages s of the construction. So,

$$\cup_{s<\omega} F_{e,i}(s) \neq \Phi_{e,m}^A \qquad \text{for all } m$$

Thus, requirement $\mathcal{R}_e$ is satisfied. $\square$

This finishes the proof of Theorem 4.4. □

By applying Corollary 2.8 to Theorem 4.3 we have

Corollary 4.4. *No c.e. degree with the aligned escape property is saturated bounding.*

Since the saturated bounding degrees are closed downward in the Turing degrees

Corollary 4.5. *No degree below a c.e. degree with the aligned escape property is saturated bounding.*

5 low_n c.e. Degrees have the Aligned Escape Property

We first prove that the low_1 c.e. degrees have the aligned escape property. This provides the core idea behind the proof that all low_n c.e. degrees have this property. We then explain how to extend the argument to the low_2 c.e. degrees, where the new jump class adds a new layer of quantifier complexity to the argument. This establishes the basic pattern for all low_n c.e. degrees. A new characterization of the Σ_n and Π_n sets, the Strong Normal Form Theorem 5.6, provides the essential bridge over the added quantifier complexity in the general low_n c.e. case to the core idea of the low_1 case.

Earlier, we suggested two ways of extending Martin's characterization of the non-high degrees: degrees for which escape functions can be effectively found, and degrees for which the aligned escape property holds. In both cases it useful to start with the problem of producing escape functions from a given non-high set, and we outline the essential idea at the start of the proof of Theorem 5.1, that all low_1 c.e. degrees have the aligned escape property. Although both problems start with the same core idea, their solutions do not appear to coincide. The low_1 degrees are characterized by having an effective procedure for producing escape functions ([Har]); but, the proof that the low_1 c.e. degrees have the aligned escape property given here depends upon being computably enumerable. The proof of Theorem 5.1 does yield an effective procedure for producing aligned escape functions for low_1 c.e. degrees.

5.1 low_1 Case

The proof that low_1 c.e. degrees have the aligned escape property captures the essence of the strategy, which is then extended to the low_n c.e. degrees.

Theorem 5.1. *Every* low_1 *c.e. degree has the aligned escape property.*

By Lemma 4.2,

Corollary 5.2. *Any degree below a* low_1 *c.e. degree has the aligned escape property.*

Central to the proof of Theorem 5.1 is the following characterization of Σ_2 and Π_2 predicates ([Soa87, Theorem IV.3.7]). Lemma 5.3 is the basis case of a new normal form theorem for Σ_n and Π_n predicates (Theorem 5.6):

Lemma 5.3. *There is a computable g such that for any Π_2 predicate V with Π_2-index v*

$$\begin{aligned} V(x) &\iff \left[W_{g(v,x)} = \omega\right] \\ \neg V(x) &\iff \left[W_{g(v,x)} \textit{ is finite }\right] \end{aligned}$$

Proof. Let $V \in \Pi_2$ with Π_2-index v. Then

$$V(x) \iff (\forall y)\left[\langle x, y\rangle \in W_v\right].$$

By the by the s-m-n Theorem ([Soa87, Theorem I.3.5] there is a computable h such that

$$V(x) \iff (\forall y)\left[y \in W_{h(v,x)}\right].$$

Define g so that

$$y \in W_{g(v,x)} \iff (\forall z \leq y)\left[z \in W_{h(v,x)}\right],$$

then

$$\begin{aligned} V(x) &\iff \left[W_{g(v,x)} = \omega\right] \\ \neg V(x) &\iff \left[W_{g(v,x)} \text{ is finite }\right] \end{aligned}$$

□

We begin the proof of Theorem 5.1.

Proof. Let A be low_1 and c.e. and Φ_e^A be total. Fix a computable approximation $\hat{\Phi}_e$ of Φ_e^A with modulus function m_e^A. For every x and $s \geq m_e^A(x)$

$$\Phi_e^A(x) = \hat{\Phi}_e(x, s)$$

and by Lemma 2.10, $m_e^A \leq_{\mathrm{T}} A$.

Consider the problem of producing an escape function for Φ_e^A, without the additional problem of escaping on some subset of ω. We would like a computable procedure which given any x produces an s_x so that for infinitely many x

$$s_x \geq m_e^A(x), \tag{1}$$

since for any x satisfying equation (1)

$$\hat{\Phi}_e(x, s_x) = \Phi_e^A(x).$$

A procedure $x \mapsto s_x$ could use a c.e. set W_j as a timer

$$s_x = \mu t\big[x \in W_{j,t}\big]$$

provided W_j satisfied two conditions:

$$(\exists^\infty x)(\exists s)\big[s \geq m_e^A(x) \ \ \& \ \ x \notin W_{j,s}\big] \tag{2}$$

$$W_j = \omega \tag{3}$$

Condition (3) ensures that for each x, s_x is defined. Condition (2) ensures that equation (1) holds for infinitely many x. Instead of condition (2), it would be sufficient to have

$$(\exists^\infty s)(\exists x)\big[s \geq m_e^A(x) \ \ \& \ \ x \notin W_{j,s}\big]. \tag{4}$$

If condition (3) holds for W_j, then for each x the following sets are finite

$$\big\{s \mid x \notin W_{j,s}\big\}$$

so that, if condition (4) holds, then condition (2) holds.

