A Characterization of the low_n Degrees Using Escape Functions

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Abstract

Martin characterized the high degrees as those which could compute a function which *dominates* every computable function. So, a degree is non-high if and only if for each function computable in the degree there is a computable function which *escapes domination*. We show that the low_1 degrees are those in which computable escape functions can be found effectively. We generalize this condition to provide a characterization for each of low_n degrees by means of how effectively escape functions can be found.

0 Introduction

Martin provided a characterized of the high Turing degrees as the degrees which can compute a *dominant* function:

- For total functions $f, g : \mathbb{N} \to \mathbb{N}$, f *dominates* g if
$$(\forall^\infty x)\ \big[f(x) > g(x)\big]$$
- f is a *dominant function* if f dominates every computable function.

Then Martin's High Domination Theorem ([Soa87, Theorem XI.1.3]) is that the high degrees are precisely the degrees which can compute a dominant function:

A Turing degree $\mathbf{a}$ is high $\big(\mathbf{a}' \geq \mathbf{0}''\big)$ *iff*
$$(\exists f \leq \mathbf{a})(\forall g \leq \mathbf{0})\ (\forall^\infty x)\big[f(x) > g(x)\big]$$

Martin's Theorem has a contrapositive form, characterizing the non-high degrees by escape functions

- For total functions $f, g : \mathbb{N} \to \mathbb{N}$ g *escapes* [1] (domination from) f if
$$(\exists^\infty x)\ \big[f(x) \leq g(x)\big]$$
- f has an *escape function* if some computable function escapes domination from f.

Then the contrapositive form of Martin's Theorem is that the non-high degrees are precisely the degrees for which any function computable in the degree has an escape function:

A Turing degree $\mathbf{a}$ is non-high $\big(\mathbf{a}' < \mathbf{0}''\big)$ *iff*
$$(\forall f \leq \mathbf{a})(\exists g \leq \mathbf{0})\ (\exists^\infty x)\big[f(x) \leq g(x)\big]$$

The starting point for this investigation is the question

[1] The terminology *escape function* was first introduced in [CHKS04]. This is now used in [Soa05].

Question: When can escape functions for a non-high degree be *effectively* produced?

Our answer to this question leads to the first definition

Definition: A degree $\mathbf{a}$ has the *Uniform Escape Property (UEP)* when for any set $A \in \mathbf{a}$:

There is a partial computable $\lambda ex.h_e(x)$ such that whenever Φ_e^A is total, then h_e is total and escapes Φ_e^A:

$$(\exists^\infty x)\ \left[\Phi_e^A(x) \leq h_e(x)\right]$$

We can provide a precise answer to which degrees have (UEP), (Theorem 2.2)

A degree $\mathbf{a}$ has (UEP) iff $\mathbf{a}$ is low ($\mathbf{a}' = \mathbf{0}'$).

We next turn to look at how we can weaken the insistence on *effectiveness*. Martin's Theorem provides one extreme

A degree $\mathbf{a}$ is non-high iff for the array of partial computable functions $\left\{\varphi_e\right\}_{e\in\omega}$ and every $f \leq \mathbf{a}$, *there exists* some e such that φ_e is total and escapes domination from f.

This suggests a weakening of the effectiveness in (UEP) which is still better than this bare existence:

Question: For what degrees $\mathbf{a}$ can we uniformly produce an array of partial computable functions $\left\{h_{e,y}\right\}_{e,y\in\omega}$ such that given any total Φ_e^A, then *for almost every* y, $h_{e,y}$ is total and escapes domination from Φ_e^A.

We discovered that any degree with $\mathbf{a}'' \leq \mathbf{0}''$ will positively answer this question, and in fact a stronger property holds of any such degree, which we call (2-UEP) (Definition 3.1.) This property characterizes the degrees which are low$_2$, (Theorem 3.2)

A degree $\mathbf{a}$ has (2-UEP) iff $\mathbf{a}$ is low$_2$ ($\mathbf{a}'' = \mathbf{0}''$).

We produce a hierarchy of properties, for each $n \in \omega$ (n-UEP), which characterize the degrees $\mathbf{a}$ for which $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$, by uniformly producing escape functions with decreasing effectiveness. Theorem 6.2 provides

For every n, a degree $\mathbf{a}$ has (n-UEP) iff $\mathbf{a}$ is low$_n$ ($\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$).

The paper is organized to first introduce the characterization of low_1, low_2 and low_3, which provide the essence of the general construction. Section 1 provides a review of notation, definitions and the basic results in computability theory needed. Section 2 proves the (UEP) characterization of the low_1 degrees, Theorem 2.2, which also introduces the basic strategies involved in each direction of the characterization. In section 3 the (UEP) property is modified to provide a characterization low_2 degrees (2-UEP) and the low_3 degrees (3-UEP). Each jump class requires one-layer of quantificational complexity, and these two cases suggest how to extend the characterization throughout the arithmetic hierarchy. The proof that the low_3 degrees are characterized by (3-UEP), Theorem 3.5 is given in full. Section 4 provides a Strong Quantifier Normal Form Theorem (4.2) for arithmetic predicates, by characterizing these predicates using the two strongest natural quantifiers $\forall$ and $\forall^\infty$ (*almost every*.) The results in this section are central for the low_n characterization. Section 5 shows that for every n, the low_n degrees have (n-UEP), Theorem 5.2. Section 6 provides the converse direction that all degrees satisfying (n-UEP) are low_n, Theorem 6.2.

1 Basic Notation and Definitions

We refer to [Soa87] for the results in computability we will need, but all these will be standard for any introductory reference of computability theory. The rest of this section collects less standard notation and definitions used in this paper, as well as the basic facts about the low_n degrees we will use.

We will use two additional quantifiers besides $\{\forall, \exists\}$

- $(\forall^\infty y)\ P$ to assert *for almost every* y, P
- $(\exists^\infty y)\ P$ to assert *for infinitely many* y, P

with the following relations to the standard quantifiers $\forall$ and $\exists$

$$\begin{aligned} (\forall^\infty y)\ P &\iff (\exists x)(\forall y)\ \big[y \geq x \implies P\big] \\ (\exists^\infty y)\ P &\iff (\forall x)(\exists y)\ \big[y \geq x\ \ \&\ \ P\big]. \end{aligned}$$

The quantifier $\forall^\infty$ behaves similarly to $\forall$ with respect to the propositional connectives, and $\exists^\infty$ behaves similarly to $\exists$. The logical relations between the quantifiers is given by

$$\forall \implies \forall^\infty \implies \exists^\infty \implies \exists.$$

The following Lemma summarizes some key logical laws for these quantifiers to which we will appeal

Lemma 1.1. *In the following P, Q are any relations, and R is a relation in which the variable x*

does not occur free.

$$(\exists^\infty x)P \iff \neg(\forall^\infty x)\,\neg P \tag{a}$$
$$(\forall^\infty x)P \wedge (\forall^\infty x)Q \iff (\forall^\infty x)\,\big[P \wedge Q\big] \tag{b}$$
$$R \rightarrow (\forall^\infty x)P \iff (\forall^\infty x)\,\big[R \rightarrow P\big] \tag{c}$$
$$(\forall^\infty x)P \rightarrow R \iff (\exists^\infty x)\,\big[P \rightarrow R\big] \tag{d}$$
$$(\forall^\infty x)\,\big[P \rightarrow Q\big] \implies (\forall^\infty x)P \rightarrow (\forall^\infty x)Q \tag{e}$$
$$(\forall^\infty x)P \wedge (\exists^\infty x)Q \implies (\exists^\infty x)\,\big[P \wedge Q\big] \tag{f}$$
$$(\forall^\infty x)P \wedge (\forall^\infty x)Q \implies (\forall^\infty x)\,\big[P \wedge Q\big] \tag{g}$$
$$(\forall^\infty x)P \wedge (\forall x)Q \implies (\forall^\infty x)\,\big[P \wedge Q\big] \tag{h}$$
$$(\exists^\infty x)P \wedge (\forall x)Q \implies (\exists^\infty x)\,\big[P \wedge Q\big] \tag{i}$$

When $\mathcal{Q}_1, \mathcal{Q}_2 \in \{\forall, \exists, \forall^\infty, \exists^\infty\}$ we will write

$$(\mathcal{Q}_1 y_n)(\mathcal{Q}_2 y_{n-1}) \ldots \; P$$

for the assertion obtained by alternating the quantifiers $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in turn, starting with $\mathcal{Q}_1$.

Let $\omega = \{0, 1, 2, \ldots\}$. We write $\{\varphi_e\}_{e\in\omega}$ for a standard enumeration of the partial computable functions, and define for $e \in \omega$

$$W_e = \{x \in \omega \mid \varphi_e(x){\downarrow}\}$$

where $\varphi_e(x){\downarrow}$ means $x \in \operatorname{dom}\varphi_e$. Fix some computable pairing function $\langle\cdot\rangle$ so that for every k-tuple of natural numbers $y_1, \ldots, y_k$, $\langle y_1, \ldots, y_k\rangle \in \omega$. We will write $\overline{y}$ for $\langle y_1, \ldots, y_k\rangle$ where the natural numbers $y_1, \ldots, y_k$ and arity k are determined by the context. Any c.e. k-ary relation R can be represented by some c.e. set W_e using the pairing function:

$$R(y_1, \ldots, y_k) \iff \langle y_1, \ldots, y_k\rangle \in W_e$$

We will say in this case that the c.e. relation R has Σ_1-*index* e.

The classes Σ_n and Π_n (standardly written Σ^0_n and Π^0_n) are the usual arithmetic classes of sets defined by quantifier complexity (see [Soa87, Section IV.1].) If A is any Σ_{2n+1} relation then there is a c.e. set W_e such that

$$A(x) \iff (\exists y_{2n-1})(\forall y_{2n-2}) \ldots (\exists y_1)(\forall z)\,\big[\langle x, z, y_1, \ldots, y_{2n-1}\rangle \in W_e\big]$$

and we will take e to be a Σ_{2n+1}-*index* of A. (The important point is that there are $2n$ alternating quantifiers starting with $\exists$ and ending with $\forall$.) If $B \in \Pi_{2n+1}$ then a Π_{2n+1}-*index* for B will be a a Σ_{2n+1}-index for $\neg B$. Similarly, if A is any k-ary Π_{2n} relation then there is a c.e. set W_e such that

$$A(x) \iff (\forall y_{2n-2})(\exists y_{2n-3}) \ldots (\exists y_1)(\forall z)\,\big[\langle x, z, y_1, \ldots, y_{2n-2}\rangle \in W_e\big]$$

and we will take e to be a Π_{2n}-*index* for A. (What is important is that there are $2n-1$ alternating quantifiers beginning with $\forall$ and ending with $\forall$.) If $B \in \Sigma_{2n}$ then a Σ_{2n}-*index* for B will be a a

Π_{2n}-index for $\neg B$. (See [Rog87, §14.2] for the *index* notation.)

Let $\{\Phi_e^X\}_{e\in\omega}$ a computable enumeration of the partial computable functionals. For any set A, the A-computably enumerable sets $\{W_e^A\}$ and the classes Σ_n^A and Π_n^A are defined by relativizing to A. (See [Soa87, Sections III.1 and IV.1].)

For a set A, $A^{(n)}$ is the nth Turing jump of A.

Definition 1.2 (low$_n$ Degrees)**.** *Let $n \geq 1$. A set A is* low$_n$ *if $A^{(n)} = 0^{(n)}$, and a Turing degree $\boldsymbol{d}$ is* low$_n$ *if any (every) set in that degree is* low$_n$*.*

Post's Theorem connects the jump hierarchy to the arithmetic hierarchy ([Soa87, Post's Theorem, IV.2.2]):

Theorem 1.3 (Post's Theorem)**.** *For every $n \geq 1$ and every A,*

(a) $A^{(n)}$ is Σ_n-complete.

