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RELATIVE CATEGORICITY IN BOOLEAN ALGEBRAS

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ABSTRACT. A computable Boolean algebra $\mathcal{A}$ is relatively arithmetically categorical if for any copy $\mathcal{B}$ of $\mathcal{A}$ there is a an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ which is arithmetically computable. We show that the Boolean algebras which are relatively arithmetically categorical are the finitary Boolean algebras introduced by Palyutin and Pierce. The key step along the way to this result is a classification of the relatively Δ^0_n-categorical Boolean algebras.

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1. INTRODUCTION

The main result of this note is the following Corollary 4.6:

Theorem 1.1. *A Boolean algebra is relatively arithmetically categorical if and only if it finitary.*

A computable Boolean algebra $\mathcal{A}$ is *relatively arithmetically categorical* if for any isomorphic copy $\mathcal{B}$ there is an isomorphism which is Δ^0_n in the diagram of $\mathcal{B}$ for some integer n.

The key component of the proof is the analysis of the back-and-forth invariants developed by Antonio Montalbán and the author in [HM]. The key step is the characterization of the relatively Δ^0_{n+1}-categorical Boolean algebras: computable algebras $\mathcal{A}$ such that for any copy $\mathcal{B}$ there is an isomorphism Δ^0_{n+1} in the diagram of $\mathcal{B}$. The result is Theorem 4.5:

Theorem 1.2. *A Boolean algebra Δ^0_{n+1}-categorical if and only if it is a direct sum of $(n+2)$-indecomposable Boolean algebras whose $(n+2)$ back-and-forth type is maximal.*

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2. Preliminaries

2.1. **Basics of categoricity.** The unrelativized notion of categoricity for general structures is:

Definition 2.1. A computable structure $\mathcal{A}$ is Δ^0_n-*categorical* if for any computable copy $\mathcal{B} \cong \mathcal{A}$, there is a Δ^0_n isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.
A computable structure $\mathcal{A}$ is *arithmetically categorical* if for each computable copy $\mathcal{B} \cong \mathcal{A}$, there is an $n \in \omega$ and a Δ^0_n isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

The relativized notion of categoricity for general structures is

Definition 2.2. A computable structure $\mathcal{A}$ is *relatively* Δ^0_n-*categorical* if for any copy $\mathcal{B} \cong \mathcal{A}$, there is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ which is $\Delta^0_n(\mathcal{B})$.

A computable structure $\mathcal{A}$ is *relatively arithmetically categorical* if for each copy $\mathcal{B} \cong \mathcal{A}$, there is an $n \in \omega$ and an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ which is $\Delta^0_n(\mathcal{B})$.

There is also a syntactic characterization of the relativized notion. A structure has a *formally* Σ^0_n *Scott family*, if the orbits of all finite tuples of elements in a structure are Σ^c_n definable from a fixed set of parameters. More specifically,

Definition 2.3. Let $\mathcal{L}$ be a computable language and $\mathcal{A}$ an $\mathcal{L}$-structure. A *formally* Σ^0_n *Scott family* for $\mathcal{A}$ is a set of Σ^c_n formulas Φ from $\mathcal{L}$ in a fixed finite set of parameters $\overline{c}$ so that

(a) For each finite tuple $\overline{a}$ from $\mathcal{A}$, there is a $\varphi(\overline{x}) \in \Phi$ such that $\mathcal{A} \models \varphi(\overline{a})$.
(b) For any $\varphi \in \Phi$ and tuples $\overline{a}$ and $\overline{b}$ of the same length from $\mathcal{A}$, if $\mathcal{A} \models \varphi(\overline{a})$ and $\mathcal{A} \models \varphi(\overline{b})$ then $\langle \mathcal{A}, \overline{a}, \overline{c} \rangle \cong \langle \mathcal{A}, \overline{b}, \overline{c} \rangle$

The following is from [AK00, Theorem 10.14]

Theorem 2.4. *For a computable structure $\mathcal{A}$, the following are equivalent:*

(i) $\mathcal{A}$ has a c.e. Scott family consisting of Σ^c_n formulas $\phi(\overline{c}, \overline{x})$ with a fixed tuple of parameters $\overline{c}$.
(ii) $\mathcal{A}$ has a Σ^0_n consisting of Σ^c_n formulas $\phi(\overline{c}, \overline{x})$ with a fixed tuple of parameters $\overline{c}$.
(iii) $\mathcal{A}$ is relatively Δ^0_n-categorical

We give some examples of what was known before this paper.

(1) A computable Boolean algebra is (relatively) computably categorical if and only if it has finitely many atoms. ([GonDz] and [Rem81])
(2) A Boolean algebra $\mathcal{A}$ is relatively Δ^0_2-categorical if and only if it can be expressed as a finite direct sum of subalgebras $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_k$ where each $\mathcal{A}_i$ is isomorphic to one of an atom, atomless or a 1-atom. (McCoy [McCoy03])
(3) A Boolean algebra $\mathcal{A}$ relatively Δ^0_3-categorical if and only if if and only if it can be expressed as a finite direct sum of subalgebras $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_k$ where each $\mathcal{A}_i$ is isomorphic to one of an atom, atomless, a 1-atom, an interval algebra $\mathrm{I}\big(2 \cdot \breve{\eta}\big)$, or an interval algebra $\mathrm{I}\big(\omega + \eta\big)$. (McCoy [McCoy02])

The proof of these results was quite difficult and required a detailed analysis about the types of Boolean algebras. Getting a better understanding of these types is precisely the reason for the characterization of the back-and-forth types found in [HM]. Rephrasing these results:

(1) A computable Boolean algebra is (relatively) computably categorical if and only if it is a finite direct sum of maximal 2-indecomposable types.
(2) A Boolean algebra is relatively Δ^0_2-categorical if and only if it is a finite direct sum of maximal 3-indecomposable types.
(3) A Boolean algebra is relatively Δ^0_3-categorical if and only if it is a finite direct sum of maximal 4-indecomposable types.

This suggested a pattern and a way of extending these results generally to the relatively Δ^0_n-categorical

> A Boolean algebra relatively Δ^0_n-categorical if and only if it can be expressed as a finite direct sum of maximal $n+1$-indecomposable bf-types.