The problem of finding a computable procedure $x \mapsto s_x$ satisfying equation (1) for infinitely many x, has been replaced with finding a c.e. set W_j such that conditions (4) and (3) hold. But condition (4) is a Π_2^A predicate (the index e), and as A is low$_1$, this condition is also Π_2. Lemma 5.3 provides the key to connect these two conditions in the same set W_j, for some j. Let g be the partial computable function given by Lemma 5.3. The problem will be to find a value $j \in \omega$ satisfying

$$(\exists^\infty s)(\exists x)\big[s \geq m_e^A(x) \ \ \& \ \ x \notin W_{g(j,e),s}\big]$$
$$W_{g(j,e)} = \omega$$

We now carry out this strategy on an aligned subset of ω. Let k be a partial computable function where the domain of $\varphi_{k(i)}$ depends upon the domain of φ_i for every i. Then there is a partial computable function r such that for every x

$$\varphi_{r(i,j,e)}(x) = \hat{\Phi}_e(\varphi_{k(i)}(x), s) \quad \text{where } s = \mu t\big[x \in W_{g(j,e),t}\big].$$

(The parameter j is the Π_2-index from Lemma 5.3 of a yet to be determined Π_2 set.) By the the Generalized Fixed Point Theorem [Soa87, Theorem II.3.5] there is a computable function f such that

$$\varphi_{r(f(j,e),j,e)} \simeq \varphi_{f(j,e)}.$$

Thus, for every x

$$\varphi_{f(j,e)}(x) = \hat{\Phi}(\varphi_{k(f(j,e))}(x), s) \quad \text{where } s = \mu t\big[x \in W_{g(j,e),t}\big]. \tag{5}$$

We will show that there is a suitable j such that $\varphi_{f(j,e)}$ is an escape function for Φ_e^A required by the aligned escape property.

Recall that for every j, $\varphi_{k(f(j,e))}$ depends on the domain of $\varphi_{f(j,e)}$, so that for every z, if $\varphi_{f(j,e)}(x)\downarrow$ for each $x < z$ then $\varphi_{k(f(j,e))}(z)\downarrow$.

Sublemma 1. *Let $j \in \omega$. Fix $z \in \omega$, and suppose $x \in W_{g(j,e)}$ for all $x < z$. Then $\varphi_{k(f(j,e))}(x)\downarrow$ for all $x \leq z$, and $\varphi_{f(j,e)}(x)\downarrow$ for all $x < z$.*

Proof. The proof is by induction on z. For $z = 0$, $\varphi_{k(f(j,e))}(0)\downarrow$ since $\varphi_{f(j,e)}(x)\downarrow$ for all $x < 0$ and $\varphi_{k(f(j,e))}$ depends upon the domain of $\varphi_{f(j,e)}$. Suppose that $x \in W_{g(j,e)}$ for all $x < z$ and that $\varphi_{k(f(j,e))}(x)\downarrow$ for all $x \leq z-1$ and $\varphi_{f(j,e)}(x)\downarrow$ for all $x < z-1$. Since

$$\varphi_{f(j,e)}(z-1) = \hat{\Phi}_e(\varphi_{k(f(j,e))}(z-1), s) \quad \text{where } s = \mu t\big[z-1 \in W_{g(j,e),t}\big],$$

so $\varphi_{f(j,e)}(z-1)\downarrow$. Also, $\varphi_{k(f(j,e))}(z)\downarrow$ since $\varphi_{f(j,e)}(x)\downarrow$ for all $x < z$ and the domain of $\varphi_{k(f(j,e))}$ depends on the domain of $\varphi_{f(j,e)}$. □

We will show that there is a j so that the following two properties are satisfied

$$(\exists^\infty s)(\exists x)\big[s \geq m_e^A(\varphi_{k(f(j,e))}(x)) \;\;\&\;\; x \notin W_{g(j,e),s}\big] \tag{6}$$
$$W_{g(j,e)} = \omega \tag{7}$$

Let $\tilde{V}^A(j,e)$ be condition (6), which is Π_2^A. Since A is low$_1$, by Lemma 2.12 it is actually Π_2, so

$$\tilde{V}^A(j,e) \iff (\forall z)\big[\langle e,j,z\rangle \in W_i\big].$$

for some index i. Using the s-m-n Theorem ([Soa87, I.3.5]), there is a computable p such that

$$\tilde{V}^A(j,e) \iff (\forall z)\big[\langle e,z\rangle \in W_{p(j)}\big].$$

By the Recursion Theorem [Soa87, Theorem II.3.1], there is a computable v such that

$$W_{p(v)} = W_v.$$

Let $V^A(e)$ be $\tilde{V}^A(v,e)$. Then v is a Π_2-index for V^A

$$\begin{aligned} V^A(e) &\iff \tilde{V}^A(v,e) \\ &\iff (\forall z)\big[\langle e,z\rangle \in W_{p(v)}\big] \\ &\iff (\forall z)\big[\langle e,z\rangle \in W_v\big]. \end{aligned}$$

Inserting v into the conditions (6) and (7) yields

$$(\exists^\infty s)(\exists x)\big[s \geq m_e^A(\varphi_{k(f(v,e))}(x)) \;\;\&\;\; x \notin W_{g(v,e),s}\big] \tag{Esc}$$
$$W_{g(v,e)} = \omega \tag{Tot}$$

and $V^A(e)$ is equivalent to (Esc).

Since v is a Π_2-index for V^A, it follows from Lemma 5.3 that

$$\begin{aligned} V^A(e) &\iff \big[W_{g(v,e)} = \omega\big] \\ \neg V^A(e) &\iff \big[W_{g(v,e)} \text{ is finite }\big] \end{aligned}$$

Sublemma 2. *$V^A(e)$ is true.*

Proof. Suppose $V^A(e)$ is false. Then

$$W_{g(v,e)} \text{ is finite .}$$

Let z be least with $z \notin W_{g(v,e)}$. By Sublemma 1, $\varphi_{k(f(v,e))}(z)\downarrow$. Since Φ_e^A is total, $m_e^A(\varphi_{k(f(v,e))}(z))\downarrow$. But

$$(\forall s)\big[z \notin W_{g(v,e),s}\big]$$

so that for infinitely many s we have both

$$\begin{aligned} s > m_e^A(\varphi_{k(f(v,e))}(z)) \quad &\text{and} \\ z \notin W_{g(v,e),s}. \end{aligned}$$

Together these imply $V^A(e)$. Thus, it is impossible that $V^A(e)$ is false. □

From Sublemma 2 $V^A(e)$ is true, so condition (Esc) holds for e. From Lemma 5.3

$$W_{g(v,e)} = \omega$$

which is condition (Tot). Thus, conditions (Esc) and (Tot) hold for $W_{g(v,e)}$.