(b) $X \in \Delta_{n+1}^A$ if and only if $X \leq_{\mathrm{T}} A^{(n)}$.

The only alternative characterization of the hierarchy of low$_n$ degrees, is the following characterization based which holds only for Δ_2 degrees (see [Soa87, Section IV.4 and Exercise IV.4.5]

Theorem 1.4. *For $n \geq 1$ and $A \leq_{\mathrm{T}} 0'$, the following are equivalent*

(A) A is low$_n$ *.*

(B) $\Sigma_n^A \subset \Pi_{n+1}$.

(C) $\Pi_n^A \subset \Sigma_{n+1}$.

Note that the equivalence $(B) \Leftrightarrow (C)$ always holds. Also, $(A) \Rightarrow (B),(C)$ always holds. Only $(B),(C) \Rightarrow (A)$ requires the set $A \leq_{\mathrm{T}} 0'$.

Proof. That $(B) \Leftrightarrow (C)$ is logic.

$(A) \Rightarrow (B)$: If A is low$_n$, then $A^{(n)} \leq_{\mathrm{T}} 0^{(n+1)}$. Since $A^{(n)}$ is Σ_n^A-complete (Theorem 1.3),

$$\Sigma_n^A \subset \Delta_{n+1} \subset \Pi_{n+1}.$$

$(B) \Rightarrow (A)$: If $A \leq_{\mathrm{T}} 0'$, then $A^{(n)} \in \Sigma_{n+1}$, so that as $A^{(n)}$ is Σ_n^A-complete (Theorem 1.3)

$$\Sigma_n^A \subset \Sigma_{n+1}.$$

But by hypothesis

$$\Sigma_n^A \subset \Pi_{n+1}$$

so that

$$\Sigma_n^A \subset \Delta_{n+1}$$

and thus

$$A^{(n)} \in \Delta_{n+1}.$$

It follows by Post's Theorem (1.3) that

$$A^{(n)} \leq_{\mathrm{T}} 0^{(n)}$$

and since $0^{(n)} \leq_{\mathrm{T}} A^n$ always holds, A is low_n □

All functions will have arity one. We will write for a function $h : \omega \to \omega$

$$h(x, y_1, \dots, y_n) \quad \text{for} \quad h(\langle x, y_1, \dots, y_n \rangle).$$

In this way the standard enumeration of partial computable functions $\{\varphi_e\}$ and functionals $\{\Phi_e^X\}$ can be taken to be multi-arity functions.

Definition 1.5. *A* uniformly enumerable (u.e.) family of functions *is a partial computable function $\lambda x, y_1, \dots, y_n . h(x, y_1, \dots, y_n)$ where the arguments $y_1, \dots, y_n$ ($\overline{y}$) are taken to be parameters. We will write this as*

$$\{h_{\overline{y}}\}_{\overline{y} \in \omega}.$$

We will always take $\overline{y}$ to be the parameters to the array of functions.
A uniformly enumerable (u.e.) array of families of functions *is a partial computable function $\lambda x, e, y_1, \dots, y_n . h(x, e, y_1, \dots, y_n)$, in which on argument e returns a u.e. family of functions:*

$$\{h_{e,\overline{y}}\}_{\overline{y} \in \omega}.$$

2 low_1 Degrees and (1-UEP)

When can escaping functions for a non-high degree be *effectively* produced?

Definition 2.1. *A set A has the* uniform escape property (UEP)*, or* (1-UEP) *when there is a partial computable $\lambda ex.h_e(x)$ such that whenever Φ_e^A is total, the function h_e is total and escapes domination from Φ_e^A:*

$$(\exists^\infty x) \left[\Phi_e^A(x) \leq h_e(x)\right].$$

A degree has the uniform escape property *when some set in the degree has the property.*

The main result of this section is

Theorem 2.2 (Characterization of low_1 Sets)**.** *A degree $\boldsymbol{a}$ is low_1 if and only if $\boldsymbol{a}$ has (UEP).*

We will first prove that the low_1 degrees have (UEP), Theorem 2.4, then prove the converse with Theorem 2.5. Both proofs isolate the key ideas that will be used for the general low_n case.

Central to the proof of the first theorem is the following characterization of Π_2 sets ([Soa87, Theorem IV.3.7]), and which is also the basis case for our Strong Quantifier Normal Form Theorem (4.2):

Lemma 2.3 (Strong Quantifier Normal Form for Π_2)**.** *There is a computable g such that for any Π_2 set A with Π_2-index e*

$$\begin{aligned} A(x) &\iff \left[W_{g(e,x)} = \omega\right] \\ \neg A(x) &\iff \left[W_{g(e,x)} \text{ is finite }\right]. \end{aligned}$$

Proof. Let $A \in \Pi_2$ with Π_2 index e. Then

$$A(x) \iff (\forall y)\ \left[\langle x, y\rangle \in W_e\right].$$

By the by the S^m_n Theorem there is a computable h such that

$$A(x) \iff (\forall y)\ \left[y \in W_{h(e,x)}\right].$$

Define g so that

$$y \in W_{g(e,x)} \iff (\forall z \leq y)\ \left[z \in W_{h(e,x)}\right].$$

Then

$$\begin{aligned} A(x) &\iff \left[W_{g(e,x)} = \omega\right] \\ \neg A(x) &\iff \left[W_{g(e,x)} \text{ is finite }\right]. \end{aligned}$$

□

We break the proof of Theorem 2.2 into the two directions, Theorem 2.4 and Theorem 2.5

Theorem 2.4. *Every* low_1 *degree has (UEP).*

Proof. For a c.e. set W, the *settling time function* for W, m_W, is the partial computable function

$$m_W(x) = (\mu s)\ \left[x \in W_s\right].$$

We will write m_e for m_{W_e}. Of course, m_e is computable if and only if $W_e = \omega$.

Let A be a low_1 set. We will show that that there is a computable k such that for each e, if Φ^A_e is total then the settling time function $m_{k(e)}$ for the c.e. set $W_{k(e)}$ is total and escapes domination from Φ^A_e:

$$(\exists^\infty x)\ \left[\Phi^A_e(x){\downarrow} \leq m_{k(e)}(x){\downarrow}\ \right] \tag{Esc}$$

$$W_{k(e)} = \omega. \tag{Tot}$$

Note that condition (Esc) is Π_2^A and condition (Tot) is Π_2 and since A is low$_1$, by Theorem 1.4, the condition (Esc) is Π_2.

Let $V^A(e)$ be the A-predicate expressing (Esc)

$$V^A(e) \iff (\exists^\infty x)(\exists s)\left[\Phi_{e,s}^A(x)\downarrow \leq\ s\ \ \&\ \ x \notin W_{g(v,e),s}\right] \tag{1}$$

where v is a Π_2-index for V^A and g is the computable function given by Lemma 2.3

$$\begin{aligned} V^A(e) &\iff W_{g(v,e)} = \omega \\ \neg V^A(e) &\iff W_{g(v,e)} \text{ is finite }. \end{aligned}$$

The computable function $\lambda e.g(v,e)$ is the computable function k we were after in the previous paragraph. By the Recursion Theorem we may assume we the index v for V^A when defining V^A. This is shown formally in the next paragraph, which the reader may skip over to Sublemma 1.

Define a Π_2^A predicate using an extra parameter i, the *index*,

$$\hat{V}^A(e,i) \iff (\exists^\infty x)(\exists s)\left[\Phi_{e,s}^A(x)\downarrow \leq\ s\ \ \&\ \ x \notin W_{g(i,e),s}\right].$$

Since A is low$_1$, $\Pi_2^A \subset \Pi_2$, so there is a u such that

$$\begin{aligned} \hat{V}^A(e,i) &\iff (\forall z)\left[\langle i,e,z\rangle \in W_u\right] \\ &\iff (\forall z)\left[\langle e,z\rangle \in W_{s(i)}\right] \end{aligned}$$

where s is computable and given by the s-m-n Theorem. By the Recursion Theorem, there is a value $v \in \omega$ such that

$$W_{s(v)} = W_v$$

so, v is the Π_2-index for $V^A(e) = \hat{V}^A(v,e)$:

$$\begin{aligned} V^A(e) &\iff (\forall z)\left[\langle e,z\rangle \in W_{s(v)}\right] \\ &\iff (\forall z)\left[z \in W_v\right]. \end{aligned}$$

Sublemma 1. *If Φ_e^A is total then*

$$\left[W_{g(v,e)} \text{ is finite }\right] \implies V^A(e)$$

Proof of Sublemma 1. Suppose Φ_e^A is total, and $W_{g(v,e)}$ is finite. Fix z so that for all $x > z$, $x \notin W_{g(v,e)}$. For each such x

$$(\exists s)\left[\Phi_{e,s}^A(x)\downarrow \leq\ s\ \ \&\ \ x \notin W_{g(v,e),s}\right]$$

and since Φ_e^A is total,

$$(\exists^\infty x)(\exists s)\left[\Phi_{e,s}^A(x)\downarrow \leq\ s\ \ \&\ \ x \notin W_{g(v,e),s}\right]$$

so that by Equation (1), $V^A(e)$ holds. □

If Φ_e^A is total we must have $V^A(e)$ since from Sublemma 1:

$$\begin{aligned} \neg V^A(e) \quad &\implies \quad W_{g(v,e)} \text{ is finite} \\ &\implies \quad V^A(e). \end{aligned}$$

Thus, when Φ_e^A is total we have $V^A(e)$, so

$$(\exists^\infty x)(\exists s)\ \left[\Phi_{e,s}^A(x)\!\downarrow\ \leq\ s\ \ \&\ \ x \notin W_{g(v,e),s}\right]$$
$$W_{g(v,e)} = \omega$$

which is precisely the conditions (Esc) and (Tot) to be established. □

Remark. The proof actually establishes that for any Φ_e^A whose domain is infinite, there is a computable h_e such that

$$(\exists^\infty x)\ \left[\Phi_e^A(x)\!\downarrow\ \leq\ h_e(x)\!\downarrow\,\right].$$

All that is needed for Sublemma 1 is that $\left|\operatorname{dom}\Phi_e^A\right| = \infty$.

We now turn to showing the converse, that any degree with (UEP) is low_1.

Theorem 2.5. *For any set A which has (UEP), $A' \leq_{\mathrm{T}} 0'$.*

Proof. Since A' is Σ_1^A-complete, it is sufficient to show that

$$A' \in \Pi_2$$
$$A' \in \Sigma_2$$

which together imply that $A' \in \Delta_2$, and thus, by Post's Theorem (1.3) that $A' \leq_{\mathrm{T}} 0'$.

We will show that there is a computable k such that for any $x \in \omega$

$$x \in A' \iff \varphi_{k(x)} \text{ total} \qquad (\Pi_2)$$

and a computable ℓ such that for any $x \in \omega$

$$x \in A' \iff \operatorname{dom}\varphi_{\ell(x)} \text{ is finite}\ . \qquad (\Sigma_2)$$

We begin describing the general strategy common to establishing both conditions (Π_2) and (Σ_2). We will construct A-partial computable functions F_x against some partial computable function h_x (a potential escape function given by (UEP).) At each stage of the construction s exactly one of two strategies is employed:

> $\mathcal{E}(s)$: Choose n least and not in the domain of F_x at stage s and let $F_x(n) = 0$. This is the *extend* strategy at stage s.

$\mathcal{A}(s)$: Choose n least and not in the domain of F_x at stage s, but in the domain of h_x by stage s and let

$$F_x(n) = 1 + h_x(n).$$

If there is no such n, then do nothing. This is the *attack* strategy at stage s.

What is essential about these strategies is summarized in the following

Sublemma 1. *Suppose that for each s, exactly one of the strategies $\mathcal{E}(s)$ or $\mathcal{A}(s)$ determines the extension of F_x. Then*

(a) If $(\exists^\infty s)\ \mathcal{E}(s)$ then F_x is total.

