Reverse Mathematics of Saturated Models

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1 Introduction

This note will investigate the proof theoretic strength in second-order arithmetic of the following classical theorems in model theorem involving saturated and universal models:

(SMT) A theory $\mathcal{T}$ has a saturated model if and only if there are countably many types of $\mathcal{T}$.

(SUT) If $\mathfrak{A}$ and $\mathfrak{B}$ are saturated models of a theory $\mathcal{T}$, then $\mathfrak{A} \cong \mathfrak{B}$.

(SAU) All saturated models are universal.

(UAS) If a theory has a universal model then it has a saturated model. (Universal models need not be saturated, see [CK90, Section 2.3].)

(UUT) If $\mathfrak{A}$ and $\mathfrak{B}$ are universal models of a theory $\mathcal{T}$, then $\mathfrak{A} \cong \mathfrak{B}$.

When properly interpreted in second-order arithmetic, the first theorem (*Vaught*), is equivalent to ATR_0. The theorems (SUT) and (SAU) are equivalent to ACA_0. The theorem (UAS) is provable in RCA_0. The theorem (UUT) is provable in ACA_0, although it is unknown whether this can be reversed.

There are also connections between prime and atomic models and atomic theories whose proof theoretic strength is investigated in [HSS], a model for this note.

2 Model Theory in Reverse Mathematics

In this note all languages $\mathcal{L}$ and models $\mathfrak{A}$ are countable; all theories $\mathcal{T}$, in a language $\mathcal{L}$, are deductively closed ($\mathcal{T} \vdash \varphi \Rightarrow \varphi \in \mathcal{T}$), complete ($\mathcal{T} \vdash \varphi$ or $\mathcal{T} \vdash \neg\varphi$ for every sentence φ of $\mathcal{L}$) and consistent (there is no sentence φ such that $\mathcal{T} \vdash \varphi$ and $\mathcal{T} \vdash \neg\varphi$.) For the purposes of reverse mathematics, we include with a model $\mathfrak{A}$ of a theory $\mathcal{T}$ the function interpreting terms and the elementary diagram $\mathrm{Th}(\mathfrak{A}_A)$ (the full satisfaction predicate for formulas with constants from the model.) Formal definitions suitable for work in RCA_0 can be found in [Sim99, II.8] along with, for example, a proof in RCA_0 that any complete consistent theory has a model in this sense.

In this note we will be concerned with trees $T \subseteq 2^{<\mathbb{N}}$. Such trees code closed subsets of $2^{\mathbb{N}}$. A tree T is *extendible* if for each $\sigma \in T$ there is a $\tau \supset \sigma$ with $\tau \in T$. For example, the Stone space of types for a theory $\mathcal{T}$, $\mathcal{S}(\mathcal{T})$, is an extendible tree. A tree T is *perfect* if for each $\sigma \in T$ there are $\tau_1, \tau_2 \in T$ such that $\sigma \subset \tau_1, \tau_2$ and $\tau_1 \,\big|\, \tau_2$ (τ_1 and τ_2 are incompatible–for some $i < |\tau_1|, |\tau_2|$, $\tau_1(i) \neq \tau_2(i)$.) We will write for a function $f \in 2^{\mathbb{N}}$, $\overline{f}(n)$ for the finite sequence $\langle f(0), \ldots, f(n)\rangle$, and $f \in T$ to mean that f is a path through T (for each $n \in \mathbb{N}$, $\overline{f}(n) \in T$.) We will say that T is *countable* if there exists a sequence $\langle f_n : n \in \mathbb{N}\rangle$ such that if $f \in T$ then there exists an $n \in \mathbb{N}$ with $f = f_n$. (We will also say that the paths of T are *enumerable* in this case, and the sequence of paths an enumeration.) We will say that T *contains a perfect set* if there exists a perfect subtree $P \subseteq T$. Finally, note that in RCA_0 we can code a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ as a tree $T^* \subseteq 2^{\mathbb{N}}$ and code $f \in \mathbb{N}^{\mathbb{N}}$ as $f^* \in 2^{\mathbb{N}}$ such that we can prove in RCA_0 that

$$\forall f \left[f \in T \longleftrightarrow f^* \in T^* \right]$$

(see [Sim99, Lemma V.5.6].)