2.2. **Basics of back-and-forth relations.** For a Boolean algebra $\mathcal{A}$, we write $\Pi_n(\mathcal{A})$ for the set of infinitary Π_n sentences true in $\mathcal{A}$. The importance of the back-and-forth hierarchy on Boolean algebras is in the following ([AK00, Theorem 15.1])

$$\mathcal{A} \leq_n \mathcal{B} \iff \Pi_n(\mathcal{A}) \subseteq \Pi_\alpha(\mathcal{B})$$

A Boolean algebra $\mathcal{A}$ is *n-indecomposable* if for any partition of $\mathcal{A}$ into subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$, there is an i with $\mathcal{A} \equiv_n \mathcal{A}_i$. In [HM] we provided, for each n, invariants $(\mathbf{INV}_n, +, \leq_n)$ for the equivalences classes of the n-back-and-forth relation $\leq_n$, and a map T_n on Boolean algebras into $\mathbf{INV}_n$ satisfying

$$\begin{aligned} T_n(\mathcal{A} \oplus \mathcal{B}) &= T_n(\mathcal{A}) + T_n(\mathcal{B}) \\ \mathcal{A} \leq_n \mathcal{B} &\iff T_n(\mathcal{A}) \leq_n T_n(\mathcal{B}). \end{aligned}$$

The invariants record the possible decompositions of an algebra into n-indecomposable subalgebras. For any $n \in \omega$, any Boolean algebra can be decomposed into a finite sum of n-indecomposable subalgebras. For each n there are finitely many distinct n-indecomposable subalgebras (up to n-back-and-forth equivalence). We write $\mathbf{BF}_n$ for the n-indecomposable types (n-bf-types). Each $\alpha \in \mathbf{BF}_n$ is a finite subset of $(n-1)$-bf-types, and for every $\beta \in \alpha$ and every n-indecomposable algebra $\mathcal{A}$ with $T_n(\mathcal{A}) \equiv_n \alpha$, there are infinitely many disjoint subalgebras of $\mathcal{A}$ which are $(n-1)$-indecomposable and whose $(n-1)$-bf-type is β. Furthermore, for any partition into $(n-1)$-indecomposable subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$ there is some i with $T_n(\mathcal{A} \restriction a_i) = T_n(\mathcal{A})$ and for each $j \neq i$, $T_{n-1}(\mathcal{A} \restriction a_j) \leq_{n-1} \beta$ for some $\beta \in \alpha$.

An indecomposable bf-type $\alpha \in \mathbf{BF}_n$ is an *isomorphism type* (short, *isotype*) if $\mathcal{A} \cong \mathcal{B}$ whenever $T_n(\mathcal{A}) = \alpha = T_n(\mathcal{B})$. A characteristic property of these isotypes is that their descendants are isolated: each descendant of the type has a unique child at the next level of the hierarchy. There is a nice characterization of the isotypes which arise in the back-and-forth hierarchy using the *finitary* isomorphism types studied in [Hei92]. A Boolean algebra is *pseudo-indecomposable* if for any partition of $\mathcal{A}$ into subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$, there is an i with $\mathcal{A} \cong \mathcal{A}_i$. The *diagram* of a Boolean algebra is the set of isomorphism types which occur as subalgebras. A *finitary* Boolean algebra is one whose diagram is finite. The finitary Boolean algebras are exactly the isomorphism bf-types which arise at the finite levels of the back-and-forth hierarchy ([Har, Section 4]).

Note that $n > 0$, unless α is the trivial bf-type 0. We list here some basic facts about indecomposable isomorphism types which follow from facts proven in [HM]

(A) If $\alpha \in \mathbf{BF}_n$ is an isomorphism type, then so is each $\beta \in \alpha$.
(B) If $\mathcal{A}$ is a Boolean algebra, $\alpha \in \mathbf{BF}_n$ an isomorphism type and $\alpha \equiv_n T_n(\mathcal{A})$, then $\mathcal{A}$ is n-indecomposable and $T_n(\mathcal{A}) = \alpha$.
(C) If $\mathcal{A}$ is a Boolean algebra, $\alpha \in \mathbf{BF}_n$ an isomorphism type and $T_n(\mathcal{A}) = \alpha$ and $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ is a partition into n-indecomposable subalgebras with $T_n(\mathcal{A}_1) = \alpha$, then $T_{n-1}(\mathcal{A}_2) \in \alpha$.

Let $\alpha \in \mathbf{BF}_{n+1}$. We define

$$\alpha^\infty \equiv_n \alpha \cup \{(\alpha)_n\}.$$

Note that $(\alpha)_n^\infty = (\alpha^\infty)_n$.

For each n and n-bf-type $\alpha \in \mathbf{BF}_n$, we define the relation $\mathrm{R}_\alpha(x)$ on Boolean algebras $\mathcal{A}$ by

$$\mathcal{A} \models \mathrm{R}_\alpha(x) \iff \alpha \leq_n T_n(\mathcal{A} \restriction x)$$

[HM, Lemma 8.9] showed that the predicates $\mathrm{R}_\alpha(x)$ are Π^c_n definable.

The value of these predicates is the following a normal form theorem for Σ^c_{n+1} formulas from [HM, Corollary 8.8]:

Theorem 2.5. *For every Σ^c_{n+1} formula $\varphi(x_0, \dots, x_k)$ which implies $x_0 \dot\vee \cdots \dot\vee x_k = 1$, there are $0^{(n)}$-computable enumerations of tuples $\langle(\alpha_{\ell,i,1}, \dots, \alpha_{\ell,i,m_{\ell,i}}) : i \in \omega\rangle$ from $\boldsymbol{BF}_n$, one for each $\ell \leq k$, such that*

$$\text{(1)}\quad \varphi(x_1, \dots, x_k) \iff x_0 \dot\vee \cdots \dot\vee x_k = 1 \ \& \ \bigwedge_{\ell=1}^{k} \bigvee_{i\in\omega} (\exists y_{i,0} \dot\vee \dots \dot\vee y_{i,m_{\ell,i}} = x_\ell) \ \Big[\bigwedge_{j=0}^{m_{\ell,i}} \mathrm{R}_{\alpha_{\ell,i,j}}(y_{i,j})\Big].$$

We will not need the fact that there is a 0^n-computable enumeration of tuples in what follows, so we isolate the key properties of the normal form theorem that will be used in this paper:

Theorem 2.6. *Let $\mathcal{A}$ be a Boolean algebra and $a_1, \dots, a_k$ a partition of $\mathcal{A}$. Then for any Σ^c_{n+1} formula $\varphi(z_1, \dots, z_k)$ where $\mathcal{A} \models \varphi(a_1, \dots, a_k)$, there are n-bftypes $\{\alpha_{j,i} : i \leq m_j, j \leq k\}$ such that*

(a)

$$\mathcal{A} \models \bigwedge_{j=1}^{k} \exists x_{j,1} \dot\vee \dots \dot\vee x_{j,m_j} = a_j \ \big(\bigwedge_{i=1}^{m_j} \mathrm{R}_{\alpha_{j,i}}(x_{j,i})\big).$$

(b) For any Boolean algebra $\mathcal{B}$ and partition $b_1, \dots b_k$, if

$$\mathcal{B} \models \bigwedge_{j=1}^{k} \exists x_{j,1} \dot\vee \dots \dot\vee x_{j,m_j} = b_j \ \big(\bigwedge_{i=1}^{m_j} \mathrm{R}_{\alpha_{j,i}}(x_{j,i})\big).$$

then $\mathcal{B} \models \varphi(b_1, \dots, b_k)$.