(b) If $(\forall^\infty s)\ \mathcal{A}(s)$ and h_x is total then F_x is total and dominates h_x

$$(\forall^\infty n)\ \big[F_x(n) > h_x(n)\big].$$

Proof of Sublemma 1. (a): The strategy $\mathcal{E}(s)$ extends F_x to the least element not yet in the domain of F_x. If $(\exists^\infty s)\ \mathcal{E}(s)$, then F_x is total.

(b): The strategy $\mathcal{A}(s)$ only extends the domain of F_x to a new element n, if n was defined on h_x, and then ensures that

$$F_x(n) > h_x(n).$$

If $(\forall^\infty s)\ \mathcal{A}(s)$ then there are only finitely many stages s on which $\mathcal{E}(s)$ was played, and this strategy only adds one element on each stage, so that F_x is defined on only finitely many elements through the Extend strategy. Let t be a stage larger than any stage s on which $\mathcal{E}(s)$. Note that for any $n \geq t$, if $F_x(n)$ is defined, it must occur on a stage s in which $\mathcal{A}(s)$. So, for any $n \geq t$, if $h_x(n)$ is defined at stage s, then $F_x(n)$ will eventually be extended to n at some attack stage, $\mathcal{A}(s)$, so that

$$F_x(n) > h_x(n)$$

Since this is true for all $n \geq t$, if h_x is total, thenF_x will be total and dominate h_x. □

Proof of Condition (Π_2):

The construction describes for each $x \in \omega$ an A-computable function F_x, which by the Relativized Kleene Fixed Point Theorem [Soa87, Theorem III.1.6], we may assume ahead has index $f(x)$. Thus, $F_x = \Phi^A_{f(x)}$. Using (UEP) there is a partial computable $\lambda n.h_{f(x)}(n)$ such that whenever F_x is total, the function $h_{f(x)}$ is total and escapes domination from F_x. Let k be the computable function which gives the index of the function $h_{f(x)}$, so that $\varphi_{k(x)} = h_{f(x)}$.

The basic strategies will be employed in the following circumstances:

$\mathcal{E}(s)$: $x \in A'_s$.

$\mathcal{A}(s)$: $x \notin A'_s$.

For each stage exactly one of $\mathcal{A}(s)$ or $\mathcal{E}(s)$ determines the extension of F_x. Since A' is c.e. in A we have

Sublemma 2. *For each x the following holds for F_x*

$$\begin{aligned} x \in A' &\implies (\forall^\infty s)\, \mathcal{E}(s) \\ x \notin A' &\implies (\forall s)\, \mathcal{A}(s) \end{aligned}$$

To verify condition (Π_2):

Suppose $x \in A'$. Then $(\forall^\infty s)\, \mathcal{E}(s)$, so by Sublemma (1a), F_x is total, and by (UEP) $\varphi_{k(x)}$ is also total.

Suppose $x \notin A'$. Then $(\forall s)\, \mathcal{A}(s)$, so if $\varphi_{k(x)}$ were total then by Lemma (1b) F_x would be total and dominate $\varphi_{k(x)}$, contradicting (UEP). Thus, $\varphi_{k(x)}$ is not total.

Proof of Condition (Σ_2):

The construction describes for each $x \in \omega$ an A-computable function G_x, which by the Relativized Kleene Fixed Point Theorem [Soa87, Theorem III.1.6], we may assume ahead has index $f(x)$. Thus, $G_x = \Phi^A_{f(x)}$. Using (UEP) there is a partial computable $\lambda n.h_{f(x)}(n)$ such that whenever G_x is total, the function $h_{f(x)}$ is total and escapes domination from G_x. We modify $h_{f(x)}$ by

$$h'_x(n) = \begin{cases} h_{f(x)}(n) & \text{if } (\forall m \leq n)\ \left[h_{f(x)}(m)\downarrow\ \right] \\ \uparrow & \text{otherwise} \end{cases}$$

Let ℓ be the computable function which gives the index of h'_x, so that $\varphi_{\ell(x)} = h'_x$. Then

$$\begin{aligned} h_{f(x)} \text{ total} &\iff \varphi_{\ell(x)} \text{ total} \\ h_{f(x)} \text{ not total} &\iff \operatorname{dom} \varphi_{\ell(x)} \text{ is finite .} \end{aligned}$$

We reverse the strategies from the previous case:

$\mathcal{A}(s)$: $x \in A'_s$.

$\mathcal{E}(s)$: $x \notin A'_s$.

For each stage exactly one of $\mathcal{E}(s)$ or $\mathcal{A}(s)$ must hold. Then since A' is c.e. in A we have

Sublemma 3. *For each x the following holds for G_x*

$$\begin{aligned} x \in A' &\implies (\forall^\infty s)\ \mathcal{A}(s) \\ x \notin A' &\implies (\forall s)\ \mathcal{E}(s). \end{aligned}$$

To verify condition (Σ_2):

Suppose $x \in A'$. Then $(\forall^\infty s)\ \mathcal{A}(s)$, so if $\varphi_{\ell(x)}$ is total, then by Sublemma (1b), G_x is total, and dominates $\varphi_{\ell(x)}$. This is impossible by (UEP), so $\varphi_{\ell(x)}$ is not total implying the domain of $\varphi_{\ell(x)}$ is finite.

Suppose $x \notin A'$. Then $(\forall s)\ \mathcal{E}(s)$ so by Sublemma (1a), G_x is total, and so by (UEP) $\varphi_{\ell(x)}$ is also total. □

3 Finding Escaping Functions Uniformly: low_2 and low_3

This section extends the characterization for low_1 degrees by (UEP) to the low_2 and low_3 degrees. The method for handling the quantifier complexity introduced by the extra jump in low_2 and again in low_3 will be the starting point for stating the family of properties (n-UEP) and proving the characterization in section 5 and 6.

3.1 low_2 Degrees

A set A is low_2 if $A'' \leq_{\mathrm{T}} 0''$, which adds one jump class and one layer of quantifier complexity to getting information about A, by Theorem 1.4. Instead of insisting on an *effective* procedure for producing escape functions, the 2-uniform enumeration property (2-UEP) posits a procedure which is *almost effective* in producing escape functions:

> There is a uniformly enumerable (u.e.) array of families of partial computable functions $\lambda e.\{h_{e,y}\}_{y\in\omega}$ such that whenever Φ_e^A is total, then *for almost every y*
>
> $$h_{e,y} \text{ is total and escapes } \Phi_e^A.$$

This condition does hold for all low_2 degrees, but a second change is also introduced. The single total function Φ_e^A is replaced with a uniformly enumerable family of A-partial computable $\{\Phi_{e,y}\}_{y\in\omega}$. If almost all of these functions are total, then for almost every y

$$h_{e,y} \text{ is total and escapes } \Phi_{e,y}^A.$$

Definition 3.1. *A set A has the* 2-Uniform Escape Property (2-UEP) *when there is a uniformly enumerable array of families of partial computable functions $\lambda e.\{h_{e,y}\}_{y\in\omega}$ satisfying:*

> *For each $e \in \omega$ with the family of functions $\{\Phi^A_{e,y}\}_{y\in\omega}$ satisfying*
> $$(\forall^\infty y)\ \big[\Phi^A_{e,y} \text{ total}\ \big]$$
> *then*
> $$(\forall^\infty y)\ \big[h_{e,y} \text{ is total and escapes } \Phi^A_{e,y}\big].$$

A degree has the 2-Uniform Escape Property *when some set of the degree has the property.*

The low_2 degrees are characterized by the 2-uniform escape property:

Theorem 3.2 (Characterization of low_2 Degrees)**.** *A degree is* low_2 *if and only if it has the 2-uniform escape property.*

We will sketch how the left-to-right implication modifies the low_1 case. Suppose A is a low_2. Recall from the proof of Theorem 2.2 that the settling time function m_e for the c.e. set W_e was defined as

$$m_e(x) = (\mu s)\ \big[x \in W_{e,s}\big].$$

The goal is to show that there is a computable $\lambda ey.k(e,y)$ such that for each e, if for almost all y, $\Phi^A_{e,y}$ is total then almost all the settling time functions $m_{k(e,y)}$ for the c.e. sets $W_{k(e,y)}$ are total and escapes domination from $\Phi^A_{e,y}$. Suppose that for $e \in \omega$ almost all of the functions $\Phi^A_{e,y}$ are total. Then it is sufficient to satisfy the following two conditions for e:

$$(\forall^\infty y)(\exists^\infty x)\ \big[\Phi^A_{e,y}(x){\downarrow} \le m_{k(e,y)}(x){\downarrow}\ \big] \qquad \text{(Esc)}$$
$$(\forall^\infty y)\ \big[W_{k(e,y)} = \omega\big]. \qquad \text{(Tot)}$$

The first conditon is Σ^A_3 and the second condition is Σ_3. But, since A is low_2, by Theorem 1.4, the condition (Esc) is Σ_3. We need a way to bridge the conditions (Esc) and (Tot). This is provided by the following Lemma, which is the familiar characterization of Σ_3 sets (see [Soa87, Theorem IV.3.7].)

Lemma 3.3 (Strong Quantifier Normal Form for Σ_3)**.** *There is a computable g such that for any Σ_3 set A with Σ_3-index e*

$$\begin{aligned} A(x) \quad &\iff \quad (\forall^\infty y)\ \big[W_{g(e,x,y)} = \omega\big] \\ \neg A(x) \quad &\iff \quad (\forall y)\ \big[W_{g(e,x,y)} \text{ is finite}\ \big]. \end{aligned}$$

Proof. Let $A \in \Sigma_3$, so that for some $B \in \Pi_2$, A is $(\exists x)\ B$. By Lemma 2.3, re-writing B, we have for some computable h

$$\begin{aligned} A(x) \quad &\iff \quad (\exists y)\big[W_{h(e,x,y)} = \omega\big] \\ \neg A(x) \quad &\iff \quad (\forall y)\big[W_{h(e,x,y)} \text{ is finite}\ \big]. \end{aligned}$$

We are given a matrix $\{W_{h(e,x,y)}\}_{y\in\omega}$ such that if $A(x)$ holds for at least one row y, $W_{h(e,x,y)} = \omega$ and if $\neg A(x)$ then all rows are finite. Define a computable g so that

$$W_{g(e,x,y)} = \bigcup_{z\leq y} W_{h(e,x,z)}.$$

Then

$$\begin{aligned} A(x) &\iff (\forall^\infty y)\left[W_{g(e,x,y)} = \omega\right] \\ \neg A(x) &\iff (\forall y)\left[W_{g(e,x,y)} \text{ is finite }\right]. \end{aligned}$$

□

We will defer the proof of Theorem 3.2 to the general theorem, Theorem 6.2, but we will provide a proof for low_3 below.