The escape function required by the aligned escape property is $\varphi_{f(v,e)}$, given by equation (5). First, $\varphi_{f(v,e)}$ is total by condition (Tot) and Sublemma 1. Next, from condition (Esc),

$$(\exists^\infty x)\big[\Phi^A(\varphi_{k(f(v,e))}(x) \leq \varphi_{f(v,e)}(x)\big].$$

□

5.2 low$_2$ Case

A set A is low$_2$ if $A'' = 0''$ and from Lemma 2.12, this implies $\Sigma_3^A \subseteq \Sigma_3$. This is the property of low$_2$ sets which will be exploited. Consider the problem of producing an escape function for an A-computable function Φ_e^A, where A is c.e. and low$_2$. Let $\hat{\Phi}_e$ be a computable approximation to Φ_e^A and m_e^A be the A-computable modulus for Φ_e^A, as in the Modulus Lemma 2.10. In the low$_1$ case we were looking for a c.e. set W_j satisfying two properties

$$(\exists^\infty x)(\exists s)\big[s \geq m_e^A(x) \ \ \& \ \ x \notin W_{j,s}\big] \tag{8}$$
$$W_j = \omega \tag{9}$$

so that the partial computable function f defined by

$$f(x) = \hat{\Phi}_e(x, s_x) \quad \text{where } s_x = \mu t\big[x \in W_{j,s}\big]$$

would be total and escape from Φ_e^A. In the low$_1$ case we exploited that the conditions (8) and (9) were Π_2; but, this need not be true of condition (8) in the low$_2$ case. The strategy here is to boost the complexity of both equations to Σ_3:

$$(\forall^\infty y)(\exists^\infty x)(\exists s)\big[s \geq m_e^A(x) \ \ \& \ \ \langle x, y\rangle \notin W_{j,s}\big] \tag{10}$$

$$(\forall^\infty y)(\forall x)\big[\langle x, y\rangle \in W_j\big] \tag{11}$$

Condition (10) is Σ_3^A, but since A is low$_2$, this property is actually Σ_3. Condition (11) is also Σ_3, and the central fact which will link these two is the following characterization of Σ_3 sets, the familiar strong characterization of Σ_3 sets ([Soa87, Theorem IV.3.7]),

Lemma 5.4 (Strong Quantifier Normal Form for Σ_3)**.** *There is a computable g such that for any Σ_3 set V with Σ_3-index v*

$$\begin{aligned} V(x) &\iff (\forall^\infty y)\ \big[W_{g(v,x,y)} = \omega\big] \\ \neg V(x) &\iff (\forall y)\ \big[W_{g(v,x,y)} \text{ is finite }\big] \end{aligned}$$

Using the computable function g from Lemma 5.4 we can properly state the conditions (10) and (11) as

$$(\forall^\infty y)(\exists^\infty x)(\exists s)\big[s \geq m_e^A(x) \ \ \& \ \ x \notin W_{g(j,e,y),s}\big] \tag{ESC}$$

$$(\forall^\infty y)\big[W_{g(j,e,y)} = \omega\big]. \tag{TOT}$$

As in the low$_1$ case, with a proper choice of j, both conditions (ESC) and (TOT) must be true. Together, these imply that there exists a $y \in \omega$ with

$$(\exists^\infty x)(\exists s)\big[s \geq m_e^A(x) \ \ \& \ \ x \notin W_{g(j,e,y),s}\big] \tag{12}$$

$$\big[W_{g(j,e,y)} = \omega\big]. \tag{13}$$

(In fact, almost every y will satisfy these two properties simultaneously.) We can then use this choice of y to define an escape function.

Proof of Lemma 5.4. Let $V \in \Sigma_3$, so that for some $U \in \Pi_2$, A is $(\exists x)\, U$. By Lemma 5.3, re-writing U, we have for some computable h

$$\begin{array}{rcl} V(x) & \iff & (\exists y)\big[W_{h(v,x,y)} = \omega\big] \\ \neg V(x) & \iff & (\forall y)\big[W_{h(v,x,y)} \text{ is finite }\big]. \end{array}$$

We are given a matrix $\{W_{h(v,x,y)}\}_{y\in\omega}$ such that if $V(x)$ holds for at least one row y, $W_{h(v,x,y)} = \omega$ and if $\neg V(x)$ then all rows are finite. Define a computable g so that

$$W_{g(v,x,y)} = \bigcup_{z\leq y} W_{h(v,x,z)}$$

then we have

$$\begin{array}{rcl} V(x) & \iff & (\forall^\infty y)\big[W_{g(v,x,y)} = \omega\big] \\ \neg V(x) & \iff & (\forall y)\big[W_{g(v,x,y)} \text{ is finite }\big]. \end{array}$$

□

5.3 Strong Normal Form Theorem

Lemmas 5.3 and 5.4 provide strong normal forms for Π_2 and Σ_3 sets, respectively. They are also the first two levels of a general strong normal form theorem for the arithmetic hierarchy. We first provide the statement and proof for Π_4 sets:

Lemma 5.5 (Strong Quantifier Normal Form for Π_4)**.** *There is a computable g such that for any Π_4 set V with Π_4-index v*

$$\begin{array}{rcl} V(x) & \iff & (\forall z)(\forall^\infty y)\big[W_{g(v,x,y,z)} = \omega\big] \\ \neg V(x) & \iff & (\forall^\infty z)(\forall y)\big[W_{g(v,x,y,z)} \textit{ is finite }\big] \end{array}$$

Proof. Let $V \in \Pi_4$, then $V(x)$ can be expressed as $(\forall z)U(x,z)$, where $U \in \Sigma_3$. By the Lemma 5.4, there is a computable h such that

$$\begin{array}{rcl} U(x) & \iff & (\forall^\infty y)\big[W_{h(v,x,y,z)} = \omega\big] \\ \neg U(x) & \iff & (\forall y)\big[W_{h(v,x,y,z)} \text{ is finite }\big] \end{array}$$

This provides an *array* of matrices

$$\Big\{\{W_{h(v,x,y,z)}\}_{y\in\omega}\Big\}_{z\in\omega}$$

with the property that if $V(x)$, then *for each* z and almost all rows y in the zth matrix, $W_{h(v,x,y,z)} = \omega$, and if $\neg V(x)$ *there exists some* z in which all the rows y in the zth matrix are finite. Define a computable g by

$$W_{g(v,x,y,z)} = \bigcap_{w \leq z} W_{h(v,x,y,w)}$$

This provides a new array of matrices

$$\left\{ \{W_{g(v,x,y,z)}\}_{y\in\omega} \right\}_{z\in\omega}$$

still with the property that if $V(x)$ then *for each* z and almost all rows y in the zth matrix, $W_{h(v,x,y,z)} = \omega$, but if $\neg V(x)$ then *for almost all* z, every row y of the zth matrix is finite. Thus,

$$\begin{array}{rcl} V(x) & \Longleftrightarrow & (\forall z)(\forall^\infty y)\left[W_{g(v,x,y,z)} = \omega\right] \\ \neg V(x) & \Longleftrightarrow & (\forall^\infty z)(\forall y)\left[W_{g(v,x,y,z)} \text{ is finite }\right] \end{array}$$

□

Theorem 5.6 (Strong Quantifier Normal Form)**.** *For $n \geq 1$:*

1. *There exists a computable g such that for any $V(x) \in \Sigma_{2n+1}$ with index v*

$$\begin{array}{rcl} V(x) & \Longleftrightarrow & (\forall^\infty y_{2n-1})(\forall y_{2n-2})\ldots(\forall y_2)(\forall^\infty y_1)\ \left[W_{g(v,x,y_1,\ldots,y_{2n-1})} = \omega\right] \\ \neg V(x) & \Longleftrightarrow & (\forall y_{2n-1})(\forall^\infty y_{2n-2})\ldots(\forall^\infty y_2)(\forall y_1)\ \left[W_{g(v,x,y_1,\ldots,y_{2n-1})} \textit{ is finite }\right] \end{array}$$

2. *There exists a computable g such that for any $V(x) \in \Pi_{2n}$ with index v*

$$\begin{array}{rcl} V(x) & \Longleftrightarrow & (\forall y_{2n-2})(\forall^\infty y_{2n-3})\ldots(\forall y_2)(\forall^\infty y_1)\ \left[W_{g(v,x,y_1,\ldots,y_{2n-2})} = \omega\right] \\ \neg V(x) & \Longleftrightarrow & (\forall^\infty y_{2n-2})(\forall y_{2n-3})\ldots(\forall^\infty y_2)(\forall y_1)\ \left[W_{g(v,x,y_1,\ldots,y_{2n-2})} \textit{ is finite }\right] \end{array}$$

We provide a proof of this theorem in [Har]. The basis case is Lemma 5.3. The general case of Σ_{2n+1} is in essence as in the proof of Lemma 5.4; the general case of Π_{2n} is in essence as in the proof of Lemma 5.5.

5.4 low_n Case

Theorem 5.7. *Every* low_n *c.e. degree has the aligned escape property.*

By the downward closure of the aligned escape property (Lemma 4.2),

Corollary 5.8. *Any degree below a* low_n *c.e. degree has the aligned escape property.*

By Corollary 4.4 we have

Corollary 5.9. *No* low_n *c.e. degree is saturated bounding.*

Proof. The proof will eventually split into two cases, low_{2n-1} and low_{2n}, following Theorem 5.6. The first part of the proof is common to both cases.

Let A be c.e. . Let Φ_e^A be total and fix a computable approximation $\hat{\Phi}_e$ of Φ_e^A with modulus function $m_e^A \leq_{\text{T}} A$, as in the Modulus Lemma 2.10.

Let k be a partial computable function where the domain of the function $\varphi_{k(i)}$ depends upon the domain of the function φ_i for every i. Let g be the appropriate computable function from the Strong Normal Form Theorem 5.6, and let $\overline{y}$ be a tuple whose length will will be determined exactly depending on the low_{2n-1} or low_{2n} case. There is a partial computable function r such that for every x

$$\varphi_{r(i,j,e,\overline{y})}(x) = \hat{\Phi}_e(\varphi_{k(i)}(x), s) \quad \text{where } s = \mu t\big[x \in W_{g(j,e,\overline{y}),t}\big].$$

By the the Generalized Fixed Point Theorem [Soa87, Theorem II.3.5] there is a computable function f such that

$$\varphi_{r(f(j,e,\overline{y}),j,e,\overline{y})} \simeq \varphi_{f(j,e,\overline{y})}.$$

Thus, for every x

$$\varphi_{f(j,e,\overline{y})}(x) = \hat{\Phi}_e(\varphi_{k(f(j,e,\overline{y}))}(x), s) \quad \text{where } s = \mu t\big[x \in W_{g(j,e,\overline{y}),t}\big]. \tag{14}$$

We will show that there is a suitable value $j \in \omega$ such that for some choice of tuple $\overline{y}$, the function $\varphi_{f(j,e,\overline{y})}$ escapes domination from Φ_e^A on the range of the function $\varphi_{k(f(j,e,\overline{y}))}$.