In fact, we will show that every formally Σ^0_{n+1} Scott family can be given as a c.e. set of formulas of the form

$$1 = z_1 \dot{\vee} \dots \dot{\vee} z_k \ \& \ \bigwedge_{j=1}^{k} \exists x_{j,1} \dot{\vee} \dots \dot{\vee} x_{j,m_j} = z_j \ \big(\bigwedge_{i=1}^{m_j} \mathrm{R}_{\alpha_{j,i}}(x_{j,i})\big).$$

where $\{\alpha_{j,i} : i \leq m_j, j \leq k\}$ is a set of n-bf-types.

We recall the relation $\gamma \lhd_n \delta$:

$$(\delta)_{n-1} \in^{\mathrm{w}} \gamma \ \& \ \gamma \leq^{\mathrm{w}}_n \delta;$$

and isolate some basic properties of $\lhd_n$. A key property of this relation is that

If $\alpha \lhd_n \beta$ the $\alpha \equiv_n \alpha + \beta$

(This follows from [HM, Lemma 7.12, 7.18(3)] and is proved in [Har, Lemma 2.2(e)].)

The indecomposable types $\mathbf{BF}_n$ are *realizable*, which means that for each $\alpha \in \mathbf{BF}_n$ there is an n-indecomposable Boolean algebra $\mathcal{A}$ with $T_n(\mathcal{A}) = \alpha$. A consequence of this is that all $\alpha \in \mathbf{BF}_n$ satisfy the following condition:

For every $\gamma \in \alpha$

$$\forall \delta \in \gamma \ \exists \beta \in \alpha \ (\gamma \leq^{w}_n \beta \ \& \ (\beta)_{n-1} = \delta).$$

(See [HM, Theorem 5.2].)

3. Necessary and sufficient conditions

3.1. **Reduction to indecomposables.** Theorem 3.3 reduces the general problem of producing formally Σ^0_n Scott families for arbitrary sequences of elements from the algebra (as in Definition 2.3), to the problem of determining (i) how the algebra decomposes into m-indecomposables (for some $m \geq n$) and (ii) showing that all algebras of that m-indecomposable type have formally Σ^0_n Scott families.

For m-indecomposable algebras the problem of finding Scott families is simpler and does not even require the introduction of parameters. Since any Boolean algebra can be expressed as a finite sum of m-indecomposable algebras, we can extend the solution for the indecomposable case to the general case by adding parameters.

Definition 3.1. An isomorphism type $\alpha \in \mathbf{BF}_m$ has a Σ^0_n *Scott family* if for each $\beta \in \alpha$ there is a Σ^c_n formula $\varphi_\beta(x)$ (without parameters) such that for each Boolean algebra $\mathcal{A}$ with $T_m(\mathcal{A}) = \alpha$:

(a) Whenever $1_{\mathcal{A}} = a_1 \dot{\vee} a_2$ and a_2 is $(m-1)$-indecomposable with $T_{m-1}(\mathcal{A} \upharpoonright a_2) = \beta \in \alpha$,

$$\mathcal{A} \models \varphi_\beta(a_2)$$

(b) For any elements a_1 and a_2 of $\mathcal{A}$, and for any $\beta \in \alpha$,

$$\mathcal{A} \models \varphi_\beta(a_1) \ \& \ \mathcal{A} \models \varphi_\beta(a_2) \quad \Rightarrow \quad \mathcal{A} \upharpoonright a_1 \cong \mathcal{A} \upharpoonright a_2.$$

Lemma 3.2. *Suppose $\alpha \in \boldsymbol{BF}_m$ is an isomorphism type and has a Σ^0_n Scott family (as in Definition 3.1). Then any Boolean algebra $\mathcal{A}$ with $T_m(\mathcal{A}) = \alpha$ has a formally Σ^0_n Scott family (without parameters).*

Proof. Suppose $\alpha \in \mathbf{BF}_m$ has a Σ^0_n Scott family as in Definition 3.1. Let $\mathcal{A}$ be a Boolean algebra with $T_m(\mathcal{A}) = \alpha$, so $\mathcal{A}$ is necessarily m-indecomposable. Consider any sequence $\langle a_1, \ldots, a_k\rangle$ of elements from $\mathcal{A}$. We may assume these elements are disjoint. Let $a_0 = 1_{\mathcal{A}} - (a_1 \dot\vee \ldots \dot\vee a_k)$, so that $\langle a_0, a_1, \ldots, a_k\rangle$ is a partition of $\mathcal{A}$. To simplify the following discussion we will assume $\mathcal{A} \cong \mathcal{A}\restriction a_0$ and we will produce a formula $\psi(a_0, \ldots, a_k)$ with the desired properties. The required formula is then obtained by $\exists$-quantifying-out the element not in the original list.

For $1 \le i \le k$, each a_i can be decomposed into $(m-1)$-indecomposable elements $b_{i,1}, \ldots, b_{i,\ell_i}$ where $T_{m-1}(\mathcal{A}\restriction b_{i,j}) = \beta_{i,j} \in \alpha$ (by condition (C) for isotypes). Let $\varphi_{\beta_{i,j}}(x)$ be the Σ^c_n formula (without parameters) as in Definition 3.1, and for $1 \le i \le k$ let $\psi_i(x)$ the formula

$$\exists x_1 \dot\vee \ldots \dot\vee x_{\ell_i} = x \bigwedge_{j \le \ell_i} \varphi_{\beta_{i,j}}(x_j).$$

So $\psi_i(x)$ is computably equivalent to a Σ^c_n formula and $\mathcal{A} \models \psi_i(a_i)$ by condition (a) of the definition. For any $c \in \mathcal{A}$, if $\mathcal{A} \models \psi_i(c)$ then $c = c_1 \dot\vee \ldots \dot\vee c_{\ell_i}$ where $\mathcal{A} \models \varphi_{\beta_{i,j}}(c_j)$ for each $j \le \ell_i$ so that $\mathcal{A}\restriction b_{i,j} \cong \mathcal{A}\restriction c_j$ by condition (b). Thus, $\mathcal{A}\restriction a_i \cong \mathcal{A}\restriction c$.