2.1 Stone Space of Types

We can code the types consistent with a theory as a subtree of $2^{<\mathbb{N}}$, the Stone space of types. See [Mar02, Chapter 4.1] for more details on the Stone space of types.

Fix a language $\mathcal{L}$. Let $x_1, x_2, \ldots$ be a canonical sequence of variables and for each $n \in \mathbb{N}$ let $\psi_0(x_1, \ldots, x_n), \psi_1(x_1, \ldots, x_n), \ldots$ a canonical sequence of formulas in n variables. Given a complete theory $\mathcal{T}$ in the language $\mathcal{L}$, let $\mathcal{S}_n(\mathcal{T}) \subseteq 2^{<\mathbb{N}}$ be the tree of n-types

$$\sigma \in \mathcal{S}_n(\mathcal{T}) \;\leftrightarrow\; (\exists x_1, \dots, x_n) \bigwedge_{i<|\sigma|} \psi_i^{\sigma(i)}(x_1, \dots, x_n) \in \mathcal{T}$$

where

$$\psi_\sigma \quad =_{\text{def}} \quad \bigwedge_{i<|\sigma|} \psi_i^{\sigma(i)}(x_1, \dots, x_n)$$

and ψ^0 is $\neg\psi$ and ψ^1 is ψ. The existence of $\mathcal{S}(\mathcal{T})$n is provable in RCA_0 for each theory $\mathcal{T}$ by Δ^0_1 comprehension. There is a correspondence between paths f through $\mathcal{S}_n(\mathcal{T})$ and types $\Gamma(x_1, \dots, x_n)$ consistent with $\mathcal{T}$ given by

$$\begin{aligned} \Gamma_f(x_1, \dots, x_n) \;&=_{\text{def}}\; \left\{\psi_{\langle f(0), \dots, f(n)\rangle} \;:\; n \in \mathbb{N}\right\} \\ f_\Gamma \;&=_{\text{def}}\; \left\{\langle i, j\rangle \;:\; j \in \{0,1\} \;\&\; \psi_i^j \in \Gamma\right\}. \end{aligned}$$

The following is provable in RCA_0 for each theory $\mathcal{T}$

> $\langle f_i \;:\; i \in \mathbb{N}\rangle$ is an enumeration of paths in $\mathcal{S}_n(\mathcal{T})$ if and only if $\langle \Gamma_{f_i} \;:\; i \in \mathbb{N}\rangle$ is an enumeration of the n-types of $\mathcal{T}$; and, $\langle \Gamma_i \;:\; i \in \mathbb{N}\rangle$ is an enumeration of the n-types of $\mathcal{T}$ if and only if $\langle f_{\Gamma_i} \;:\; i \in \mathbb{N}\rangle$ is an enumeration of paths in $\mathcal{S}_n(\mathcal{T})$.

Let $\mathcal{S}(\mathcal{T}) \subseteq 2^{<\mathbb{N}}$ be the tree formed by attaching $\mathcal{S}_n(\mathcal{T}) \subseteq 2^{<\mathbb{N}}$ to the node $0^{n+1}1$. It is provable in RCA_0 that $\mathcal{S}(\mathcal{T})$ exists. Using the above correspondence between paths in $\mathcal{S}(\mathcal{T})$n and n-types of $\mathcal{T}$, RCA_0 proves that there is an enumeration of paths in $\mathcal{S}(\mathcal{T})$ if and only if there is an enumeration of the types of $\mathcal{T}$.

2.2 Coding trees as theories

The goal in this subsection is to show that for each tree $T \subseteq 2^{<\mathbb{N}}$, we can prove in RCA_0 that there exists a theory $\mathcal{T}$ such that the following are provable (in RCA_0)

(i) T contains a perfect subtree if and only if $\mathcal{S}(\mathcal{T})$ contains a perfect subtree.

(ii) There are countably many paths of T if and only if there are countably many types of $\mathcal{S}(\mathcal{T})$.