Recall that for every j and every $\overline{y}$, the domain of $\varphi_{k(f(j,e,\overline{y}))}$ depends on the domain of $\varphi_{f(j,e,\overline{y})}$, and that our hypothesis is that Φ_e^A is total.

Sublemma 1. *For any $j \in \omega$ and tuple of natural numbers $\overline{y}$ the following holds. If $z \in \omega$ such that $x \in W_{g(j,e,\overline{y})}$ for all $x < z$, then $\varphi_{k(f(j,e,\overline{y}))}(x)\downarrow$ for all $x \leq z$, and $\varphi_{f(j,e,\overline{y})}(x)\downarrow$ for all $x < z$.*

Proof. The proof is by induction on z. For $z = 0$, $\varphi_{k(f(j,e,\overline{y}))}(0)\downarrow$ since $\varphi_{f(j,e,\overline{y})}(x)\downarrow$ for all $x < 0$ and $\varphi_{k(f(j,e,\overline{y}))}$ depends upon the domain of $\varphi_{f(j,e,\overline{y})}$. Suppose that $x \in W_{g(j,e,\overline{y})}$ for all $x < z$ and that $\varphi_{k(f(j,e,\overline{y}))}(x)\downarrow$ for all $x \leq z-1$ and $\varphi_{f(j,e,\overline{y})}(x)\downarrow$ for all $x < z-1$. Since

$$\varphi_{f(j,e,\overline{y})}(z-1) = \hat{\Phi}_e(\varphi_{k(f(j,e,\overline{y}))}(z-1), s) \quad \text{where } s = \mu t\big[z-1 \in W_{g(j,e,\overline{y}),t}\big]$$

thus $\varphi_{f(j,e,\overline{y})}(z-1)\downarrow$. Also, $\varphi_{k(f(j,e,\overline{y}))}(z)\downarrow$ since $\varphi_{f(j,e,\overline{y})}(x)\downarrow$ for all $x < z$ and the domain of $\varphi_{k(f(j,e,\overline{y}))}$ depends on the domain of $\varphi_{f(j,e,\overline{y}))}$. □

Sublemma 2. *For any $j \in \omega$ and tuple of natural numbers $\overline{y}$ satisfying the following two conditions*

$$(\exists^\infty x)(\exists s)\big[s \geq m_e^A(\varphi_{k(f(j,e,\overline{y}))}(x)) \;\;\&\;\; x \notin W_{g(j,e,\overline{y}),s}\big] \tag{15}$$

$$W_{g(j,e,\overline{y})} = \omega \tag{16}$$

the function $\varphi_{f(j,e,\overline{y}))}$ escapes domination from the function Φ_e^A on the range of the function $\varphi_{k(f(j,e,\overline{y}))}$:

$$(\exists^\infty x)\;\big[\Phi^A(\varphi_{k(f(j,e,\overline{y}))}(x)) \leq \varphi_{f(j,e,\overline{y}))}(x)\big].$$

Proof. If follows from condition (16) and Sublemma 1, that the function $\varphi_{f(j,e,\overline{y}))}$ is total. Fix $x \in \omega$ and suppose that there is an $s \in \omega$ with

$$s \geq m_e^A(\varphi_{k(f(j,e,\overline{y}))}(x)) \;\;\&\;\; x \notin W_{g(j,e,\overline{y}),s}\big];$$

it then follows from equation (14) that

$$\Phi_e^A(\varphi_{k(f(j,e,\overline{y}))}(x)) = \varphi_{f(j,e,\overline{y}))}(x).$$

Thus, by condition (15),

$$(\exists^\infty x)\;\big[\Phi_e^A(\varphi_{k(f(j,e,\overline{y}))}(x)) \leq \varphi_{f(j,e,\overline{y}))}(x)\big].$$

□

Sublemma 3. *For any $j \in \omega$ and tuple of natural numbers $\overline{y}$ satisfying the following two conditions*

$$(\exists^\infty s)(\exists x)\big[s \geq m_e^A(\varphi_{k(f(j,e,\overline{y}))}(x)) \ \ \& \ \ x \notin W_{g(j,e,\overline{y}),s}\big] \tag{17}$$

$$W_{g(j,e,\overline{y})} = \omega \tag{18}$$

the function $\varphi_{f(j,e,\overline{y}))}$ escapes domination from the function Φ_e^A on the range of the function $\varphi_{k(f(j,e,\overline{y}))}$.

Proof. If condition (18) holds then for each x the following sets are finite

$$\big\{s \mid x \notin W_{g(j,e,\overline{y}),s}\big\}.$$

So, if condition (17) holds, there must be infinitely many x such that for some s

$$s \geq m_e^A(\varphi_{k(f(j,e,\overline{y}))}(x)) \ \ \& \ \ x \notin W_{g(j,e,\overline{y}),s}.$$

Thus, condition (15) holds from Sublemma 2. The conclusion follows from Sublemma 2. □

Case low_{2n-1} :

Suppose A be low_{2n-1}, so that by Lemma 2.12, $\Pi_{2n}^A = \Pi_{2n}$.