What about a_0 for which we supposed that $\mathcal{A} \cong \mathcal{A}\restriction a_0$? If $T_{m-1}(\mathcal{A}) \in \alpha$, then there is a formula $\varphi_{(\alpha)_{m-1}}$ provided by the definition, so let $\psi_0(x)$ be $\varphi_{(\alpha)_{m-1}}(x)$; otherwise, let $\psi_0(x)$ be $x = x$. In either case $\mathcal{A} \models \varphi_0(a_0)$, and in the first case, if $\mathcal{A} \models \varphi_0(c)$ then $\mathcal{A}\restriction a_0 \cong \mathcal{A}\restriction c$.

Let $\psi(x_0, \ldots, x_k)$ be the formula

$$\bigwedge_{i \le k} \psi_i(x_i),$$

so that ψ has no parameters, is computably equivalent to a Σ^c_n formula and satisfied by $\mathcal{A}$, $\mathcal{A} \models \psi(a_0, \ldots, a_k)$. Suppose that $\mathcal{A} \models \psi(c_0, \ldots, c_k)$. If $\psi_0(x) = \varphi_{(\alpha)_{m-1}}(x)$ then $\mathcal{A}\restriction a_i \cong \mathcal{A}\restriction c_i$ for each i and so

$$(\mathcal{A}, a_0, \ldots, a_k) \cong (\mathcal{A}, c_0, \ldots, c_k).$$

Otherwise, $T_{m-1}(A) \notin T_m(\mathcal{A})$ and $\mathcal{A} \not\cong \mathcal{A}\restriction a_i \cong \mathcal{A}\restriction c_i$ for each $i \ge 1$. Then $\mathcal{A} \cong \mathcal{A}\restriction c_0$ and so $\mathcal{A}\restriction a_0 \cong \mathcal{A}\restriction c_0$.

Let $\{(\beta_{i,j} : i \le k, j \le \ell_i) : k, \ell \in \omega, \beta_{i,j} \in \alpha^\infty\}$ be a computable listing of finite sequences of back-and-forth types and Ψ a computable listing of all formulas in k variables (for each $k \in \omega$) constructed as ψ was constructed in the previous paragraph from the $(\beta_{i,j} : i \le k, j \le \ell_i)$. Then, Ψ is a formally Σ^0_n Scott family, since for any $a_1, \ldots, a_k$ there will be a $\psi(x_1, \ldots, x_k)$ constructed from some $(\beta_{i,j} : i \le k, j \le \ell_i)$ as in the previous paragraph, and satisfying conditions (a) and (b) in Definition 2.3. □

Theorem 3.3. *Let $\mathcal{A}$ be a Boolean algebra which has a partition into m-indecomposable elements $1_{\mathcal{A}} = a_1 \dot\vee \ldots \dot\vee a_k$ where each $T_m(\mathcal{A}\restriction a_i)$ has a formally Σ^0_n Scott family. Then $\mathcal{A}$ has a formally Σ^0_n Scott family with parameters $a_1, \ldots, a_k$.*

Proof. Let $\mathcal{A}$ be a Boolean algebra and suppose $\mathcal{A}$ can be partitioned into m-indecomposable elements $1_{\mathcal{A}} = a_1 \dot\vee \ldots \dot\vee a_k$ where each $T_m(\mathcal{A}\restriction a_i)$ has a formally Σ^0_n Scott family. Let $\langle b_1, \ldots, b_\ell\rangle$ be any sequence of elements from $\mathcal{A}$. Let $\langle c_{i,j} : i \leq k, j \leq \ell\rangle$ be a partition with

$$a_i = c_{i,1} \dot\vee \ldots \dot\vee c_{i,\ell} \quad \text{and} \quad b_j = c_{1,j} \dot\vee \ldots \dot\vee c_{k,j}.$$

and let $\psi_{i,j}(x)$ be a Σ^c_n formula for which $\mathcal{A} \models \psi_{i,j}(c_{i,j})$, as given by Lemma 3.2. Let $\psi(x_1, \ldots, x_\ell)$ be the formula

$$\exists y_{1,1} \dot\vee \ldots \dot\vee y_{1,\ell} = a_1 \ldots \exists y_{k,1} \dot\vee \ldots \dot\vee y_{k,\ell} = a_k$$
$$\bigwedge_{i\leq k, j\leq \ell} \psi_{i,j}(y_{i,j}) \wedge \bigwedge_{j\leq\ell} \left(y_{1,j} \dot\vee \ldots \dot\vee y_{k,j} = x_j\right),$$

so that ψ is computably equivalent to a Σ^c_n formula in parameters $a_1, \ldots a_k$. Furthermore, it follows from Lemma 3.2 and properties (a) and (b) of Definition 3.1 that $\mathcal{A} \models \psi(b_1, \ldots, b_\ell)$ and whenever $\mathcal{A} \models \psi(d_1, \ldots, d_\ell)$, $(\mathcal{A}, b_1, \ldots, b_\ell) \cong (\mathcal{A}, d_1, \ldots, d_\ell)$.

Let Ψ_i be the formally Σ^0_n Scott family from Lemma 3.2, and $\big\{(\psi_{i,j} : i \leq k, j \leq \ell) : k, \ell \in \omega, \psi_{i,j} \in \Psi_i\big\}$ be a computable listing of formulas from these families. Let Ψ a computable listing of all formulas in ℓ variables (for each $\ell \in \omega$) constructed as ψ was constructed in the previous paragraph from the $(\psi_{i,j} : i \leq k, j \leq \ell)$. Then, Ψ is a formally Σ^0_n Scott family in parameters $a_1, \ldots, a_k$, since for any $b_1, \ldots, b_\ell$, there will be a $\psi(x_1, \ldots, x_\ell)$ constructed from a $(\psi_{i,j} : i \leq k, j \leq \ell)$ as in the previous paragraph, and satisfying conditions (a) and (b) in Definition 2.3.

□

3.2. **Sufficient conditions.** The aim in the next couple subsections is to provide a single necessary and sufficient condition for when an m-bf-type has a Σ^0_n Scott family as given by Definition 3.1, and which depends only on general structural features of the back-and-forth invariant.