Let $T \subseteq 2^{<\mathbb{N}}$. Let $\mathcal{L}$ consist of the binary relation of equality $=$, two propositional constants $\top$ and $\bot$ (alternatively, we can take $\top$ to be the formula $x = x$ and $\bot$ to be the formula $x \neq x$) and the following unary predicates

$$\{P_\sigma \;:\; \sigma \in 2^{<\mathbb{N}}\}.$$

The axioms of $\mathcal{T}$ will be given by the following

(a) $\forall x \, \neg P_\sigma(x)$, for each $\sigma \notin T$,

(b) $\exists^{\geq n} x \; P_\sigma(x)$, for each $\sigma \in T$ and $n \in \mathbb{N}$ (that is, axioms stating that there are at least n distinct elements in the domain satisfying $P_\sigma(x)$),

(c) $\forall x \; \big[(P_\sigma(x) \;\&\; \neg P_{\sigma^\frown\langle i\rangle}(x)) \;\rightarrow\; P_{\sigma^\frown\langle(1-i)\rangle}(x)\big]$, for each $\sigma^\frown\langle(1-i)\rangle \in T$ (that is axioms that ensure that if $P_\sigma(x)$ and σ is not a leaf, then either $P_{\sigma^\frown\langle i\rangle}(x)$ or $P_{\sigma^\frown\langle(1-i)\rangle}(x)$),

(d) $\forall x \; \big[P_\sigma(x) \;\rightarrow\; P_\tau(x)\big]$ whenever $\tau \subseteq \sigma$,

(e) $\forall x \; \big[P_\sigma(x) \;\rightarrow\; \neg P_\tau(x)\big]$ whenever $\tau \,|\, \sigma$,

(f) $\forall x \; \big[\neg P_\sigma(x) \;\rightarrow\; \neg P_\tau(x)\big]$ whenever $\sigma \subseteq \tau$.

The existence of a set of axioms A_T satisfying (a) through (e) is provable in RCA_0 by Δ^0_1 comprehension for each tree T. We will show that the set of consequences of A_T is complete and this is provable in RCA_0, so that we can define $\mathcal{T}$ as the set of consequences of A_T:

$$\forall x \; \big[x \in \mathcal{T} \;\leftrightarrow x \text{ is a sentence and } A_T \vdash x\big]$$
$$\forall x \; \big[x \notin \mathcal{T} \;\leftrightarrow x \text{ is not a sentence or } A_T \vdash \neg x\big]$$

and RCA_0 proves the existence of $\mathcal{T}$ by Δ^0_1 comprehension.

We will show that A_T is complete by proving quantifier elimination holds in RCA_0. The procedure described will show that for every sentence φ, RCA_0 proves:

$$A_T \vdash \varphi \quad \Longrightarrow \quad A_T \vdash (\varphi \;\leftrightarrow\; \top)$$
$$A_T \nvdash \varphi \quad \Longrightarrow \quad A_T \vdash (\varphi \;\leftrightarrow\; \bot).$$

The proof is by induction on the structural complexity of *formulas*. Let $\exists \overline{x} \;\; \psi(\overline{x}, \overline{y})$ be a formula where $\psi(\overline{x}, \overline{y})$ is an atomic formula in disjunctive normal form. Then we can write ψ as a disjunction whose conjuncts have the form

$$\epsilon(\overline{x}, \overline{y}); \& \ \gamma_1(z_1) \ \& \ldots \& \ \gamma_k(z_k)$$

where $\epsilon(\overline{x}, \overline{y})$ are a conjunction of identities and $z_1 \ldots z_k$ are all among $\overline{x}, \overline{y}$. If either ϵ is inconsistent with the identity laws, or $z_i = z_j$ is an identity of ϵ but $\gamma_i(z_i)$ & $\gamma_j(z_i)$ is inconsistent with A_T, or for some $i \leq k$ $\gamma_i(z_i)$ is inconsistent with A_T, then replace then entire conjunction with $\perp$. Otherwise, eliminate all identities of ϵ which contain some variable from $\overline{x}$ and replace all $\gamma(z_i)$ with $\top$ where z_i is among $\overline{x}$. Note that if $\overline{y}$ is empty, the reduction above produces either $\top$ or $\perp$ after simplification. The procedure can be carried-out in RCA_0 using Σ^0_1 induction.

It is provable in RCA_0 that for the theory $\mathcal{T}$,

> $\mathcal{S}(\mathcal{T})$ is countable iff $\mathcal{S}_1(\mathcal{T})$ is countable, and $\mathcal{S}(\mathcal{T})$ contains a perfect subset iff $\mathcal{S}_1(\mathcal{T})$ contains a perfect subset.