Let $\overline{y}$ be $\langle y_1, \ldots, y_{2n-2}\rangle$, and

$$\tilde{U}^A(j,e,\overline{y}) \iff (\exists^\infty s)(\exists x)\big[s \geq m_e^A(\varphi_{k(f(j,e,\overline{y}))}(x)) \ \ \& \ \ x \notin W_{g(j,e,\overline{y}),s}\big]$$

We want to show that there is a j so that the following two properties are satisfied

$$(\exists^\infty y_{2n-2})(\forall^\infty y_{2n-3})\ldots(\exists^\infty y_2)(\forall^\infty y_1)\ \tilde{U}^A(j,e,\overline{y}) \tag{19}$$

$$(\forall y_{2n-2})(\forall^\infty y_{2n-3})\ldots(\forall y_2)(\forall^\infty y_1)\ \big[W_{g(j,e,\overline{y})} = \omega\big] \tag{20}$$

Let $\tilde{V}^A(j,e)$ be condition (19), which is Π_{2n}^A. Since A is low_{2n-1}, by Lemma 2.12 $\tilde{V}^A(j,e)$ is actually Π_{2n}, so

$$\tilde{V}^A(j,e) \iff (\forall y_{2n-2})(\exists y_{2n-3})\ldots(\forall y_2)(\exists y_1)(\forall z)\big[\langle e,j,\overline{y},z\rangle \in W_i\big].$$

for some index i. By the s-m-n Theorem ([Soa87, I.3.5]) there is a computable p such that

$$\tilde{V}^A(j,e) \iff (\forall y_{2n-2})(\exists y_{2n-3})\dots(\forall y_2)(\exists y_1)(\forall z)\big[\langle e,\overline{y},z\rangle \in W_{p(j)}\big].$$

By the Recursion Theorem, there is a computable v such that

$$W_{p(v)} = W_v.$$

Let $U^A(e,\overline{y})$ be $\tilde{U}^A(v,e,\overline{y})$:

$$U^A(e,\overline{y}) \iff (\exists^\infty s)(\exists x)\big[s \geq m_e^A(\varphi_{k(f(v,e,\overline{y}))}(x)) \ \& \ x \notin W_{g(v,e,\overline{y}),s}\big]. \quad (21)$$

and $V^A(e)$ be $\tilde{V}^A(v,e)$. Then v is a Π_2-index for V^A

$$\begin{aligned} V^A(e) &\iff \tilde{V}^A(v,e) \\ &\iff (\forall y_{2n-2})(\exists y_{2n-3})\dots(\forall y_2)(\exists y_1)(\forall z)\big[\langle e,\overline{y},z\rangle \in W_{p(v)}\big] \\ &\iff (\forall y_{2n-2})(\exists y_{2n-3})\dots(\forall y_2)(\exists y_1)(\forall z)\big[\langle e,\overline{y},z\rangle \in W_v\big]. \end{aligned}$$

Inserting v into conditions (19) and (20)

$$(\exists^\infty y_{2n-2})(\forall^\infty y_{2n-3})\dots(\exists^\infty y_2)(\forall^\infty y_1)\ U^A(v,e,\overline{y}) \quad \text{(Esc)}$$
$$(\forall y_{2n-2})(\forall^\infty y_{2n-3})\dots(\forall y_2)(\forall^\infty y_1)\ \big[W_{g(v,e,\overline{y})} = \omega\big] \quad \text{(Tot)}$$

and V^A is equivalent to (Esc) for all e.

Since v is a Π_{2n}-index for V^A, it follows from Theorem 5.6 that

$$\begin{aligned} V^A(e) &\iff (\forall y_{2n-2})(\forall^\infty y_{2n-3})\dots(\forall y_2)(\forall^\infty y_1)\ \big[W_{g(v,e,\overline{y})} = \omega\big] \\ \neg V^A(e) &\iff (\forall^\infty y_{2n-2})(\forall y_{2n-3})\dots(\forall^\infty y_2)(\forall y_1)\ \big[W_{g(v,e,\overline{y})} \text{ is finite }\big] \end{aligned}$$

Sublemma 4. *For every $y_1,\dots,y_{2n-2}$*

$$W_{g(v,e,\overline{y})} \textit{ is finite } \implies U^A(v,e,\overline{y}).$$

Proof. Fix $y_1,\dots,y_{2n-2}$ and suppose that $W_{g(v,e,\overline{y})}$ is finite . Let z be least with $z \notin W_{g(v,e,\overline{y})}$. By Sublemma 1, $\varphi_{k(f(v,e,\overline{y}))}(z)\downarrow$. Since Φ_e^A is total, the modulus m_e^A is total, so $m_e^A(\varphi_{k(f(v,e,\overline{y}))}(z))\downarrow$. But

$$(\forall s)\big[z \notin W_{g(v,e,\overline{y}),s}\big]$$

so that for infinitely many s we have both

$$s > m_e^A(\varphi_{k(f(v,e,\overline{y}))}(z)) \quad \text{and}$$
$$z \notin W_{g(v,e,\overline{y}),s}.$$

Together these imply $U^A(v,e,\overline{y})$ (condition (21)). $\square$

Sublemma 5. *$V^A(e)$ (Property (Esc)) is true.*

Proof. Suppose $V^A(e)$ is false. Then we have the following two conditions holding jointly

$$(\forall^\infty y_{2n-2})(\exists^\infty y_{2n-3})\dots(\forall^\infty y_2)(\exists^\infty y_1)\ \neg U^A(v,e,\overline{y}) \tag{22}$$
$$(\forall^\infty y_{2n-2})(\forall y_{2n-3})\dots(\forall^\infty y_2)(\forall y_1)\ \big[W_{g(v,e,\overline{y})} \text{ is finite }\big] \tag{23}$$

where the first condition expresses the negation of (Esc), and the second condition is an application of Theorem 5.6. The quantifiers of these two conditions line-up as $\{\forall^\infty,\forall^\infty\}$ and $\{\exists^\infty,\forall\}$. Applying Lemma (2.9g,i), the two conditions imply

$$(\forall^\infty y_{2n-2})(\exists^\infty y_{2n-3})\dots(\forall^\infty y_2)(\exists^\infty y_1)\ \big[\neg U^A(v,e,\overline{y})\ \ \&\ \ W_{g(v,e,\overline{y})} \text{ is finite }\big].$$