3.2 low_3 Degrees

A set A is low_3 if $A''' \leq_{\text{T}} 0'''$, which adds another jump class and another layer of quantifier complexity to getting information about A, by Theorem 1.4. This will require another parameter in the family of potential escape functions

$$\{h_{e,y_1,y_2}\}_{y_1,y_2\in\omega}.$$

Definition 3.4. *A set A has the* 3-Uniform Escape Property (3-UEP) *when there is a uniformly enumerable array of families of partial computable functions $\lambda e.\{h_{e,y_1,y_2}\}_{y_1,y_2\in\omega}$ satisfying:*

> *For each $e \in \omega$ with the family of functions $\{\Phi^A_{e,y_1,y_2}\}_{y_1,y_2\in\omega}$ satisfying*
>
> $$(\exists^\infty y_2)(\forall^\infty y_1)\ \left[\Phi^A_{e,y_1,y_2} \textit{ total }\right]$$
>
> *then*
>
> $$(\exists^\infty y_2)(\forall^\infty y_1)\ \left[h_{e,y_1,y_2} \textit{ is total and escapes } \Phi^A_{e,y_1,y_2}\right].$$

A degree has the 3-Uniform Escape Property *when some set of the degree has the property.*

The general family of escape properties (n-UEP) follows the (3-UEP) pattern of adding another parameter to the uniformly enumerable family of escaping functions and weakening the effectiveness condition by alternating $\{\forall^\infty, \exists^\infty\}$ quantifiers. The 3-uniform escape property characterizes the low_3 degrees:

Theorem 3.5 (Characterization of low_3 Degrees)**.** *A degree is* low_3 *if and only if it has the 3-uniform escape property.*

Central to the proof of the of this theorem is the following characterization of Π_4 sets:

Lemma 3.6 (Strong Quantifier Normal Form for Π_4)**.** *There is a computable g such that for any Π_4 set A with Π_4-index e*

$$\begin{array}{rcl} A(x) & \iff & (\forall z)(\forall^\infty y)\big[W_{g(e,x,y,z)} = \omega\big] \\ \neg A(x) & \iff & (\forall^\infty z)(\forall y)\big[W_{g(e,x,y,z)} \textit{ is finite }\big] \end{array}$$

Proof. Let $A \in \Pi_4$, then $A(x)$ can be expressed as $(\forall z)B(x,z)$, where $B \in \Sigma_3$. By the Lemma 3.3, there is a computable h such that

$$\begin{array}{rcl} B(x) & \iff & (\forall^\infty y)\big[W_{h(e,x,y,z)} = \omega\big] \\ \neg B(x) & \iff & (\forall y)\big[W_{h(e,x,y,z)} \text{ is finite }\big]. \end{array}$$

Now we are given an *array* of matrices

$$\Big\{\{W_{h(e,x,y,z)}\}_{y\in\omega}\Big\}_{z\in\omega}.$$

If $A(x)$, then *for each* z and almost all rows y in the zth matrix, $W_{h(e,x,y,z)} = \omega$. But if $\neg A(x)$ *there exists some* z in which all the rows y in the zth matrix are finite. Define a computable g by

$$W_{g(e,x,y,z)} = \bigcap_{w\leq z} W_{h(e,x,y,w)}.$$

We now have a new array of matrices

$$\Big\{\{W_{g(e,x,y,z)}\}_{y\in\omega}\Big\}_{z\in\omega}$$

still with the property that if $A(x)$ then *for each* z and almost all rows y in the zth matrix, $W_{h(e,x,y,z)} = \omega$, but now if $\neg A(x)$ then *for almost all* z, every row y of the zth matrix is finite. Thus,

$$\begin{array}{rcl} A(x) & \iff & (\forall z)(\forall^\infty y)\big[W_{g(e,x,y,z)} = \omega\big] \\ \neg A(x) & \iff & (\forall^\infty z)(\forall y)\big[W_{g(e,x,y,z)} \text{ is finite }\big]. \end{array}$$

□

The proof of Theorem 3.5 is broken into the two directions, Theorem 3.7 and Theorem 3.9

Theorem 3.7. *Every* low_3 *degree has (3-UEP).*

Proof. Let A be a set of low_3 degree.

The strategy will be for each $e \in \omega$ and u.e. family of A-partial computable functions $\{\Phi^A_{e,y_1,y_2}\}$, to produce a uniformly enumerable (u.e.) family of c.e. sets, $\{W_{g(e,y_1,y_2)}\}$ such that sufficiently many of the corresponding settling time functions, $m_{g(e,y_1,y_2)}$ are total and escape domination from Φ^A_{e,y_1,y_2}, provided sufficiently many of the latter are total. For each $y_1, y_2 \in \omega$ this means tthat

$$(\exists^\infty x)\ \left[\Phi^A_{e,y_1,y_2}(x)\downarrow \leq\ m_{g(e,y_1,y_2)}(x)\downarrow\ \right] \tag{Esc}$$

$$W_{g(e,y_1,y_2)} = \omega. \tag{Tot}$$

Note that condition (Esc) is Π^A_2 and condition (Tot) is Π_2, but unlike the low_1 case, we cannot conclude condition (Esc) is Π_2. To remedy this, the strategy is to pump (Esc) with $\{\forall^\infty, \exists^\infty\}$ quantifiers to a Π^A_4 condition, which is Π_4, since A is low_3. We then appeal to the Strong Quantifier Normal Form Theorem (4.2) for a corresponding Π_4 condition with (Tot).

Define the A-predicate

$$V^A(e, y_1, y_2) \iff (\exists^\infty x)(\exists s)\ \left[\Phi^A_{e,y_1,y_2,s}(x)\downarrow \leq\ s\ \ \&\ \ x \notin W_{g(v,e,y_1,y_2),s}\right] \tag{2}$$

where v is a Π_4-index for the Π^A_4 predicate

$$(\exists^\infty y_2)(\forall^\infty y_1)\ V^A(e, y_1, y_2)$$

and g is the computable function given by Theorem 4.2

$$\begin{aligned} (\exists^\infty y_2)(\forall^\infty y_1)\ V^A(e, y_1, y_2) \quad &\iff \quad (\forall y_2)(\forall^\infty y_1)\ \left[W_{g(v,e,y_1,y_2)} = \omega\right] \\ \neg(\exists^\infty y_2)(\forall^\infty y_1)\ V^A(e, y_1, y_2) \quad &\iff \quad (\forall^\infty y_2)(\forall y_1)\ \left[W_{g(v,e,y_1,y_2)} \text{ is finite }\right]. \end{aligned}$$

$V^A(e, y_1, y_2)$ expresses the condition (Esc) for e, y_1, y_2. It is Π^A_2, so that, as A is low_3,

$$(\exists^\infty y_2)(\forall^\infty y_1)\ V^A(e, y_1, y_2) \in \Pi^A_4 \subset \Pi_4.$$

(Theorem 1.4.) We use the Recursion Theorem to provide the index v ahead of the definition of V^A, just as with the low_1 case. We will skip the argument here, but we do provide it for the general case in Theorem 5.2.

Sublemma 1. *If Φ^A_{e,y_1,y_2} is total then*

$$\left[W_{g(v,e,y_1,y_2)} \textit{ is finite }\right] \implies V^A(e, y_1, y_2).$$

The proof is exactly like the corresponding Sublemma 1 from the low_1 proof of Theorem 2.4.

Sublemma 2. *If*

$$(\exists^\infty y_2)(\forall^\infty y_1)\ \left[\Phi^A_{e,y_1,y_2} \textit{ is total }\right]$$

then

$$(\exists^\infty y_2)(\forall^\infty y_1)\ V^A(e, y_1, y_2).$$

Proof of Sublemma 2. Suppose that $\neg(\exists^\infty y_2)(\forall^\infty y_1)\ V^A(e,y_1,y_2)$. Then the following three conditions holding simultaneously

$$\begin{gathered}(\exists^\infty y_2)(\forall^\infty y_1)\ \big[\Phi^A_{e,y_1,y_2} \text{ is total}\,\big],\\ (\forall^\infty y_2)(\exists^\infty y_1)\ \neg V^A(e,y_1,y_2) \text{ and}\\ (\forall^\infty y_2)(\forall y_1)\ \big[W_{g(v,e,y_1,y_2)} \text{ is finite}\,\big].\end{gathered}$$

The last condition by the Strong Quantifier Normal Form. The quantifiers of these three expressions line-up as $\{\exists^\infty,\forall^\infty,\forall^\infty\}$ and $\{\forall^\infty,\exists^\infty,\forall\}$, so by Lemma 1.1

$$(\exists^\infty y_2)(\exists^\infty y_1)\ \Big[\big[\Phi^A_{e,y_1,y_2} \text{ is total}\,\big]\ \ \&\ \neg V^A(e,y_1,y_2)\ \ \&\ \ \big[W_{g(v,e,y_1,y_2)} \text{ is finite}\,\big]\Big]$$

which implies that for some y_1,y_2

$$\big[\Phi^A_{e,y_1,y_2} \text{ is total}\,\big]\ \ \&\ \neg V^A(e,y_1,y_2)\ \ \&\ \ \big[W_{g(v,e,y_1,y_2)} \text{ is finite}\,\big].$$

This is impossible by Sublemma 1. □

If

$$(\exists^\infty y_2)(\forall^\infty y_1)\ \big[\Phi^A_{e,y_1,y_2} \text{ is total}\,\big]$$

then from Sublemma 2 and the Strong Quantifier Normal Form both

$$\begin{gathered}(\exists^\infty y_2)(\forall^\infty y_1)\ V^A(e,y_1,y_2) \text{ and}\\ (\forall y_2)(\forall^\infty y_1)\ \big[W_{g(v,e,y_1,y_2)}=\omega\big].\end{gathered}$$

These quantifiers line-up as $\{\exists^\infty,\forall\}$ and $\{\forall^\infty,\forall^\infty\}$, so by Lemma 1.1

$$(\exists^\infty y_2)(\forall^\infty y_1)\ \Big[V^A(e,y_1,y_2)\ \ \&\ \ \big[W_{g(v,e,y_1,y_2)}=\omega\big]\Big]$$

The inner matrix to this formula just expresses that (Esc) and (Tot) hold of the settling time function $m_{g(e,y_1,y_2)}$ for $W_{g(e,y_1,y_2)}$, which is equivalent to

$$(\exists^\infty y_2)(\forall^\infty y_1)\ \big[m_{g(e,y_1,y_2)} \text{ is total and escapes } \Phi^A_{e,y_1,y_2}\big].$$

□

For the converse, we will use the following re-organization and relativization of Lemma 3.3 for Σ^A_3 sets

Lemma 3.8. *There is a computable g such that for any set A and Σ^A_3 set C with Σ^A_3-index e*

$$\begin{aligned} C(x) &\iff (\forall^\infty z)(\forall y)\big[\langle y,z\rangle\in W^A_{g(e,x)}\big]\\ \neg C(x) &\iff (\forall z)(\forall^\infty y)\big[\langle y,z\rangle\notin W^A_{g(e,x)}\big].\end{aligned}$$

Theorem 3.9. *For any set A which has (3-UEP), $A'''\leq_{\mathrm{T}} 0'''$.*

Proof. Since A''' is Σ_3^A-complete, it is sufficient to show that

$$A''' \in \Pi_4$$
$$A''' \in \Sigma_4$$

which together imply that $A''' \in \Delta_4$, and thus, by Post's Theorem (1.3) that $A''' \leq_{\mathrm{T}} 0'''$, so that A is low$_3$.

We will show that there is a computable k such that for any $x \in \omega$

$$x \in A''' \iff (\exists^\infty z)(\forall^\infty y)\ \left[\varphi_{k(x,y,z)} \text{ is total }\right] \tag{Π_4}$$

and a computable ℓ such that for any $x \in \omega$

$$x \in A''' \iff (\forall^\infty z)(\exists^\infty y)\ \left[\operatorname{dom}\varphi_{\ell(x,y,z)} \text{ is finite }\right]. \tag{Σ_4}$$

The central fact about A''' we will use is Corollary 3.8 (where we suppress the Σ_3^A-index for A''')

$$\begin{aligned} A'''(x) &\iff (\forall^\infty z)(\forall y)\left[\langle y,z\rangle \in W^A_{g(x)}\right] \\ \neg A'''(x) &\iff (\forall z)(\forall^\infty y)\left[\langle y,z\rangle \notin W^A_{g(x)}\right] \end{aligned}$$

which provides an A-c.e. condition,

$$\langle y,z\rangle \in W^A_{g(x)}.$$

inside two quantifiers. The basic strategy is just as in the low$_1$ case: we will construct an A-partial computable functions $F_{x,y,z}$ against some partial computable function $h_{x,y,z}$ provided by (3-UEP). At each stage of the construction s we will employ exactly one of two strategies at s:

$\mathcal{E}(y,z;s)$: Choose n least and not in the domain of $F_{x,y,z}$ at stage s and let $F_{x,y,z}(n) = 0$. This is the *extend* strategy at stage s.