To see this: For an n-type $\Gamma(x_1, \ldots, x_n)$ let $\Gamma_1(x_1), \ldots, \Gamma_k(x_1)$ be the distinct 1-types and $\Gamma^*(x_1, \ldots, x_n)$ be all formulas of the form $x_i = x_j$ and $x_i \neq x_j$ where $1 \leq i, j \leq n$. Then $\Gamma(x_1, \ldots, x_n)$ "says" there are n_1 terms of type $\Gamma_1(x_1)$ and n_2 terms of type $\Gamma_2(x_1)$ and ... and n_k terms of type $\Gamma_k(x_1)$. Thus, there are countably many 1-types iff there are countably many n-types for each $n \in \mathbb{N}$, and there is a perfect subtree of 1-types iff there is a perfect subtree of n-types for some $n \in \mathbb{N}$.

So, to prove (i) and (ii) we need only consider 1-types. For each $\Gamma(x_1) \in \mathcal{S}_1(\mathcal{T})$ and each $m \in \mathbb{N}$ there is at most one σ with $|\sigma| = m$ and $P_\sigma(x_1) \in \Gamma(x_1)$. If for each $m \in \mathbb{N}$ there is exactly one σ_m with $|\sigma_m| = m$ and $P_{\sigma_m}(x_1) \in \Gamma(x_1)$, then $\{\sigma_m \ : \ m \in \mathbb{N}\}$ defines a path in T. On the other hand if there is an $m \in \mathbb{N}$ such that for all σ with $|\sigma| = m$, $\neg P_\sigma(x_1) \in \Gamma(x_1)$, then $\Gamma(x_1)$ is isolated in $\mathcal{S}_1(\mathcal{T})$. Note that the maximum σ such that $P_\sigma(x_1) \in \Gamma(x_1)$ must be a leaf in T by axiom (c). It is straightforward to verify (i) and (ii) hold for $\mathcal{S}(\mathcal{T})$, and this is provable in RCA_0.

2.3 Saturated and Universal Models

See [CK90, Sections 2.3] or [Mar02, Chapter 4] for more details on the relevant definitions.

Definition 1. A model $\mathfrak{A}$ of a theory $\mathcal{T}$ is *saturated* if for every type $\Gamma(x_1, \ldots, x_n, x_{n+1})$ consistent with $\mathcal{T}$ and all $a_1, \ldots, a_n$ from $\mathfrak{A}$, if $\Gamma(a_1, \ldots, a_n, x_{n+1})$ is consistent with $\mathrm{Th}(\mathfrak{A}_A)$ then for some d in $\mathfrak{A}$, $\mathfrak{A} \models \Gamma(a_1, \ldots, a_n, d)$.

A model $\mathfrak{A}$ of a theory $\mathcal{T}$ is *weakly saturated* if for every type $\Gamma(x_1, \ldots, x_n)$ consistent with $\mathcal{T}$ there are $a_1, \ldots, a_n$ from $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma(a_1, \ldots, a_n)$.

A model $\mathfrak{A}$ of a theory $\mathcal{T}$ is *universal* if for all models $\mathfrak{B}$ of $\mathcal{T}$ there is an elementary embedding $F : \mathfrak{B} \to \mathfrak{A}$.

Vaught's theorem is that a model is saturated if and only if there are countably many types. We will investigate the proof theoretic strength of this theorem in Subsection 3.1.

There are also connections between saturated and homogeneous models. The classical equivalence is (see [Mar02, Proposition 4.3.4])

Theorem 2. *A model is saturated if and only if it is weakly saturated and homogeneous.*

The proof theoretic strength of this theorem depends upon the definition of *homogeneous*. We will take the following as defining homogeneous: (see [CK90, Sections 2.4] or [Mar02, Chapter 4].)

Definition 3. For any model $\mathfrak{A}$ and tuples of elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ from $\mathfrak{A}$ we will write

$$\big(\mathfrak{A}, a_1, \ldots, a_n\big) \equiv \big(\mathfrak{A}, b_1, \ldots, b_n\big)$$

to mean that the tuples $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ realize the same type in $\mathfrak{A}$.