This implies that there exist $y_1,\dots,y_{2n-2}$ such that

$$\big[\neg U^A(v,e,\overline{y})\ \ \&\ \ W_{g(v,e,\overline{y})} \text{ is finite }\big],$$

which is impossible by Sublemma 4. $\square$

From Sublemma 5 $V^A(e)$ is true, so condition (Esc) holds for e, and from Theorem 5.6, condition (Tot) holds. Thus, the following hold conjointly

$$(\exists^\infty y_{2n-2})(\forall^\infty y_{2n-3})\dots(\exists^\infty y_2)(\forall^\infty y_1)\ U^A(v,e,\overline{y})$$
$$(\forall y_{2n-2})(\forall^\infty y_{2n-3})\dots(\forall y_2)(\forall^\infty y_1)\ \big[W_{g(v,e,\overline{y})} = \omega\big].$$

The quantifiers of these two conditions line-up as $\{\exists^\infty,\forall\}$ and $\{\forall^\infty,\forall^\infty\}$. Applying Lemma (2.9gi), the two conditions imply

$$(\exists^\infty y_{2n-2})(\forall^\infty y_{2n-3})\dots(\exists^\infty y_2)(\forall^\infty y_1)\ \big[U^A(v,e,\overline{y})\ \ \&\ \ W_{g(v,e,\overline{y})} = \omega\big].$$

Thus, for some $y_1,\dots,y_{2n-2}$

$$(\exists^\infty s)(\exists x)\big[s \geq m_e^A(\varphi_{k(f(v,e,\overline{y}))}(x))\ \ \&\ \ x \notin W_{g(v,e,\overline{y}),s}\big] \tag{24}$$
$$W_{g(v,e,\overline{y})} = \omega \tag{25}$$

The escape function required by the aligned escape property is $\varphi_{f(v,e,\overline{y})}$, given by Equation (14). This follows from conditions (24) and (25) by Sub-lemma 3.

Case low_{2n} :

Suppose A be low_{2n}, so that by Lemma 2.12, $\Sigma^A_{2n+1} = \Sigma_{2n+1}$.

Let $\overline{y}$ be $\langle y_1, \ldots, y_{2n-1}\rangle$, and

$$\tilde{U}^A(j,e,\overline{y}) \iff (\exists^\infty s)(\exists x)\big[s \geq m^A_e(\varphi_{k(f(j,e,\overline{y}))}(x)) \;\;\&\;\; x \notin W_{g(j,e,\overline{y}),s}\big].$$

We will show that there is a value $j \in \omega$ so that the following two properties are satisfied

$$(\forall^\infty y_{2n-1})(\exists^\infty y_{2n-2})\ldots(\forall^\infty y_3)(\exists^\infty y_2)(\forall^\infty y_1)\;\tilde{U}^A(j,e,\overline{y}) \tag{26}$$

$$(\forall^\infty y_{2n-1})(\forall y_{2n-2})\ldots(\forall^\infty y_3)(\forall y_2)(\forall^\infty y_1)\;\big[W_{g(j,e,\overline{y})} = \omega\big]. \tag{27}$$

Let $\tilde{V}^A(j,e)$ be Equation (26), which is Σ^A_{2n+1}. Since A is low_{2n}, by Lemma 2.12 $\tilde{V}^A(j,e)$ is actually Σ_{2n+1}, so

$$\tilde{V}^A(j,e) \iff (\exists y_{2n-1})(\forall y_{2n-2})\ldots(\exists y_2)(\forall y_1)(\forall z)\big[\langle e,j,\overline{y},z\rangle \in W_i\big]$$

for some index i. By the s-m-n Theorem ([Soa87, I.3.5]) there is a computable p such that

$$\tilde{V}^A(j,e) \iff (\exists y_{2n-1})(\forall y_{2n-2})\ldots(\forall y_2)(\exists y_1)(\forall z)\big[\langle e,\overline{y},z\rangle \in W_{p(j)}\big].$$

By the Recursion Theorem, there is a computable v such that

$$W_{p(v)} = W_v.$$

Let $U^A(e,\overline{y})$ be $\tilde{U}^A(v,e,\overline{y})$:

$$U^A(e,\overline{y}) \iff (\exists^\infty s)(\exists x)\big[s \geq m^A_e(\varphi_{k(f(v,e,\overline{y}))}(x)) \;\;\&\;\; x \notin W_{g(v,e,\overline{y}),s}\big].$$

and $V^A(e)$ be $\tilde{V}^A(v,e)$. Then v is a Σ_{2n+1}-index for V^A

$$\begin{aligned} V^A(e) &\iff \hat{V}^A(v,e) \\ &\iff (\exists y_{2n-1})(\forall y_{2n-2})\ldots(\forall y_2)(\exists y_1)(\forall z)\big[\langle e,\overline{y},z\rangle \in W_{p(v)}\big] \\ &\iff (\exists y_{2n-1})(\forall y_{2n-2})\ldots(\forall y_2)(\exists y_1)(\forall z)\big[\langle e,\overline{y},z\rangle \in W_v\big]. \end{aligned}$$

Inserting v into the conditions (26) and (27)

$$(\forall^\infty y_{2n-1})(\exists^\infty y_{2n-2})\dots(\forall^\infty y_3)(\exists^\infty y_2)(\forall^\infty y_1)\ U^A(v,e,\overline{y}) \tag{Esc}$$
$$(\forall^\infty y_{2n-1})(\forall y_{2n-2})\dots(\forall^\infty y_3)(\forall y_2)(\forall^\infty y_1)\ \left[W_{g(v,e,\overline{y})}=\omega\right] \tag{Tot}$$

and $V^A(e)$ is equivalent to (Esc) for all e.