$\mathcal{A}(y,z;s)$: Choose n least and not in the domain of $F_{x,y,z}$ at stage s, but in the domain of h_x by stage s and let

$$F_{x,y,z}(n) = 1 + h_{x,y,z}(n).$$

If there is no such n, then do nothing. This is the *attack* strategy at stage s.

What is essential about these strategies is summarized in the following, whose proof is exactly in the corresponding Sublemma 1 from Theorem 2.5

Sublemma 1. *Suppose that for each x, y, z and s, exactly one of the strategies $\mathcal{E}(y,z;s)$ or $\mathcal{A}(y,z;s)$ determines the extension of $F_{x,y,z}$. Then*

(a) If $(\exists^\infty s)\ \mathcal{E}(y,z;s)$ then $F_{x,y,z}$ is total.

(b) If $(\forall^\infty s)\ \mathcal{A}(y,z;s)$ and $h_{x,y,z}$ is total then $F_{x,y,z}$ is total and dominates $h_{x,y,z}$

$$(\forall^\infty n)\ \left[F_{x,y,z}(n) > h_{x,y,z}(n)\right].$$

Proof of Condition (Π_4):

The construction describes for each $x \in \omega$ a u.e. array of A-partial computable functions $\{F_{x,y,z}\}_{y,z\in\omega}$, which by the Relativized Kleene Fixed Point Theorem [Soa87, Theorem III.1.6] we may assume ahead has index $f(x)$. Thus, $F_{x,y,z} = \Phi^A_{f(x),y,z}$. By (3-UEP) there is a uniformly enumerable family of partial computable functions $\lambda x.\{h_{f(x),y,z}\}_{y,z\in\omega}$ such that:

For each $x \in \omega$ with the u.e. family of functions $\{F^A_{x,y,z}\}_{y,z\in\omega}$ satisfying

$$(\exists^\infty z)(\forall^\infty y)\ \left[F^A_{x,y,z} \text{ total }\right]$$

then

$$(\exists^\infty z)(\forall^\infty y)\ \left[h_{f(x),y,z} \text{ is total and escapes } F^A_{x,y,z}\right].$$

Let k be the computable function which gives the index of $h_{f(x),y,z}$, so that $\varphi_{k(x,y,z)} = h_{f(x),y,z}$.

The basic strategies will be employed in the following circumstances:

$\mathcal{E}(y,z;s)$: $\langle y,z\rangle \in W^A_{g(x),s}$.

$\mathcal{A}(y,z;s)$: $\langle y,z\rangle \notin W^A_{g(x),s}$.

For each stage exactly one of $\mathcal{A}(y,z;s)$ or $\mathcal{E}(y,z;s)$ determines the extension of $F_{x,y,z}$. Since $W^A_{g(x)}$ is c.e. in A,

Sublemma 2. *For each x, y, z the following holds for $F_{x,y,z}$*

$$\begin{aligned} \langle y,z\rangle \in W^A_{g(x)} &\implies (\forall^\infty s)\ \mathcal{E}(y,z;s) \\ \langle y,z\rangle \notin W^A_{g(x)} &\implies (\forall s)\ \mathcal{A}(y,z;s). \end{aligned}$$

To verify condition (Π_4):

Suppose $x \in A'''$. Then

$$(\forall^\infty z)(\forall y)\ \left[\langle y,z\rangle \in W^A_{g(e,x)}\right]$$

so that

$$(\forall^\infty z)(\forall y)\ \left[(\forall^\infty s)\ \mathcal{E}(y,z;s)\right].$$

Thus, by Sublemma 1

$$(\forall^\infty z)(\forall y)\ \left[F_{x,y,z} \text{ total }\right],$$

and since ($\forall^\infty \Rightarrow \exists^\infty$) and ($\forall \Rightarrow \forall^\infty$) this implies that

$$(\exists^\infty z)(\forall^\infty y)\ \left[F_{x,y,z} \text{ total }\right].$$

Applying (3-UEP)

$$(\exists^\infty z)(\forall^\infty y)\ \big[\varphi_{k(x,y,z)} \text{ total }\big].$$

Suppose $x \notin A'''$. Then

$$(\forall z)(\forall^\infty y)\ \big[\langle y, z\rangle \notin W^A_{g(e,x)}\big]$$

so that

$$(\forall z)(\forall^\infty y)\ \big[(\forall^\infty s)\ \mathcal{A}(y,z;s)\big].$$

Suppose that for contradiction,

$$(\exists^\infty z)(\forall^\infty y)\ \big[\varphi_{k(x,y,z)} \text{ is total }\big]. \tag{3}$$

The quantifiers in the previous two conditions line-up as $\{\forall, \exists^\infty\}$ and $\{\forall^\infty, \forall^\infty\}$ so by Lemma 1.1(g,i)

$$(\exists^\infty z)(\forall^\infty y)\ \big[(\forall^\infty s)\ \mathcal{A}(y,z;s)\ \&\ \varphi_{k(x,y,z)} \text{ is total }\big].$$

By Sublemma 1 this implies

$$(\exists^\infty z)(\forall^\infty y)\ \big[F_{x,y,z} \text{ is total }\big]$$

so that by (3-UEP)

$$(\exists^\infty z)(\forall^\infty y)\ \big[\varphi_{k(x,y,z)} \text{ is total and escapes } F_{x,y,z}\big].$$

Thus, the following two conditions hold simultaneously

$$(\forall z)(\forall^\infty y)\ \big[(\forall^\infty s)\ \mathcal{A}(y,z;s)\big]$$
$$(\exists^\infty z)(\forall^\infty y)\ \big[\varphi_{k(x,y,z)} \text{ is total and escapes } F_{x,y,z}\big].$$

By Lemma 1.1(g,i)

$$(\exists^\infty z)(\forall^\infty y)\ \big[(\forall^\infty s)\ \mathcal{A}(y,z;s)\ \&\ \varphi_{k(x,y,z)} \text{ is total and escapes } F_{x,y,z}\big].$$

But by Sublemma 1 it is impossible to have the following conditions holding simultaneously

$$(\forall^\infty s)\ \mathcal{A}(y,z;s)$$
$$\varphi_{k(x,y,z)} \text{ is total and escapes } F_{x,y,z}.$$

Thus, Condition (3) must fail, so that

$$\neg(\exists^\infty z)(\forall^\infty y)\ \big[\varphi_{k(x,y,z)} \text{ is total }\big].$$

Proof of Condition (Σ_4):

The construction describes for each $x \in \omega$ a u.e. array of A-partial computable functions $\{G_{x,y,z}\}_{y,z\in\omega}$, which by the Relativized Kleene Fixed Point Theorem [Soa87, Theorem III.1.6], we may assume ahead has index $f(x)$. Thus, $G_{x,y,z} = \Phi^A_{f(x),y,z}$. By (3-UEP) there is a uniformly enumerable family of partial computable functions $\lambda x.\{h_{f(x),y,z}\}_{y,z\in\omega}$ such that:

For each $x \in \omega$ with the u.e. family of functions $\{G_{x,y,z}\}_{y,z\in\omega}$ satisfying

$$(\exists^\infty z)(\forall^\infty y)\ \big[G_{x,y,z} \text{ total }\big]$$

then

$$(\exists^\infty z)(\forall^\infty y)\ \big[h_{f(x),y,z} \text{ is total and escapes } G_{x,y,z}\big].$$

We modify $h_{f(x),y,z}$:

$$h'_{x,y,z}(n) = \begin{cases} h_{f(x),y,z}(n) & \text{if } (\forall m \leq n)\ \big[h_{f(x),y,z}(m)\downarrow\ \big] \\ \uparrow & \text{otherwise} \end{cases}$$

Let ℓ be the computable function which gives the index of $h'_{x,y,z}$, so that $\varphi_{\ell(x,y,z)} = h'_{x,y,z}$. Then

$$\begin{aligned} h_{f(x),y,z} \text{ total} &\iff \varphi_{\ell(x,y,z)} \text{ total} \\ h_{f(x),y,z} \text{ not total} &\iff \varphi_{\ell(x,y,z)} \text{ finite .} \end{aligned}$$

We reverse the strategies from the previous case:

$\mathcal{A}(y,z;s)$: $\langle y,z\rangle \in W_{g(x),s}$.

$\mathcal{E}(y,z;s)$: $\langle y,z\rangle \notin W_{g(x),s}$.

For each stage exactly one of $\mathcal{A}(y,z;s)$ or $\mathcal{E}(y,z;s)$ determines the extension of $G_{x,y,z}$. Since $W^A_{g(x)}$ is c.e. in A,

Sublemma 3. *For each x, y, z the following holds for $G_{x,y,z}$*

$$\begin{aligned} \langle y,z\rangle \in W^A_{g(x)} &\implies (\forall^\infty s)\ \mathcal{A}(y,z;s) \\ \langle y,z\rangle \notin W^A_{g(x)} &\implies (\forall s)\ \mathcal{E}(y,z;s). \end{aligned}$$

To verify condition (Σ_4):

Suppose $x \notin A'''$. Then

$$(\forall z)(\forall^\infty y)\ \big[\langle y,z\rangle \notin W^A_{g(e,x)}\big]$$

so that

$$(\forall z)(\forall^\infty y)\ \big[(\forall^\infty s)\ \mathcal{E}(y,z;s)\big].$$

Thus by Sublemma 1

$$(\forall z)(\forall^\infty y)\ \big[G_{x,y,z} \text{ is total }\big]$$

and since $(\forall \Rightarrow \exists^\infty)$ this implies that

$$(\exists^\infty z)(\forall^\infty y)\ \big[G_{x,y,z} \text{ is total }\big].$$

Applying (3-UEP)

$$(\exists^\infty z)(\forall^\infty y)\ \left[\varphi_{\ell(x,y,z)} \text{ is total }\right]$$

so that

$$\neg(\forall^\infty z)(\exists^\infty y)\ \left[\operatorname{dom}\varphi_{\ell(x,y,z)} \text{ is finite }\right].$$

Suppose $x \in A'''$. Then

$$(\forall^\infty z)(\forall y)\ \left[\langle y, z\rangle \in W^A_{g(e,x)}\right]$$

so that

$$(\forall^\infty z)(\forall y)\ \left[(\forall^\infty s)\ \mathcal{A}(y, z; s)\right]. \tag{4}$$

Suppose for contradiction,

$$\neg(\forall^\infty z)(\exists^\infty y)\ \left[\operatorname{dom}\varphi_{\ell(x,y,z)} \text{ is finite }\right]. \tag{5}$$

Then since either $\operatorname{dom}\varphi_{\ell(x,y,z)}$ is finite or $\varphi_{\ell(x,y,z)}$ is total, it must be the case that

$$(\exists^\infty z)(\forall^\infty y)\ \left[\varphi_{\ell(x,y,z)} \text{ is total }\right]$$

We have condition (4) together with the previous condition holding

$$(\forall^\infty z)(\forall y)\ \left[(\forall^\infty s)\ \mathcal{A}(y, z; s)\right]$$
$$(\exists^\infty z)(\forall^\infty y)\ \left[\varphi_{\ell(x,y,z)} \text{ is total }\right].$$

Applying Lemma 1.1(f,g)

$$(\exists^\infty z)(\forall^\infty y)\ \left[(\forall^\infty s)\ \mathcal{A}(y, z; s)\ \&\ \varphi_{\ell(x,y,z)} \text{ is total }\right].$$