A model $\mathfrak{A}$ of a theory $\mathcal{T}$ is *homogeneous* if for any tuples $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfying

$$\big(\mathfrak{A}, a_1, \ldots, a_n\big) \equiv \big(\mathfrak{A}, b_1, \ldots, b_n\big)$$

the following back-and-forth condition holds

$$\forall c \in A \exists d \in A \left[\big(\mathfrak{A}, a_1, \ldots, a_n, c\big) \equiv \big(\mathfrak{A}, b_1, \ldots, b_n, d\big)\right].$$

Note that the relation of elementary equivalence between finite tuples in a model, $\equiv$, is an arithmetic condition. A classical alternative characterization of homogeneity is given by the following (see [Mar02, Proposition 4.2.13])

Theorem 4. *A model $\mathfrak{A}$ is homogeneous if and only if for every pair of tuples $\overline{a}$ and $\overline{b}$ of $\mathfrak{A}$ such that*

$$\big(\mathfrak{A}, a_1, \ldots, a_n\big) \equiv \big(\mathfrak{A}, b_1, \ldots, b_n\big)$$

there is an automorphism F on $\mathfrak{A}$ such that $F(\overline{a}) = \overline{b}$.

This equivalence is provable in ACA by formalizing the usual back-and-forth argument. (The direction ($\Leftarrow$) is provable in RCA_0.)

> $EQUIV(\mathfrak{A}, \overline{a}, \overline{b})$ says: $\mathfrak{A}$ is a model (of the language $\mathcal{L}$, which is suppressed from the predicate), $\overline{a}$ and $\overline{b}$ are tuples of the same length from $\mathfrak{A}$ and $\big(\mathfrak{A}, \overline{a}\big) \equiv \big(\mathfrak{A}, \overline{b}\big)$.

There are two ways of interpreting a relation like $\equiv$ in reverse mathematics. First, as a ternary predicate in the language of second-order arithmetic (as $EQUIV$), which I have done above in the definition of homogeneous. Second, as an explicitly defined relation in the domain of discourse, which we will formalize as

$$(EE) \qquad \forall \mathfrak{A} \exists Z \forall \overline{a}, \overline{b} \, \big[\langle \overline{a}, \overline{b} \rangle \in Z \;\leftrightarrow\; EQUIV(\mathfrak{A}, \overline{a}, \overline{b})\big].$$

We have the following

Theorem 5 (ACA_0, or $\mathrm{RCA}_0 + (EE)$)**.** *A model $\mathfrak{A}$ is homogeneous if and only if for every pair of tuples $\overline{a}$ and $\overline{b}$ of $\mathfrak{A}$ such that*

$$\big(\mathfrak{A}, a_1, \ldots, a_n\big) \equiv \big(\mathfrak{A}, b_1, \ldots, b_n\big)$$

there is an automorphism F on $\mathfrak{A}$ such that $F(\overline{a}) = \overline{b}$.

The effect of adding (EE) is to allow us to use Δ^0_1 comprehension in constructing the automorphism in the direction ($\Rightarrow$). We will show below that $\mathrm{RCA}_0 + (EE)$ is equivalent to ACA_0.

3 Reverse Mathematics of Saturated Models

3.1 Vaught's Theorem

The classical theorem on the existence of saturated models is

Theorem 6 (Vaught's Saturated Model Theorem)**.** *A theory $\mathcal{T}$ has a saturated model if and only if there are countably many types consistent with $\mathcal{T}$.*

In order to investigate the implications of this theorem for reverse mathematics, we must interpret the statement that *A theory $\mathcal{T}$ has countably many consistent types*. The class of types of a theory can be expressed in second-order arithmetic as the paths through the Stone space of types $\mathcal{S}(\mathcal{T})$. This is a Σ^1_1 class of reals. Classically, the continuum hypothesis holds for Σ^1_1 classes of reals: every Σ^1_1 class of reals is either countable or contains a perfect subset. (See [Sac, III.6.2] for an even stronger statement of this then will be needed here.) The reverse mathematical content of this theorem is given as follows [Sim99, Theorem V.5.5]

Theorem 7. *The following are equivalent over ACA_0*

(1) ATR_0

(2) For every analytic code A, if A is uncountable then A has a nonempty perfect subset.

(3) For every tree $T \subseteq 2^{<\mathbb{N}}$, if T has uncountably many paths, then T has a nonempty perfect subtree.