Lemma 3.4. *Let $n < m$ and $\alpha \in \boldsymbol{BF}_m$ be an isomorphism type which satisfies the following additional conditions*

- *(i) Whenever there are $\beta, \gamma_1, \ldots, \gamma_k \in \alpha$ with $(\beta)_n \leq_n (\gamma_1 + \ldots + \gamma_k)_n$, it is also true that $\beta \leq_{m-1} \gamma_1 + \ldots + \gamma_k$.*
- *(ii) If $(\alpha)_n \in (\alpha)_{n+1}$ then $(\alpha)_{m-1} \in \alpha$.*

Then α has a formally Σ^0_{n+1} Scott family.

Proof. Let $\alpha \in \mathbf{BF}_m$ be an isomorphism type that satisfies the both conditions (i) and (ii) for some $n < m$. The formally Σ^0_{n+1} Scott family for α can be generated from the predicates

$$\big\{\mathrm{R}_{(\beta)_n} : \beta \in \alpha\big\},$$

using Lemma 3.2. Since these predicates are Π^c_n they are also Σ^c_{n+1}.

By (i), for each $\beta \in \alpha$, $(\beta)_n \in (\alpha)_{n+1}$, otherwise there would be some $\gamma \in \alpha$ with $(\beta)_n <_n (\gamma)_n \in (\alpha)_n$, so that $\beta \not\leq_{m-1} \gamma$. (Note that $m > n + 1$ since $\beta, \gamma \in \alpha$.) It follows that condition (a) of Definition 3.1 holds. For condition (b), let $\mathcal{A}_1$ and $\mathcal{B}_1$ be subalgebras of $\mathcal{A}$ and $\beta \in \alpha$ satisfying

$$(\beta)_n \leq_n T_n(\mathcal{A}_1) \quad \text{and} \quad (\beta)_n \leq_n T_n(\mathcal{B}_1).$$

The aim is to show that $T_{m-1}(\mathcal{A}_1) = \beta = T_{m-1}(\mathcal{B}_1)$. We prove this for $\mathcal{A}_1$, but the argument is the same for $\mathcal{B}_1$. Suppose $\mathcal{A}_1$ is a sum of $(m-1)$-indecomposable algebras with bf-types $\gamma_1, \ldots, \gamma_k$ so that $T_{m-1}(\mathcal{A}_1) = \gamma_1 + \ldots + \gamma_k$. Note that this implies that $(\beta)_n \leq_n (\gamma_1 + \ldots + \gamma_k)_n$. It follows by (C) that each of $\gamma_i \in \alpha$, except perhaps some $\gamma_j = (\alpha)_{m-1}$. In this case however $(\alpha)_{m-1} = \gamma_1 + \ldots + \gamma_k$ so that $(\alpha)_n = (\beta)_n \in (\alpha)_{n+1}$ (reason: $(\beta)_n \leq_n (\alpha)_n$ by hypothesis but $(\alpha)_n \leq^w_n (\beta)_n$, so $(\alpha)_n = (\beta)_n$.) So, if $\gamma_j = (\alpha)_{m-1}$, then by (ii) $(\alpha)_{m-1} \in \alpha$ as well. Since $(\beta)_n \leq_n T_n(\mathcal{A}_1) = (\gamma_1 + \ldots + \gamma_k)_n$ where each $\gamma_i \in \alpha$, it follows by (i) that $\beta \leq_{m-1} \gamma_1 + \ldots + \gamma_k$. But then $\beta \leq_{m-1} \gamma_i$ for some i and $\beta + \gamma_j = \beta$ for all $j \neq i$. Now $\beta = \gamma_i$ as both are in α and so $\beta = \gamma_1 + \ldots + \gamma_k$. Thus, $T_{m-1}(\mathcal{A}_1) = \beta$.

Since both $T_{m-1}(\mathcal{A}_1) = \beta = T_{m-1}(\mathcal{B}_1)$ and β is an isomorphism type (as α was assumed to be), it follows that $\mathcal{A}_1 \cong \mathcal{B}_1$.

□

The second condition can be eliminated from Lemma 3.4.

Lemma 3.5. *Let $n < m$ and $\alpha \in \boldsymbol{BF}_m$ be an isomorphism type which satisfies the following condition*

(i) Whenever there are $\beta, \gamma_1, \ldots, \gamma_k \in \alpha$ with $(\beta)_n \leq_n (\gamma_1 + \ldots + \gamma_k)_n$, it is also true that $\beta \leq_{m-1} \gamma_1 + \ldots + \gamma_k$.

Then α has a formally Σ^0_{n+1} Scott family.

Proof. We prove (ii). Let $\alpha \in \mathbf{BF}_m$ with $(\alpha)_n \in (\alpha)_{n+1}$, but that $(\alpha)_{m-1} \notin \alpha$. Let m be minimal among the ancestors of α with this property, so that $(\alpha)_{m-2} \in (\alpha)_{m-1}$. Let $\beta \in \alpha$ with $(\beta)_{m-2} = (\alpha)_{m-2}$, such a β must exist by the definition of $(\alpha)_{m-1}$:

$$(\alpha)_{m-1} = \max\left\{(\gamma)_{m-2} : \gamma \in \alpha\right\}.$$

Since $(\alpha)_{m-1} \leq^{\mathrm{w}}_{m-1} \beta$, if $\beta \leq^{\mathrm{w}}_{m-1} (\alpha)_{m-1}$ were true as well, then $\beta = (\alpha)_{m-1}$, contradicting our hypothesis about α. So, $(\alpha)_{m-1} <^{\mathrm{w}}_{m-1} \beta$. Let $\delta \in (\alpha)_{m-1}$ with no $\xi \in \beta$ satisfying $\delta \leq_{m-2} \xi$. Let $\delta^* \in \alpha$ with $(\delta^*)_{m-2} = \delta$, such a δ^* exists by the definition of $(\alpha)_{m-1}$. We have $(\beta)_n = (\alpha)_n = (\beta + \delta^*)_n$, but $\beta \not\leq_{m-1} \beta + \delta^*$, since $\delta \notin^{\mathrm{w}} \beta$ (and thus, $\beta \not\leq_{m-1} \delta^*$). This contradicts (i). □

3.3. **Necessary conditions.** We turn to establishing the necessity of condition (i) from Lemma 3.5. The next lemma makes use of the normal form theorem 2.6 for Σ^c_{n+1} predicates.