By Sublemma 1 this implies

$$(\exists^\infty z)(\forall^\infty y)\ \left[G_{x,y,z} \text{ is total }\right]$$

so that by (3-UEP)

$$(\exists^\infty z)(\forall^\infty y)\ \left[\varphi_{\ell(x,y,z)} \text{ is total and escapes } G_{x,y,z}\right].$$

Thus the following two conditions hold simultaneously

$$(\forall^\infty z)(\forall y)\ \left[(\forall^\infty s)\ \mathcal{A}(y, z; s)\right]$$
$$(\exists^\infty z)(\forall^\infty y)\ \left[\varphi_{\ell(x,y,z)} \text{ is total and escapes } G_{x,y,z}\right].$$

By Lemma 1.1(f,g)

$$(\exists^\infty z)(\forall^\infty y)\ \left[(\forall^\infty s)\ \mathcal{A}(y, z; s)\ \&\ \varphi_{\ell(x,y,z)} \text{ total and escapes } G_{x,y,z}\right].$$

But by Sublemma 1 it is impossible to have the following conditions holding simultaneously

$$(\forall^\infty s)\ \mathcal{A}(y, z; s)$$
$$\varphi_{\ell(x,y,z)} \text{ is total and escapes } G_{x,y,z}$$

Thus, Condition (5) must fail, so that

$$(\forall^\infty z)(\exists^\infty y)\ \left[\operatorname{dom}\varphi_{\ell(x,y,z)} \text{ is finite }\right].$$

□

4 Strong Quantifier Normal Form for Arithmetic Relations

Let $n \geq 1$. Then any Σ_{2n+1} set A can be expressed using alternating $\{\forall, \exists\}$ quantifiers by

$$A(x) \iff (\exists y_{2n-1})(\forall y_{2n-2}) \dots (\exists y_1)(\forall z) \left[\langle x, z, y_1, \dots, y_{2n-1}\rangle \in W_e\right]$$

(e is the Σ_{2n+1}-index for A); and any Π_{2n} set B can be expressed using alternating $\{\forall, \exists\}$ quantifiers by

$$A(x) \iff (\forall y_{2n-2})(\exists y_{2n-3}) \dots (\exists y_1)(\forall z) \left[\langle x, z, y_1, \dots, y_{2n-1}\rangle \in W_e\right]$$

(e is the Π_{2n}-index for B.)

For $x, \overline{y} \in \omega$ let g be a computable index function satisfying

$$z \in W_{g(e,x,\overline{y})} \iff (\forall w \leq z)\left[\langle x, w, \overline{y}\rangle \in W_e\right].$$

The *Quantifier Normal Form Theorem* for arithmetic predicates using $\{\forall, \exists\}$ is

Theorem 4.1 (Quantifier Normal Form)**.** *Let $n \geq 1$. Then there is a computable g such that*

1. *For any $A(x) \in \Sigma_{2n+1}$ with index e*

$$\begin{aligned} A(x) &\iff (\exists y_{2n-1})(\forall y_{2n-2}) \dots (\forall y_2)(\exists y_1) \left[W_{g(e,x,y_1,\dots,y_{2n-1})} = \omega\right] \\ \neg A(x) &\iff (\forall y_{2n-1})(\exists y_{2n-2}) \dots (\exists y_2)(\forall y_1) \left[W_{g(e,x,y_1,\dots,y_{2n-1})} \textit{ is finite }\right]. \end{aligned}$$

2. *For any $A(x) \in \Pi_{2n}$ with index e*

$$\begin{aligned} A(x) &\iff (\forall y_{2n-2})(\exists y_{2n-3}) \dots (\forall y_2)(\exists y_1) \left[W_{g(e,x,y_1,\dots,y_{2n-2})} = \omega\right] \\ \neg A(x) &\iff (\exists y_{2n-2})(\forall y_{2n-3}) \dots (\exists y_2)(\forall y_1) \left[W_{g(e,x,y_1,\dots,y_{2n-2})} \textit{ is finite }\right]. \end{aligned}$$

The *Strong Quantifier Normal Form Theorem* trades in the weakest quantifier, $\exists$, in Theorem 4.1 for the stronger $\forall^\infty$ quantifier. The simplest cases of the *Strong Quantifier Normal Form Theorem* are Theorems 2.3, 3.3, 3.6 for Π_2, Σ_3, and Π_4 sets. These examples can be generalized to obtain a characterization of the arithmetic sets in terms of the two strongest quantifiers, $\{\forall, \forall^\infty\}$:

Theorem 4.2 (Strong Quantifier Normal Form)**.** *For $n \geq 1$:*

1. *There exists a computable g such that for any $A(x) \in \Sigma_{2n+1}$ with index e*

$$\begin{aligned} A(x) &\iff (\forall^\infty y_{2n-1})(\forall y_{2n-2}) \dots (\forall y_2)(\forall^\infty y_1) \left[W_{g(e,x,y_1,\dots,y_{2n-1})} = \omega\right] \\ \neg A(x) &\iff (\forall y_{2n-1})(\forall^\infty y_{2n-2}) \dots \forall y_2)(\forall y_1) \left[W_{g(e,x,y_1,\dots,y_{2n-1})} \textit{ is finite }\right]. \end{aligned}$$

2. *There exists a computable g such that for any $A(x) \in \Pi_{2n}$ with index e*

$$\begin{aligned} A(x) &\iff (\forall y_{2n-2})(\forall^\infty y_{2n-3}) \dots (\forall^\infty y_1) \left[W_{g(e,x,y_1,\dots,y_{2n-2})} = \omega\right] \\ \neg A(x) &\iff (\forall^\infty y_{2n-2})(\forall y_{2n-3}) \dots (\forall y_1) \left[W_{g(e,x,y_1,\dots,y_{2n-2})} \textit{ is finite }\right]. \end{aligned}$$

A second characterization will also be used:

Corollary 4.3. *For $n \geq 1$:*

1. *There exists a computable f such that for any $A(x) \in \Sigma_{2n+1}$ with index e*

$$\begin{aligned} A(x) &\iff (\forall^\infty y_{2n})(\forall y_{2n-1}) \ldots (\forall^\infty y_2)(\forall y_1) \left[\langle y_1, \ldots, y_{2n} \rangle \in W_{f(e,x)}\right] \\ \neg A(x) &\iff (\forall y_{2n})(\forall^\infty y_{2n-1}) \ldots (\forall y_2)(\forall^\infty y_1) \left[\langle y_1, \ldots, y_{2n} \rangle \notin W_{f(e,x)}\right]. \end{aligned}$$

2. *There exists a computable f such that for any $A(x) \in \Pi_{2n}$ with index e*

$$\begin{aligned} A(x) &\iff (\forall y_{2n-1})(\forall^\infty y_{2n-2}) \ldots (\forall y_3)(\forall^\infty y_2)(\forall y_1) \left[\langle y_1, \ldots, y_{2n-1} \rangle \in W_{f(e,x)}\right] \\ \neg A(x) &\iff (\forall^\infty y_{2n-1})(\forall y_{2n-2}) \ldots (\forall^\infty y_3)(\forall y_2)(\forall^\infty y_1) \left[\langle y_1, \ldots, y_{2n-1} \rangle \notin W_{f(e,x)}\right]. \end{aligned}$$

Both Theorem 4.2 and Corollary 4.3 can be relativized.

Proof of Corollary 4.3. Note that

$$\begin{aligned} W_e = \omega &\iff (\forall z) \left[z \in W_e\right] \\ W_e \text{ is finite} &\iff (\forall^\infty z) \left[z \notin W_e\right] \end{aligned}$$

and apply Theorem 4.2. □

Proof of Theorem 4.2. The proof is by induction on n. The case where $n = 1$, Π_2 and Σ_3, were previously given.

Case Π_{2n+2}:

Suppose $A \in \Pi_{2n+2}$ with index e, so that $A(x)$ is $(\forall z)B(x,z)$ where $B(x,z) \in \Sigma_{2n+1}$ with index e as well. By the induction hypothesis for Σ_{2n+1} there is a computable h such that

$$\begin{aligned} B(x,z) &\iff (\forall^\infty y_{2n-1})(\forall y_{2n-2}) \ldots (\forall^\infty y_1) \left[W_{h(e,x,z,y_1,\ldots,y_{2n-1})} = \omega\right] \\ \neg B(x,z) &\iff (\forall y_{2n-1})(\forall^\infty y_{2n-2}) \ldots (\forall y_1) \left[W_{h(e,x,z,y_1,\ldots,y_{2n-1})} \text{ is finite }\right]. \end{aligned}$$

Let g be a computable function such that

$$W_{g(e,x,y_1,\ldots,y_{2n})} = \bigcap_{z \leq y_{2n}} W_{h(e,x,z,y_1,\ldots,y_{2n-1})}.$$

First, suppose $\neg A(x)$, so that for some z, $\neg B(x,z)$. For every $y_{2n} \geq z$,

$$W_{g(e,x,y_1,\ldots,y_{2n})} \subseteq W_{h(e,x,z,y_1,\ldots,y_{2n-1})}$$

so that for every $y_{2n} \geq z$,

$$(\forall y_{2n-1})(\forall^\infty y_{2n-2})\ldots(\forall y_1)\ \left[W_{g(e,x,y_1,\ldots,y_{2n})} \text{ is finite }\right].$$

Thus,

$$(\forall^\infty y_{2n})(\forall y_{2n-1})(\forall^\infty y_{2n-2})\ldots(\forall y_1)\ \left[W_{g(e,x,y_1,\ldots,y_{2n})} \text{ is finite }\right].$$

Next, suppose $A(x)$, so that for all z, $B(x,z)$. Then for all z

$$(\forall^\infty y_{2n-1})(\forall y_{2n-2})\ldots(\forall^\infty y_1)\ \left[W_{h(e,x,z,y_1,\ldots,y_{2n-1})} = \omega\right]. \tag{6}$$

The goal is to show that for all y_{2n}

$$(\forall^\infty y_{2n-1})(\forall y_{2n-2})\ldots(\forall^\infty y_1)\ \left[W_{g(e,x,y_1,\ldots,y_{2n-1},y_{2n})} = \omega\right] \tag{7}$$

By condition (6)

$$\begin{aligned}(\forall^\infty y_{2n-1})(\forall y_{2n-2})\ldots(\forall^\infty y_1)\ \left[W_{h(e,x,0,y_1,\ldots,y_{2n-1})} = \omega\right]\ \ \&\ \ \ldots\ \ \& \\ (\forall^\infty y_{2n-1})(\forall y_{2n-2})\ldots(\forall^\infty y_1)\ \left[W_{h(e,x,y_{2n},y_1,\ldots,y_{2n-1})} = \omega\right]\end{aligned} \tag{8}$$

so applying Lemma 1.1b (conjunction) to condition (8)

$$(\forall^\infty y_{2n-1})(\forall y_{2n-2})\ldots(\forall^\infty y_1)\ \left[W_{h(e,x,0,y_1,\ldots,y_{2n-1})} = \omega\ \ \&\ \ \ldots\ \ \&\ \ W_{h(e,x,y_{2n},y_1,\ldots,y_{2n-1})} = \omega\right]. \tag{9}$$

But

$$W_{g(e,x,y_1,\ldots,y_{2n-1},y_{2n})} = \omega \iff \left[W_{h(e,x,0,y_1,\ldots,y_{2n-1})} = \omega\ \ \&\ \ \ldots\ \ \&\ \ W_{h(e,x,y_{2n},y_1,\ldots,y_{2n-1})} = \omega\right] \tag{10}$$

so, applying condition (10) to condition (9), we obtain the goal that condition (7) holds for all y_{2n}.