(4) For every tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, if T has uncountably many paths, then T has a nonempty perfect subtree.

(The numbering corresponds to the numbering of [Sim99, Theorem V.5.5].)

This gives us two options for interpreting Vaught's Theorem, which turn out to have very different proof-theoretic strength:

(*Vaught*) A theory $\mathcal{T}$ has a saturated model if and only if $\mathcal{S}(\mathcal{T})$ is countable.

(*SMT*) A theory $\mathcal{T}$ has a saturated model if and only if $\mathcal{S}(\mathcal{T})$ contains no perfect subset of types.

Morley and Millar (see [Mor76] or [Mil78].) independently proved the following computability theoretic version of Vaught's theorem on the existence of saturated models:

> A complete, decidable theory has a decidable saturated model if and only if there is a computable enumeration of the types consistent with the theory. (This theorem relativizes to Turing degrees.)

This construction can be carried-out in RCA_0, yielding a proof of (*Vaught*) and the ($\Leftarrow$) of (*SMT*). But Millar [Mil78]also produced a complete and decidable theory, where all types of the theory are in fact decidable, and which has no decidable saturated model (although there is a saturated model.) Thus, in the minimal model for RCA_0, where set existence is over all computable sets, there exist theories with saturated models, but no saturated models in the universe of the model. So, (*Vaught*) comes-up short of capturing Vaught's theorem, although it most directly captures the meaning of *countable*.

Theorem 8. *In ACA_0, (SMT) if and only if ATR_0.*

This result follows the equivalence of (1) and (3) from Theorem 7, together with the following

Lemma 9. *In RCA_0, (SMT) if and only if (V.5.5.3) for every tree $T \subseteq 2^{<\mathbb{N}}$, if T has uncountably many paths, then T has a nonempty perfect subtree. (This is condition (3) of Theorem 7.)*

Proof of Lemma. ($\Rightarrow$). In RCA_0: Suppose $\mathcal{S}(\mathcal{T})$ has no perfect subset. Then by (V.5.5.3) $\mathcal{S}(\mathcal{T})$ is countable. Since $\mathrm{RCA}_0 \vdash$ (*Vaught*), so $\mathcal{T}$ has a saturated model.

($\Leftarrow$). In RCA_0: We showed in Subsection 2.2 that for each tree $T \subseteq 2^{<\mathbb{N}}$, there exists a theory $\mathcal{T}$ such that the following are provable (in RCA_0)

(i) T contains a perfect subtree if and only if $\mathcal{S}(\mathcal{T})$ contains a perfect subtree.

(ii) There are countably many paths of T if and only if there are countably many types of $\mathcal{S}(\mathcal{T})$.

Let $T \subseteq 2^{<\mathbb{N}}$ and suppose that T has no perfect subtree (the antecedent of (V.5.5.3).) Let $\mathcal{T}$ be a theory satisfying (i) and (ii). By (i) $\mathcal{S}(\mathcal{T})$ contains no perfect subtree, so by (*SMT*) there exists a saturated model $\mathfrak{A}$ of $\mathcal{T}$. By (*Vaught*) (provable in RCA_0) there are countably many types of $\mathcal{S}(\mathcal{T})$, so by (ii) it follows that there are countably many paths of T. Thus, (V.5.5.3). $\square$

There are connections with atomic theories, atomic models and prime models through the classic theorem: *If a theory has a saturated model then the theory is atomic.* But this result is provable in RCA_0.

3.2 Computability theoretic analysis of Vaught's Theorem

Let $\mathcal{M} \models \mathrm{ATR}_0$ and $\mathcal{T}$ a theory in the domain of $\mathcal{M}$ and suppose that $\mathcal{T}$ has a saturated model; we will show that there is a saturated model of $\mathcal{T}$ in $\mathcal{M}$. Since there are countably many types for $\mathcal{T}$, $\mathcal{S}(\mathcal{T})$ does not contain a perfect subset. By Theorem 7 (3),

$$\mathcal{M} \models \mathcal{S}(\mathcal{T}) \text{ countable or } \mathcal{S}(\mathcal{T}) \text{ contains a perfect subset,}$$

so, $\mathcal{M} \models \mathcal{S}(\mathcal{T})$ is countable, and by (*Vaught*)

$$\mathcal{M} \models \textit{There is a saturated model of } \mathcal{T}.$$

It does not yet follow that this model for $\mathcal{T}$ which is saturated according to $\mathcal{M}$, is really a saturated model.