Since v is a Σ_{2n+1}-index for V^A, it follows from Theorem 5.6 that

$$\begin{aligned} V^A(e) &\iff (\forall^\infty y_{2n-1})(\forall y_{2n-2})\dots(\forall^\infty y_3)(\forall y_2)(\forall^\infty y_1)\ \left[W_{g(v,e,\overline{y})}=\omega\right] \\ \neg V^A(e) &\iff (\forall y_{2n-1})(\forall^\infty y_{2n-2})\dots(\forall y_3)(\forall^\infty y_2)(\forall y_1)\ \left[W_{g(v,e,\overline{y})} \text{ is finite }\right]. \end{aligned}$$

Sublemma 6. *$V^A(e)$ (Property (Esc)) is true.*

Proof. Suppose $V^A(e)$ is false. Then we have the following two conditions holding jointly

$$(\exists^\infty y_{2n-1})(\forall^\infty y_{2n-2})\dots(\exists^\infty y_3)(\forall^\infty y_2)(\exists^\infty y_1)\ \neg U^A(e,\overline{y}) \tag{28}$$
$$(\forall y_{2n-1})(\forall^\infty y_{2n-2})\dots(\forall y_3)(\forall^\infty y_2)(\forall y_1)\ \left[W_{g(v,e,\overline{y})} \text{ is finite }\right] \tag{29}$$

where the first condition expresses the negation of (Esc), and the second condition is an application of Theorem 5.6. The quantifiers of these two conditions line-up as $\{\exists^\infty,\forall\}$ and $\{\forall^\infty,\forall^\infty\}$. Applying Lemma (2.9g,i), the two conditions imply

$$(\exists^\infty y_{2n-1})(\forall^\infty y_{2n-2})\dots(\exists^\infty y_3)(\forall^\infty y_2)(\exists^\infty y_1)\ \left[\neg U^A(e,\overline{y})\ \ \&\ \ W_{g(v,e,\overline{y})} \text{ is finite }\right].$$

This implies that there exist $y_1,\dots,y_{2n-1}$ such that

$$\left[\neg U^A(e,\overline{y})\ \ \&\ \ W_{g(v,e,\overline{y})} \text{ is finite }\right]$$

which is impossible by Sublemma 4. □

From Sublemma 6 $V^A(e)$ is true, so condition (Esc) holds, and from Theorem 5.6, condition (Tot) holds. Thus, the following hold conjointly

$$(\forall^\infty y_{2n-1})(\exists^\infty y_{2n-2})\dots(\forall^\infty y_3)(\exists^\infty y_2)(\forall^\infty y_1)\ U^A(v,e,\overline{y})$$
$$(\forall^\infty y_{2n-1})(\forall y_{2n-2})\dots(\forall^\infty y_3)(\forall y_2)(\forall^\infty y_1)\ \left[W_{g(v,e,\overline{y})}=\omega\right]$$

The quantifiers of these two conditions line-up as $\{\forall^\infty, \forall^\infty\}$ and $\{\exists^\infty, \forall\}$. Applying Lemma (2.9gi), the two conditions imply

$$(\forall^\infty y_{2n-1})(\exists^\infty y_{2n-2}) \ldots (\forall^\infty y_2)(\exists^\infty y_1) \left[U^A(v, e, \overline{y}) \;\;\&\;\; W_{g(v,e,\overline{y})} = \omega\right]$$

Thus, for some $y_1, \ldots, y_{2n-1}$

$$(\exists^\infty s)(\exists x)\left[s \geq m_e^A(\varphi_{k(f(v,e,\overline{y}))}(x)) \;\;\&\;\; x \notin W_{g(v,e,\overline{y}),s}\right] \tag{30}$$

$$W_{g(v,e,\overline{y})} = \omega \tag{31}$$

The escape function required by the aligned escape property is $\varphi_{f(v,e,\overline{y})}$, given by Equation (14). This follows from conditions (30) and (31) by Sublemma 3.

□

References

[Ehr61] A. Ehrenfeucht. Separable theories. *Bull. Acad. Polon. Scie. Ser. Sci. Math. Astronom. Phys.*, 9:17–19, 1961.

[GN73] S.S. Goncharov and A.T. Nurtazin. Constructive models of complete decidable theories (russian). *Algebra i Logika*, 12:125–42, 1973. [English Translation: Algebra and Logic, **12** (1973) 67-77].

[Har] Kenneth Harris. A characterization of the low_n degrees by escape functions. At **http://people.cs.uchicago.edu/~kaharris/papers/computability/lown.pdf**.

[Har98] V.S. Harizanov. Pure computable model theory. In Y.L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel, editors, *Handbook of Recursive Mathematics*, volume 138 of *Studies in Logic and the Foundations of Mathematics*, chapter 1. Elsevier, 1998.

[Joc72] Carl Jockusch. Degrees in which the recursive sets are uniformly recursive. *Canadian Journal of Mathematics*, 24:1092–1099, 1972.

[Mil78] T.S. Millar. Foundations of recursive model theory. *Annals of Mathematical Logic*, 13:305–320, 1978.

[MM84] A.J. Macintyre and D. Marker. Degrees of recursively saturated models. *Transactions of the American Mathematical Society*, 282:539–54, 1984.

[Mor76] M. Morley. Decidable models. *Israel Journal of Mathematics*, 25:233–40, 1976.

[Rog87] Hartley Rogers. *Theory of Recursive Functions and Effective Computability*. The MIT Press, 1987.

[Sch60] J. Schoenfield. Degrees of models. *Journal of Symbolic Logic*, 252:233–7, 1960.

[Soa87] R.I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematics. Springer-Verlag, 1987.