Case Σ_{2n+3} :

Suppose $A \in \Sigma_{2n+3}$ with index e and assume (b) of the Theorem holds for Π_{2n+2}. Then $A(x)$ is $(\exists z)B(x,z)$ also with index e and where $B(x,z) \in \Pi_{2n+2}$. By the induction hypothesis for Π_{2n+2} there is a computable h such that

$$\begin{aligned} B(x,z) &\iff (\forall y_{2n})(\forall^\infty y_{2n-1})\ldots(\forall^\infty y_1)\ \left[W_{h(e,x,z,y_1,\ldots,y_{2n})} = \omega\right] \\ \neg B(x,z) &\iff (\forall^\infty y_{2n})(\forall y_{2n-1})\ldots(\forall y_1)\ \left[W_{h(e,x,z,y_1,\ldots,y_{2n})} \text{ is finite }\right]. \end{aligned}$$

Now let g be a computable function such that

$$W_{g(e,x,y_1,\ldots,y_{2n})} = \bigcup_{z \leq y_{2n}} W_{h(e,x,z,y_1,\ldots,y_{2n-1})}.$$

First, suppose $A(x)$, so that for some z, $B(x,z)$. For every $y_{2n} \geq z$,

$$W_{h(e,x,z,y_1,\ldots,y_{2n-1})} \subseteq W_{g(e,x,y_1,\ldots,y_{2n})}$$

so that for every $y_{2n+1} \geq z$

$$(\forall y_{2n})(\forall^\infty y_{2n-1})\dots(\forall^\infty y_1)\ \left[W_{g(e,x,y_1,\dots,y_{2n+1})} = \omega\right].$$

Thus,

$$(\forall^\infty y_{2n+1})(\forall y_{2n})(\forall^\infty y_{2n-1})\dots(\forall^\infty y_1)\ \left[W_{g(e,x,y_1,\dots,y_{2n+1})} = \omega\right].$$

Next, suppose $\neg A(x)$, so that for all z, $\neg B(x,z)$. So, for every y_{2n+1}

$$(\forall^\infty y_{2n})(\forall y_{2n-1})\dots(\forall y_1)\ \left[W_{h(e,x,y_{2n+1},y_1,\dots,y_{2n})} \text{ is finite }\right]. \tag{11}$$

The goal is to show that for every y_{2n+1}

$$(\forall^\infty y_{2n})(\forall y_{2n-1})\dots(\forall y_1)\ \left[W_{g(e,x,y_1,\dots,y_{2n+1})} \text{ is finite }\right]. \tag{12}$$

By condition (11)

$$\begin{array}{r}(\forall^\infty y_{2n})(\forall y_{2n-1})\dots(\forall^\infty y_1)\ \left[W_{h(e,x,0,y_1,\dots,y_{2n})} \text{ is finite }\right]\ \&\ \dots\ \&\\ (\forall^\infty y_{2n})(\forall y_{2n-1})\dots(\forall^\infty y_1)\ \left[W_{h(e,x,y_{2n+1},y_1,\dots,y_{2n})} \text{ is finite }\right]\end{array} \tag{13}$$

so applying Lemma 1.1b (conjunction) to condition (13)

$$(\forall^\infty y_{2n})(\forall y_{2n-1})\dots(\forall^\infty y_1)\ \Big[W_{h(e,x,0,y_1,\dots,y_{2n-1})} \text{ is finite }\ \&\ \dots\ \&\ W_{h(e,x,y_{2n+1},y_1,\dots,y_{2n})} \text{ is finite }\Big]. \tag{14}$$

But

$$W_{g(e,x,y_1,\dots,y_{2n},y_{2n})} \text{ is finite} \iff \Big[W_{h(e,x,0,y_1,\dots,y_{2n})} \text{ is finite }\ \&\ \dots\ \&\ W_{h(e,x,y_{2n+1},y_1,\dots,y_{2n})} \text{ is finite }\Big] \tag{15}$$

so, applying condition (15) to condition (14) we obtain the goal that condition (12) holds for all y_{2n+1}.

□

5 Finding Escape Functions Uniformly for low_n Sets

The pattern for the properties (n-UEP) and the proof that all low_n degrees have (n-UEP) follows the example of low_3. We write $\overline{y}$ for $\langle y_1, \dots, y_k \rangle$, where k will be understood in the context.

Definition 5.1 (Characterization of (n-UEP))**.** *A set A has the* n-Uniform Escape Property (n-UEP) *when there is a uniformly enumerable array of families of partial computable functions $\lambda e.\{h_{e,\overline{y}}\}_{\overline{y}\in\omega}$ satisfying:*

For each $e \in \omega$ with the family of functions $\left\{\Phi^A_{e,\overline{y}}\right\}_{\overline{y}\in\omega}$ satisfying

$$(\mathcal{Q}_1 y_{n-1})(\mathcal{Q}_2 y_{n-2})\ldots\left[\Phi^A_{e,\overline{y}} \textit{ total }\right]$$

then

$$(\mathcal{Q}_1 y_{n-1})(\mathcal{Q}_2 y_{n-2})\ldots\left[h_{e,\overline{y}} \textit{ is total and escapes } \Phi^A_{e,\overline{y}}\right].$$

where $\mathcal{Q}_1, \mathcal{Q}_2 \in \left\{\exists^\infty, \forall^\infty\right\}$ by

- *For odd n: alternate $\exists^\infty\forall^\infty$*
- *For even n: alternate $\forall^\infty\exists^\infty$.*

A degree has the n-Uniform Escape Property (n-UEP) *when some set in the degree has the property.*

The main result of this section is the following:

Theorem 5.2 (low$_n$ Implies (n-UEP))**.** *All* low$_n$ *degrees have (n-UEP).*

Proof.

Let $\overline{y} = \langle y_1, \ldots, y_n\rangle$. We will show that we can effectively find c.e. sets $W_{g(e,\overline{y})}$ such that a sufficient density settle slower than $\{\Phi^A_{e,\overline{y}}\}$ in the following sense

$$(\exists^\infty x)(\exists s)\left[s \geq \Phi^A_e(\langle \overline{y}, x\rangle) \;\;\&\;\; x \notin W_{g(e,\overline{y}),s}\right] \tag{16}$$

but we will also want simultaneously

$$W_{g(e,\overline{y})} = \omega. \tag{17}$$

Define a computable function $\lambda x.h_{e,\overline{y}}(x)$

$$h_{e,\overline{y}}(x) = (\mu s)\left[x \in W_{g(e,\overline{y}),s}\right]$$

so that if condition (16) and (17) hold for a tuple $\overline{y}$ then the function $h_{e,\overline{y}}$ is total and escapes domination from $\Phi^A_{e,\overline{y}}$. The heart of the proof is to find the function g and show it works often enough (as required by the property n-UEP.)

The cases break down into low$_{2n-1}$ and low$_{2n}$, but are very similar.

Case 1: low$_{2n-1}$:

Suppose A is low$_{2n-1}$, so that

$$\Pi^A_{2n} = \Pi_{2n}.$$

Let g be the computable function given by case (2) of Theorem 4.2 (for Π_{2n} relations.) We write $\overline{y}$ for $\langle y_1, \ldots, y_{2n-2}$ in the remainder of the proof, and drop references to the variables in quantifiers which quantify over the parameters in $\overline{y}$: writing $\forall^\infty$ for $(\forall^\infty y_i)$, and similarly for the other quantifiers.

The remainder of this paragraph is intended to show that we can define a Π_{2n} predicate $U^A(e)$ by reference to its Π_{2n}-index, using the Recursion Theorem, and may be skipped to the definition of V^A below if this is clear to the reader. For every i and $\overline{y}$ let $\hat{V}^A(e, i, \overline{y})$ be defined by

$$(\exists^\infty x)(\exists s)\big[s \geq \Phi^A_{e,\overline{y}}(x) \;\;\&\;\; x \notin W_{g(i,e,\overline{y}),s}\big]$$

so that $\hat{V}^A$ is Π^A_2. Let

$$\hat{U}^A(e,i) \iff \exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \; \hat{V}^A(e,i,\overline{y})$$

so,

$$\hat{U}^A(e,i) \in \Pi^A_{2n} = \Pi_{2n}.$$

By the s-m-n theorem ([Soa87, Theorem I.3.5]) there is a computable p providing an index for $U^A_{e,i}$ such that (see section 2)

$$\hat{U}^A(e,i) \iff \forall\exists\ldots\forall\exists(\forall z)\big[\langle z, \overline{y}\rangle \in W_{p(i,e)}\big].$$

By the Recursion Theorem with Parameters ([Soa87, Theorem II.3.5]) there is a computable function $f(e)$ such that

$$W_{p(f(e),e)} = W_{f(e)}$$

and so

$$\hat{U}^A(e, f(e)) \iff \forall\exists\ldots\forall\exists(\forall z)\big[\langle z, \overline{y}\rangle \in W_{f(e)}\big].$$

Let $V^A(e, \overline{y})$ be $\hat{V}^A(e, f(e), \overline{y})$, so

$$V^A(e,\overline{y}) \iff (\exists^\infty x)(\exists s)\big[s \geq \Phi^A_{e,\overline{y}}(x) \;\;\&\;\; x \notin W_{g(f(e),e,\overline{y}),s}\big]. \tag{18}$$

Let $U^A(e)$ be $\hat{U}^A(e, f(e))$, so that

$$U^A(e) \iff \exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \; V^A(e,\overline{y})$$

with a Π_{2n}-index for U^A given by $f(e)$. By Theorem 4.2 case (2)

$$\begin{aligned} U^A(e) &\iff \forall\forall^\infty \ldots \forall\forall^\infty \; \big[W_{g(f(e),e,\overline{y})} = \omega\big] \\ \neg U^A(e) &\iff \forall^\infty\forall \ldots \forall^\infty\forall \; \big[W_{g(f(e),e,\overline{y})} \text{ is finite }\big]. \end{aligned}$$

Now, to verify $W_{g(f(e),e,\overline{y})}$ satisfies conditions (17) and (16):

Sublemma 1. *If $\Phi^A_{e,\overline{y}}$ is total then*

$$\big[W_{g(f(e),e,\overline{y})} \text{ is finite } \big] \implies V^A(e,\overline{y}).$$

Proof of Sublemma 1. Suppose $\Phi^A_{e,\overline{y}}$ is total, and $W_{g(f(e),e,\overline{y})}$ is finite. Fix z so that for all $x > z$, $x \notin W_{g(f(e),e,\overline{y})}$. For each such x

$$(\exists s)\big[s \geq \Phi^A_e(x) \ \ \& \ \ x \notin W_{g(f(e),e,\overline{y}),s}\big]$$

and since this holds for almost every x, by Equation (18) $V^A(e,\overline{y})$ holds. □

Sublemma 2.