Lemma 3.6. *Let $\alpha \in \boldsymbol{BF}_m$ and $n < m$. Suppose there are $\beta, \gamma_1, \ldots, \gamma_k \in \alpha$ with $(\beta)_n \leq_n (\gamma_1 + \ldots + \gamma_k)_n$ but $\beta \not\leq_{m-1} \gamma_1 + \ldots + \gamma_k$. Then no m-indecomposable Boolean algebra $\mathcal{A}$ with $T_m(\mathcal{A}) = \alpha$ has a formally Σ^0_{n+1} Scott family.*

Proof. Suppose $\alpha \in \mathbf{BF}_m$ and that there are $\beta, \gamma_1, \dots, \gamma_k \in \alpha$ with $(\beta)_n \leq_n (\gamma_1 + \dots + \gamma_k)_n$ but $\beta \not\leq_{m-1} \gamma_1 + \dots + \gamma_k$. Let $\mathcal{A}$ be an m-indecomposable algebra with $T_m(\mathcal{A}) = \alpha$. We will show that there is a partition of $1_{\mathcal{A}} = a_1 \dot{\vee} a_2$ such that for any Σ^c_{n+1} formula $\varphi(x)$ where $\mathcal{A} \models \varphi(a_1)$, there is a $b \in \mathcal{A}$ with $\mathcal{A} \models \varphi(b)$ but $\mathcal{A} \restriction a_1 \ncong \mathcal{A} \restriction b$.

Let $1_{\mathcal{A}} = a_1 \dot{\vee} a_2$ where a_2 is m-indecomposable with $T_m(a_2) = \alpha$ and let a_1 be $(m-1)$-indecomposable with $T_{m-1}(a_1) = \beta$. Fix any Σ^c_{n+1} formula $\varphi(x)$ such that $\mathcal{A} \models \varphi(a_1)$. By Theorem 2.6 we may assume $\varphi(x)$ is of the form

$$\exists x_1 \dot{\vee} \dots \dot{\vee} x_\ell = x \;\; \exists y_1 \dot{\vee} \dots \dot{\vee} y_m = -x \; \Big(\bigwedge_{i=1}^{\ell} \mathrm{R}_{\delta_i}(x_i) \;\; \& \;\; \bigwedge_{i=1}^{m} \mathrm{R}_{\epsilon_i}(y_i) \Big),$$

where each of the δ_i and ϵ_j are n-bf-types.

Fix a partition of $a_1 = b_1 \dot{\vee} \dots \dot{\vee} b_\ell$ so that $\mathcal{A} \models \mathrm{R}_{\delta_i}(b_i)$, and a partition $a_2 = c_1 \dot{\vee} \dots \dot{\vee} c_m$ with $\mathcal{A} \models \mathrm{R}_{\epsilon_j}(c_j)$. We may assume that $T_{m-1}(b_1) = \beta$ and $T_m(c_1) = \alpha$. Let $\delta_1, \dots, \delta_p$ be all the bf-types with $\delta_i \leq_n (\beta)_n$. We will replace each b_i (for $i \leq p$) by disjoint elements $d_i = d_{i,1} \dot{\vee} \dots \dot{\vee} d_{i,k}$ from below c_1, where $T_{m-1}(d_{i,j}) = \gamma_j$. Let $c_1' = c_1 - (d_1 \dot{\vee} \dots \dot{\vee} d_p)$ and $c_1'' = c_1' \dot{\vee} (b_1 \dot{\vee} \dots \dot{\vee} b_p)$ (swapping b_i for d_i). Then $T_m(c_1'') = \alpha$ and for each i $T_n(b_i) \leq_n T_n(d_i)$ by hypothesis. Let $b = d_1 \dot{\vee} \dots d_p \dot{\vee} b_{p+1} \dot{\vee} b_\ell$, so that $\mathcal{A} \models \varphi(b)$. It remains to show that $\mathcal{A} \restriction b \ncong \mathcal{A} \restriction a_1$.

Suppose that $\mathcal{A} \restriction b \cong \mathcal{A} \restriction a_1$. By our swap we have

$$\beta \leq_{m-1} \sigma_1 + \dots + \sigma_p + \delta^*_{p+1} + \dots \delta^*_\ell \qquad \text{where } \sigma_i = \gamma_1 + \dots + \gamma_k \text{ and } (\delta^*_j)_n = \delta_j.$$

Since $(\beta)_n \not\leq_n \delta_j$, we cannot have $\beta \leq_{m-1} \delta^*_j$ for any j. So, by [HM, Lemma 7.12] we must have for some $i \leq p$,

$$\beta \leq_{m-1} \sigma_i = \gamma_1 + \dots + \gamma_k,$$

contradicting our hypothesis about $\beta, \gamma_1, \dots, \gamma_n$. So, $\mathcal{A} \restriction b \ncong \mathcal{A} \restriction a_1$.

□

We record a simple corollary.

Corollary 3.7. *If an algebra $\mathcal{A}$ can be partitioned into a finite direct sum of m-indecomposable subalgebras $\mathcal{A}_1 \oplus \dots \mathcal{A}_k$ and for one of these algebras $T_m(\mathcal{A}_i)$ condition (i) of Theorem 3.4 fails. Then $\mathcal{A}$ does not have a formally Σ^0_{n+1} Scott family.*

Proof. Suppose there is a partition of $\mathcal{A}$ into $\mathcal{A} = \mathcal{A}_1 \oplus \dots \mathcal{A}_k$ into m-indecomposable subalgebras and that $T_m(\mathcal{A}_i) = \alpha$ is a m-bf-type on which (i) fails. Consider any partitioning of $\mathcal{A}$ into subalgebras $\mathcal{A} = \mathcal{B}_1 \oplus \dots \oplus \mathcal{B}_\ell$. We can refine the two partitions and write $\mathcal{A}$ as

$$\mathcal{A} = \sum_{i=1}^{\ell} \sum_{j=1}^{k} \mathcal{C}_{i,j} \quad \text{where} \quad \mathcal{A}_j = \sum_{i=1}^{\ell} \mathcal{C}_{i,j} \qquad \mathcal{B}_i \sum_{j=1}^{k} \mathcal{C}_{i,j}.$$