If $\mathscr{C}$ is a class of reals which is Σ^1_1 in a set Z and contains no perfect subset, then there is an enumeration of the class $\mathscr{C}$ which is hyperarithmetic in Z. (This is a relativization of [Sac, III.6.3].) Let the predicate $\mathcal{O}(a, Z)$ say that a is an ordinal notation relative to Z ([Sim99, Definition VIII.3.2]) and H^Z_a the H-set relative to Z obtained by iterating the Turing jump along the ordinal notations relative to Z below a ([Sim99, Definition VIII.3.5]). So, if X is hyperarithmetic relative to Z, then there is an a with $\mathcal{O}(a, Z)$ and $X \leq_{\mathrm{T}} H^Z_a$. Then

$$\mathrm{ATR}_0 \vdash \forall Z \forall a \left[\mathcal{O}(a, Z) \ \rightarrow\ H^Z_a \text{ exists}\right]$$

([Sim99, Theorem VIII.3.15].) So, for any model $\mathcal{M}$ of ATR_0 and Z in the domain of $\mathcal{M}$, if X is hyperarithmetic in Z then X is in the domain of $\mathcal{M}$. If $\mathcal{T}$ is a theory in the domain of $\mathcal{M} \models \mathrm{ATR}_0$ with a saturated model, then there is an enumeration of the paths in $\mathcal{S}(\mathcal{T})$ hyperarithmetic in $\mathcal{T}$, and thus an enumeration of the paths of $\mathcal{S}(\mathcal{T})$ in $\mathcal{M}$. Therefore, by (*Vaught*) there is a saturated model of $\mathcal{T}$ in the domain of $\mathcal{M}$.

The correspondence between ATR_0 and (SMT) is also tight, in that the entire hyperarithmetic hierarchy is needed to compute saturated models. For each ordinal notation a there is a theory $\mathcal{T}_a$ with a saturated model computable in H_a, and for any degree $\mathbf{d}$ which can compute a saturated model for $\mathcal{T}_a$, $H_a \leq \mathbf{d}$. (H_a is the H-set corresponding to the ordinal $|a|$–see [Sac, §II.1] or [Sim99, §VIII.3].) There is a Π^0_2 predicate $H(a, X)$ such that for each ordinal notation a, H_a is the unique set X such that $H(a, X)$ (see [Sac, Theorem II.4.2], although we really only need that $H(a, X)$ is arithmetic, see [Sim99, Definition VIII.3.6].) There is a computable relation R such that $H(a, X)$ is $(\forall x)(\exists y)\ R(x, y, a, X)$. Define $Q(x, g, a, X)$ by

$$R(x, g(x), a, X)\ \&\ \forall y < g(x)\ \neg R(x, y, a, X).$$

(This idea for producing unique witnesses is generalized to arithmetic predicates by [Sim99, Lemma V.5.4].) Then

$$H(a, X) \longleftrightarrow (\exists! g)(\forall x)\ Q(x, g, a, X),$$

and this unique g witnessing $H(a, H_a)$ is computable in H_a. Let $A_a \subseteq \mathbb{N}^{<\mathbb{N}}$ be the analytic code for $H(a, X)$. By [Sim99, Lemma V.5.6] we can code A_a as a tree in $2^{<\mathbb{N}}$ with a unique path corresponding to $\langle g, H_a \rangle$ and Turing equivalent to H_a. We can code this tree in $2^{<\mathbb{N}}$ as a theory $\mathcal{T}_a$ whose only nonprincipal type corresponds to $\langle g, H_a \rangle$, and whose principal types have a computable enumeration. Any degree $\mathbf{d}$ which can enumerate the types of $\mathcal{T}_a$ can compute H_a, and H_a can enumerate the types of $\mathcal{T}_a$. Thus, from (*Vaught*) any degree $\mathbf{d}$, $\mathbf{d}$ computes a saturated model for $\mathcal{T}_a$ if and only if $\mathbf{d} \geq H_a$. The argument can be relatived to any set Z to obtain for any $\mathcal{O}(a, Z)$ a theory $\mathcal{T}^Z_a$ such that for any degree $\mathbf{d}$, $\mathbf{d}$ computes a saturated model for $\mathcal{T}^Z_a$ if and only if $\mathbf{d} \geq H^Z_a$.

3.3 Saturated and Universal models

In this subsection I will investigate the proof theoretic complexity of the following classical theorems:

(*SUT*) If $\mathfrak{A}$ and $\mathfrak{B}$ are saturated models of a theory $\mathcal{T}$, then $\mathfrak{A} \cong \mathfrak{B}$.

(*SAU*) All saturated models are universal.

(*UAS*) If a theory has a universal model then it has a saturated model. (Universal models need not be saturated, see [CK90, Section 2.3].)

(*UUT*) If $\mathfrak{A}$ and $\mathfrak{B}$ are universal models of a theory $\mathcal{T}$, then $\mathfrak{A} \cong \mathfrak{B}$.

It is straightforward to verify $\mathrm{RCA}_0 \vdash (UAS)$ and $\mathrm{ACA}_0 \vdash (SUT), (SAU), (UUT)$. I will sketch the proof that the following principles are equivalent to ACA_0: (SUT), (SAU) and (EE). All constructions are based on the same theory used in the proof of [HSS, Theorem 2.3], but will require different models.

I will first show that (EE) is equivalent to ACA_0. The model I will construct uses a similar idea as [HSS, Theorem 2.3]. Let the domain of $\mathfrak{A}$ be $\mathbb{N}$, and the interpretation of R_i be $\{\langle i, n\rangle, \ : \ n \in \mathbb{N}\}$. If $\varphi_{i,s}(i)\uparrow$, then let $\neg R_{i,s}(\langle i, n\rangle)$ hold for all $n \in \mathbb{N}$. If $\varphi_{i,s}(i)\downarrow$ and s is least, then let

- $R_{i,s}(\langle i, 0\rangle)$,
- $\neg R_{i,s}(\langle i, t\rangle)$ for $1 \leq t < s$,
- $R_{i,s}(\langle i, 2t\rangle)$ for $s \leq 2t$ and $R_{i,s}(\langle i, 2t+1\rangle)$ for $s \leq 2t+1$.

Otherwise assign $R_{i,s}$ according to the rules of the theory $\mathcal{T}$ given in [HSS, Theorem 2.3]. Now verify that

$$i \in K \iff \varphi_{i,s}(i)\downarrow \text{ where } s > 0 \text{ least with } EQUIV(\mathfrak{A}, \langle i, 0\rangle, \langle i, s\rangle).$$

(EE) allows the replacement of the arithmetic predicate $EQUIV$ with a Σ^0_0 condition, so that K exists by Δ^0_1 comprehension.

Showing (SUT) and (SAU) are equivalent to ACA_0 will use the same models, these will be nearly identical to the models as used in [HSS, Theorem 2.3]. Just add a new row to each of those models, $\{\langle -1, n\rangle \ : \ n \in \mathbb{N}\}$, and the condition that $\neg R_i(\langle -1, n\rangle)$ for all $n \in \mathbb{N}$. These models are saturated, and this is provable in RCA_0. To see this note that every type $\Gamma(x)$ either has exactly one term $R_i(x) \in \Gamma(x)$ for some $i \in \mathbb{N}$ or for every $i \in \mathbb{N}$, $\neg R_i(x)$. All types of the first instance are principle, and there is only one type of the second instance. So, by adding an extra row realizing this nonprinciple type we realize every type. Now, note that types with more than one variable $\Gamma(x_1, \ldots, x_n)$ can be built by saying with $x_i = x_j$ and the 1-types of the distinct terms. Since every 1-type is realized by infinitely many elements, it is easy to verify that these models are homogeneous. All this can be carried-out in RCA_0. Call the two models of the construction $\mathfrak{A}$ and $\mathfrak{B}$. Since it is provable in RCA_0 that $\mathfrak{A}$ and $\mathfrak{B}$ are saturated, there is an isomorphism $F : \mathfrak{A} \to \mathfrak{B}$. Now verify that

$$i \in K \iff \varphi_{i,s}(i)\downarrow \text{ where } F(\langle i, 0\rangle) = \langle i, s\rangle$$

and note that the right-hand side of the biconditional is Δ^0_1.

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