Then for some $j \leq k$, $\mathcal{C}_j$ is m-indecomposable and $T_m(\mathcal{C}_j) = \alpha$. Since (i) fails on $\mathcal{C}_j$, by the previous lemma 3.5, there is a partition of $\mathcal{C}_j$ into $c_1 \dot{\vee} c_2$ so that for any Σ^0_{n+1} formula $\phi(x_1, x_2, \overline{y})$ (without loss of generality we may assume that $x_1, x_2, \overline{y}$ are a partition of $\mathcal{A}$) for which $\mathcal{A} \models \phi(c_1, c_2, \overline{a})$, there is a partition of $\mathcal{C}_j$ into d_1, d_2 for which $\mathcal{A} \models \phi(d_1, d_2, \overline{a})$ but $\mathcal{C}_j \restriction c_1 \ncong \mathcal{C}_j \restriction d_1$, and so $\mathcal{A} \restriction c_1 \ncong \mathcal{A} \restriction d_1$, and so there can be no isomorphism taking $c_1, c_2, \overline{a}$ to $d_1, d_2, \overline{a}$. Thus, $\mathcal{A}$ cannot have a formally Σ^0_{n+1} Scott family with constants that partition it into $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_\ell$. Since the partitioning was arbitrary, $\mathcal{A}$ can have no formally Σ^0_{n+1} Scott family. □

4. Characterizations of the relatively Δ^0_n-categorical algebras

The goal in this section is to prove that the relatively Δ^0_n-categorical Boolean algebras are finite disjoint sums of *maximal* $(n+1)$-indecomposable Boolean algebras. We will need some basic facts about maximal bf-types found in [Har].

For each $\alpha \in \mathbf{BF}_n$ there is a largest $(n+1)$-indecomposable descendant $\overline{\alpha}$ and a smallest $(n+1)$-indecomposable descendant $\underline{\alpha}$. This means that these two bftypes satisfy the property:

For all $\beta \in \mathbf{BF}_{n+1}$ with $(\beta)_n = \alpha$, $\underline{\alpha} \leq_{n+1} \beta \leq_{n+1} \overline{\alpha}$.

We list some of the properties of these two types which will be used here. All references (A)-(D) in this are to this list.

(A) The bf-type $\underline{\alpha}$ should include everything which is consistent with $(\underline{\alpha})_n = \alpha$. This is simple to describe (from [Har, Theorem 2.4])

$$\underline{\alpha} = \left\{ \overline{\gamma} \,:\, \alpha \lhd_n \overline{\gamma} \right\}.$$

(B) $\overline{\alpha}$ should have only what it is required to have and should include only the smallest n-bf-types which it can consistently contain. This means each $\gamma \in \overline{\alpha}$ satisfies $(\gamma)_{n-1} \in \alpha$ and is the smallest n-bf-type δ with (a) $(\delta)_{n-1} = (\gamma)_{n-1}$ and (b) $\alpha \lhd_n \delta$. It is not apparent there are such n-bf-types γ, but there is an alternate characterization which explicitly provides the members of $\overline{\alpha}$:

$$\gamma = \max\{\xi \in \alpha \,:\, (\gamma)_{n-1} \lhd_{n-1} \xi\}$$

In [Har, Lemma 3.3] it is shown that γ defined in this way indeed satisfies (i) and (ii). In [Har, Theorem 2.4] it is shown that $\overline{\alpha}$ is the maximal immediate descendant of α.

(C) From the "minimality" of $\gamma \in \overline{\alpha}$, one would expect that if $\gamma_1 \in \gamma \in \overline{\alpha}$ then γ_1 is maximal. This is indeed the case, which follows from the characterization of B: if $\gamma_1 \leq_{n-1} \delta$ then from the fact that $(\gamma)_{n-1} \lhd_{n-1} \gamma_1$ it follows that $(\gamma)_{n-1} \lhd_{n-1} \delta$, so that $\delta \in \gamma$ by B.

(D) Every maximal bf-type is an isomorphism type, proven in [Har, Theorem 3.5]. The converse however is not true: there are isomorphism bf-types which are not maximal. An example is given in [Har, Example 3.10].

We recall the necessary and sufficient condition from the last section:

(i) Whenever there are $\beta, \gamma_1, \ldots, \gamma_k \in \alpha$ with $(\beta)_n \leq_n (\gamma_1 + \ldots + \gamma_k)_n$, it is also true that $\beta \leq_{m-1} \gamma_1 + \ldots + \gamma_k$.

Lemma 4.1. *If $\alpha \in \boldsymbol{BF}_{n+2}$ is maximal, then α has a Σ^0_{n+1} Scott family.*

Proof. We apply the condition (i) with $m = n + 2$.

(i). Suppose $\beta, \gamma_1, \dots, \gamma_k \in \alpha$ and $(\beta)_n \leq_n (\gamma_1 + \dots + \gamma_k)_n$. Then $(\beta)_n \leq_n (\gamma_i)_n$ for some i and $(\beta)_n + (\gamma_j)_n = (\beta)_n$ for all $j \neq i$. Since $\beta \in \alpha$, it follows by the property (B) of the maximal α that $(\beta)_n \in (\alpha)_{n+1}$, and so it must be that $(\beta)_n = (\gamma_i)_n$.

We show that $\gamma_i \leq_{n+1} \gamma_1 + \dots + \gamma_k$. By [HM, Lemma 7.12], it is sufficient to show that $\gamma_i \triangleleft_{n+1} \gamma_j$ when $j \neq i$.

First, $(\gamma_j)_n \in^{\mathrm{w}}{}_n \gamma_i$ when $j \neq i$. Since both (a) $(\gamma_j)_n \in (\alpha)_{n+1}$ and (b) $(\gamma_i)_n \triangleleft_n (\gamma_j)_n$ by hypothesis, it follows from the characterization in (B) of the members of α that $(\gamma_j)_n \in^{\mathrm{w}} \gamma_i$.

Second, $\gamma_i \leq^{\mathrm{w}}_{n+1} \gamma_j$. Suppose $\delta \in \gamma_j$. Then (a) $\delta \in (\alpha)_{n+1}$ and (b) $(\gamma_i)_n \triangleleft_n (\gamma_j)_n \triangleleft_n \delta$, so again by the characterization in (B) of the members of α it follows that $\delta \in^{\mathrm{w}} \gamma_i$.

Thus, $\gamma_i \triangleleft_{n+1} \gamma_j$. □

Lemma 4.2. *If $\beta \in \boldsymbol{BF}_{n+2}$ is not maximal, then no descendant of β has a Σ^0_{n+1} Scott family.*

Proof. Suppose $\beta \in \mathbf{BF}_{n+2}$ is not maximal and that $\alpha \in \mathbf{BF}_{n+2}$ is maximal with $(\alpha)_{n+1} = (\beta)_{n+1}$, so that $\beta <_{n+2} \alpha$. There are two ways this could happen and each leads to the failure of condition (i) for β.

(1). There a $\delta \in^{\mathrm{w}} (\beta)_{n+1}$, but $\delta \notin (\beta)_{n+1}$. Then there is a $\delta' \in \beta$ with $(\delta')_n = \delta$ and a $\gamma \in \beta$ with $\delta <_n (\gamma)_n$. This contradicts (i) because $(\delta')_n <_n (\gamma)_n$ implies that $\delta' \not\leq_{n+1} \gamma$. Thus condition (i) fails.

(2). There is a $\xi \in (\beta)_{n+1}$ with some $\delta \in \beta$ and $\gamma \in \alpha$ satisfying $(\gamma)_n = \xi = (\delta)_n$, but $\gamma \neq \delta$. By the minimality of the members of α from property (B), we must have $\gamma <_{n+1} \delta$. There are two possibilities.

(2a). There are $\gamma_1 \in \gamma \in \alpha$ and $\delta_1 \in \delta \in \beta$ with $(\gamma_1)_{n-1} = (\delta_1)_{n-1}$, but $\gamma_1 \neq \delta_1$. Then, $\delta_1 <_n \gamma_1$ by maximality from property (C). By realizability, there is some $\gamma_1^* \in \alpha$ with $(\gamma_1^*)_n = \gamma_1$, so that $\gamma_1 \in (\alpha)_{n+1} = (\beta)_{n+1}$. Thus, $\delta_1 \notin (\beta)_{n+1}$ as $\gamma_1 \in (\beta)_{n+1}$. Since $\delta_1 \in \delta \in \beta$, there is a $\delta_1^* \in \beta$ with $(\delta_1^*)_n = \delta_1$, by realizability. So, there is $\delta_1^* \in \beta$ but $(\delta_1^*)_n = \delta_1 \notin (\beta)_{n+1}$. Thus (i) fails as in case (1).

(2b). There is a $\gamma_1 \in \gamma \in \alpha$ with $(\gamma_1)_{n-1} \in^{\mathrm{w}} (\gamma)_n$, but no $\delta_1 \in \delta$ with $(\delta_1)_{n-1} = (\gamma_1)_{n-1}$. Thus, $\gamma_1 \notin^{\mathrm{w}} \delta$, since $\gamma_1 \in^{\mathrm{w}} \delta$ implies there is a $\delta_1 \in \delta$ with $\gamma_1 \leq_n \delta_1$ and so $(\delta_1)_{n-1} = (\gamma_1)_{n-1}$. By realizability, there is some $\gamma_1^* \in \alpha$ with $(\gamma_1^*)_n = \gamma_1$. But $\gamma_1 \in (\alpha)_{n+1}$ by property (B) and $(\alpha)_{n+1} = (\beta)_{n+1}$, so that $\gamma_1 \in (\beta)_{n+1}$. By realizability, there is some $\delta_1^* \in \beta$ with $(\delta_1^*)_n = \gamma_1$. So,

$$(\delta)_n = (\gamma)_n = (\gamma)_n + \gamma_1 = (\delta)_n + (\delta_1^*)_n = (\delta + \delta_1^*)_n.$$

But $\delta \not\leq_{n+1} \delta + \delta^*$ since $\delta \ntriangleleft \delta_1^*$ (as $(\delta_1^*)_n = \gamma_1 \notin^{\mathrm{w}} \delta$). Thus, (i) fails. □

Theorem 4.3. *For $\alpha \in \boldsymbol{BF}_{n+2}$, the following are equivalent:*

(a) α is maximal.

(b) α has a Σ^0_{n+1} Scott family.

(c) α satisfies the condition

- *(i) Whenever there are $\beta, \gamma_1, \dots, \gamma_k \in \alpha$ with $(\beta)_n \leq_n (\gamma_1 + \dots + \gamma_k)_n$, it is also true that $\beta \leq_{n+1} \gamma_1 + \dots + \gamma_k$.*

Proof. $(a) \Rightarrow (b)$ is Lemma 4.1.
$(b) \Rightarrow (c)$ is Lemma 3.6.
$(c) \Rightarrow (a)$ is by contraposition with Lemma 4.2. □

We can now provide a simple characterization of the relatively Δ^0_n-categorical Boolean algebras.

Theorem 4.4. *A Boolean algebra is relatively Δ^0_{n+1}-categorical if and only if it is a finite direct sum of maximal $(n+2)$-back-and-forth types.*

Proof. $(\Leftarrow)$. By Theorem 3.3 and 4.3.

$(\Rightarrow)$. Let $\mathcal{A}$ be a Boolean algebra and suppose it is not a finite direct sum of maximal $(n+2)$-bf-types. There is a partition of $\mathcal{A}$ into $(n+2)$-indecomposable subalgebras: $\mathcal{A} = \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_k$ by the compactness theorem [HM, Theorem 3.13], and so one of these subalgebras must not be a maximal $(n+2)$-bf-type. Suppose $T_{n+2}(\mathcal{A}_i) = \alpha$ is not maximal. Then condition (i) in Theorem 4.3 fails for α and so by Corollary 3.7 $\mathcal{A}$ can have no formally Σ^0_n Scott family. □

We also have a complete characterization of the relatively arithmetically categorical Boolean algebras.

Theorem 4.5. *A Boolean algebra is relatively arithmetically categorical if and only if it is a finite direct sum of indecomposable isomorphism types in the finite back-and-forth hierarchy* ***BF****.*

Proof. Let $\mathcal{A}$ be a relatively arithmetically categorical Boolean algebra. Let $\mathcal{B}$ be a generic copy in the forcing language introduced in [AK00, Chapter 10]. Then for some $n+1$ there is a $\Delta^0_{n+1}(\mathcal{B})$ isomorphism. The proof of [AK00, Theorem 10.14] can now be carried-out to produce a c.e. Σ^0_{n+1} Scott family using the forcing language. Now apply Theorem 4.5, so that $\mathcal{A}$ can be expressed as a finite direct sum of $(n+2)$-indecomposable subalgebras whose bf-type is maximal and so an isomorphism type. □

The equivalence of Heindorff's characterization of the finitary Boolean algebras in [Hei92] and the isomorphism types which occur at finite levels of the back-and-forth hierarchy (which are just finite sums of indecomposable algebras) was proved in [Har, Section 4].

Corollary 4.6. *A Boolean algebra is relatively arithmetically categorical if and only if it finitary.*

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