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \ \big[\Phi^A_{e,\overline{y}} \text{ is total } \big] \implies U^A(e).$$

Proof of Sublemma 2. Suppose $\neg U^A(e)$. Then we have the following true:

$$\begin{aligned}
&\forall^\infty\exists^\infty \ldots \forall^\infty\exists^\infty \ \neg V^A(e,\overline{y}),\\
&\forall^\infty\forall \ \ldots \ \forall^\infty\forall \ \big[W_{g(f(e),e,\overline{y}} \text{ is finite } \big] \text{ and}\\
&\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \ \big[\Phi^A_{e,\overline{y}} \text{ is total } \big]
\end{aligned}$$

where the first line is $\neg U^A(e)$, and the second line comes from the implication

$$\neg U^A(e) \implies \forall^\infty\forall \ldots \forall^\infty\forall \ \big[W_{g(f(e),e,\overline{y}} \text{ is finite } \big]$$

and the third line is antecedent of the Lemma we are proving. These are a conjunction of three conditions with $2n-2$ initial quantifiers which line-up as either $\big\{\forall^\infty, \forall^\infty, \exists^\infty\big\}$ or $\big\{\exists^\infty, \forall, \forall^\infty\big\}$. By Lemma 1.1(fghi),

$$\exists^\infty\exists^\infty \ldots \exists^\infty \left[\neg V^A(e,\overline{y}) \ \ \& \ \ \big[W_{g(f(e),e,\overline{y}} \text{ is finite } \big] \ \ \& \ \ \big[\Phi^A_{e,\overline{y}} \text{ is total } \big]\right].$$

This is impossible by Sublemma 1. □

Thus, from Sublemma 2, for each u.e. family $\big\{\Phi^A_{e,\overline{y}}\big\}$ satisfying

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \ \big[\Phi^A_{e,\overline{y}} \text{ is total } \big]$$

the following holding

$$\begin{aligned}
&\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \ V^A(e,\overline{y}) \text{ and}\\
&\forall \ \ \forall^\infty \ldots \ \forall\forall^\infty \ \big[W_{g(f(e),e,\overline{y})} = \omega\big]
\end{aligned}$$

where the first line is $U^A(e)$ and the second line comes from the implication

$$U^A(e) \implies \forall\forall^\infty \ldots \forall\forall^\infty \ \big[W_{g(f(e),e,\overline{y})} = \omega\big]$$

This is a conjunction of two conditions with $2n-2$ initial quantifiers which line-up as either $\big\{\forall, \exists^\infty\big\}$ or $\big\{\forall^\infty, \forall^\infty\big\}$. By Lemma 1.1(gi),

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[V^A(e,\overline{y}) \ \ \& \ \ \big[W_{g(f(e),e,\overline{y})} = \omega\big]\right]. \tag{19}$$

Define $\lambda x.h_{e,\overline{y}}(x)$ as follows:

$$h_{e,\overline{y}}(x) := (\mu s)\ \big[x \in W_{g(f(e),e,\overline{y})}\big].$$

Sublemma 3. *For every $\overline{y}$ for which*

$$V^A(e,\overline{y})\ \ \&\ \ \big[W_{g(f(e),e,\overline{y})} = \omega\big]$$

the function $h_{e,\overline{y}}$ is total and escapes domination from $\Phi^A_{e,\overline{y}}$:

$$(\exists^\infty x)\ \big[\Phi^A_{e,\overline{y}}(x) \leq h_{e,\overline{y}}(x)\big].$$

Proof of Sublemma 3. That the function $h_{e,\overline{y}}$ is total follows from

$$\big[W_{g(f(e),e,\overline{y})} = \omega\big].$$

That $h_{e,\overline{y}}$ escapes domination from $\Phi^A_{e,\overline{y}}$,

$$(\exists^\infty x)\ \big[\Phi^A_{e,\overline{y}}(x) \leq h_{e,\overline{y}}(x)\big],$$

follows from $V^A(e,\overline{y})$

$$(\exists^\infty x)(\exists s)\big[s \geq \Phi^A_{e,\overline{y}}(x)\ \ \&\ \ x \notin W_{g(f(e),e,\overline{y}),s}\big].$$

□

Case (1) of the Theorem now follows from condition (19) and Sublemma 3.

Case 2: low_{2n}:

Suppose A be low_{2n}, so that

$$\Sigma^A_{2n+1} = \Sigma_{2n+1}.$$

Let g be the computable function given by case (1) of Theorem 4.2 (for Σ_{2n+1}.) We write $\overline{y}$ for $\langle y_1, \ldots, y_{2n-1}$ in the remainder of the proof, and drop references to the variables in quantifiers which quantify over the parameters in $\overline{y}$: writing $\forall^\infty$ for $(\forall^\infty y_i)$, and similarly for the other quantifiers.

Just as in the case (1), we may define $U^A(e)$ assuming we have a Σ_{2n+1}-index $f(e)$ for $U^A(e)$. We skip this argument in case (2). For every $\overline{y}$ let $V^A(e,\overline{y})$ be

$$(\exists^\infty x)(\exists s)\big[s \geq \Phi^A_{e,\overline{y}}(x)\ \ \&\ \ x \notin W_{g(e,f(e),\overline{y}),s}\big] \tag{20}$$

so that V^A is Π^A_2. Let

$$U^A(e) := \forall^\infty\exists^\infty \ldots \forall^\infty\exists^\infty\forall^\infty\ V^A(e,\overline{y})$$

so that

$$U^A(e,i) \in \Sigma^A_{2n+1} = \Sigma_{2n+1}$$

with Σ_{2n+1}-index $f(e)$. By Theorem 4.2, case (1),

$$\begin{aligned} U^A(e) &\implies \forall^\infty\forall\ldots\forall^\infty\forall\forall^\infty \left[W_{g(f(e),e,\overline{y})} = \omega\right] \\ \neg U^A(e) &\implies \forall\forall^\infty\ldots\forall\forall^\infty\forall \left[W_{g(f(e),e,\overline{y})} \text{ is finite }\right] \end{aligned}$$

Now, to verify $W_{g(f(e),e,\overline{y})}$ satisfies conditions (16) and (17). Sublemma 1 holds here as well.

Sublemma 4.

$$\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\forall^\infty \left[\Phi^A_{e,\overline{y}} \text{ is total }\right] \implies U^A(e).$$

Proof of Sublemma 4. Suppose $\neg U^A(e)$. Then we have the following true:

$$\begin{aligned} &\exists^\infty\forall^\infty\ldots\exists^\infty\forall^\infty\exists^\infty\ \neg V^A(e,\overline{y}), \\ &\forall\ \forall^\infty\ \ldots\ \forall\ \forall^\infty\forall \left[W_{g(f(e),e,\overline{y})} \text{ is finite }\right] \text{ and} \\ &\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\forall^\infty \left[\Phi^A_{e,\overline{y}} \text{ is total }\right] \end{aligned}$$

where the first line is $\neg U^A(e)$, and the second line comes from the implication

$$\neg U^A(e) \implies \forall\forall^\infty\ldots\forall\forall^\infty\forall \left[W_{g(f(e),e,\overline{y})} \text{ is finite }\right]$$

and the third line is the antecedent of the Lemma we are proving. This is a conjunction of three propositions with $2n-1$ initial quantifiers which line-up as either $\{\exists^\infty,\forall,\forall^\infty\}$ or $\{\forall^\infty,\forall^\infty,\exists^\infty\}$. By Lemma 1.1(fghi) ,

$$\exists^\infty\exists^\infty\ldots\exists^\infty \left[\neg V^A(e,\overline{y})\ \ \&\ \ \left[W_{g(f(e),e,\overline{y}} \text{ is finite }\right]\ \ \&\ \ \left[\Phi^A_{e,\overline{y}} \text{ is total }\right]\right].$$

This is impossible by Sublemma 1. ☐

Thus, from Sublemma 4 , if

$$\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\forall^\infty \left[\Phi^A_{e,\overline{y}}\text{is total }\right]$$

then the following holds

$$\begin{aligned} &\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\forall^\infty\ V^A(e,\overline{y}) \text{ and} \\ &\forall^\infty\ \forall\ \ \ldots\ \forall^\infty\ \forall\forall^\infty \left[W_{g(f(e),e,\overline{y}} = \omega\right] \end{aligned}$$

where the first line is $U^A(e)$ and the second line comes from the implication

$$\hat{U}^A(e) \implies \forall^\infty\forall\ldots\forall^\infty\forall\forall^\infty \left[W_{g(f(e),e,\overline{y}} = \omega\right]$$

This is a conjunction of two propositions with $2n-1$ initial quantifiers which line-up as either $\{\forall^\infty,\forall^\infty\}$ or $\{\forall,\exists^\infty\}$. By Lemma 1.1(gi) ,

$$\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\forall^\infty \left[\hat{V}^A(e,\overline{y})\ \ \&\ \ \left[W_{g(f(e),e,\overline{y})} = \omega\right]\right]. \tag{21}$$

Define $\lambda x.h_{e,\overline{y}}(x)$ as follows:

$$h_{e,\overline{y}}(x) := (\mu s)\left[x \in W_{g(f(e),e,\overline{y})}\right].$$

Case (2) of the Theorem now follows from condition (21) and Sublemma 3.

$\square$

6 A Characterization of the low_n Degrees

In this section we will establish the converse of Theorem 5.2. This section highlights the basic argument used for low_1 (Theorem 2.5) and for low_3 (Theorem 3.9). The general case requires only the additional care of handling the quantifiers.

6.1 Basic Strategy

Let A be a set for which the property n-UEP holds. By Post's Theorem it is sufficient to show $\Sigma_n^A \subset \Delta_{n+1}$ or that $\Pi_n^A \subset \Delta_{n+1}$, to conclude $A^{(n)} \leq_{\mathrm{T}} 0^{(n)}$, and thus that A is low_n.

For odd n, we will show that $\Sigma_n^A \subset \Delta_{n+1}$ by showing:

$$\Sigma_n^A \subset \Sigma_{n+1} \text{ and}$$
$$\Sigma_n^A \subset \Pi_{n+1}$$

using the potential escape functions from n-UEP to produce computable k and ℓ such that

$$x \in A^{(n)} \iff \forall^\infty \exists^\infty \ldots \left[\varphi_{k(x,\overline{y})} \text{ is total }\right] \text{ and}$$
$$x \in A^{(n)} \iff \exists^\infty \forall^\infty \ldots \left[\operatorname{dom} \varphi_{\ell(x,\overline{y})} \text{ is finite }\right]$$

then use the fact that $A^{(n)}$ is Σ_n-complete.

For even n, we will show that $\Pi_n^A \subset \Delta_{n+1}$ by showing:

$$\Pi_n^A \subset \Pi_{n+1} \text{ and}$$
$$\Pi_n^A \subset \Sigma_{n+1}$$

using the potential escape functions from n-UEP to produce computable k and ℓ such that

$$x \in \overline{A^{(n)}} \iff \exists^\infty \forall^\infty \ldots \left[\varphi_{k(x,\overline{y})} \text{ is total }\right] \text{ and}$$
$$x \in \overline{A^{(n)}} \iff \forall^\infty \exists^\infty \ldots \left[\operatorname{dom} \varphi_{\ell(x,\overline{y})} \text{ is finite }\right]$$

and using the fact that $\overline{A^{(n)}}$ is Π_n-complete.

The basic strategy is exactly like the low$_1$ case from Theorem 2.2. Let f be the computable indexing function from the relativization of Corollary 4.3 to A, for the set $A^{(n)}$ (or $\overline{A^{(n)}}$.) We will a construct function $F_{x,\overline{y}}$ based on the condition

$$\overline{y} \in W^A_{f(x)},$$

and may assume by n-UEP we are given potential computable escape functions $h_{x,\overline{y}}$. The constuction of $F_{x,\overline{y}}$ is in stages s, by two mutually exclusive strategies:

- $\mathcal{E}(\overline{y}; s)$: Choose the least n not in the domain of $F_{x,\overline{y}}$ at stage s and let $F_{x,\overline{y}}(n) = 0$. This will be called the *extend strategy.*
- $\mathcal{A}(\overline{y}; s)$: Choose the least n in the domain of $h_{x,\overline{y}}$ by stage s, but not yet in the domain of $F_{x,\overline{y}}$ at stage s, and let

  $$F_{x,\overline{y}}(n) = 1 + h_{x,\overline{y}}.$$

  If there is no such n, then do nothing. This will be called the *attack strategy.*

The following Lemma captures the key properties of these strategies.

Lemma 6.1. *Fix $x, \overline{y} \in \omega$. Suppose that for each s, exactly one of the strategies $\mathcal{E}(\overline{y}; s)$ or $\mathcal{A}(\overline{y}; s)$ is played by $F_{x,\overline{y}}$. Then*

(a) If $(\exists^\infty s)\ \mathcal{E}(\overline{y}; s)$ then $F_{x,\overline{y}}$ is total.

(b) If $(\forall^\infty s)\ \mathcal{A}(\overline{y}; s)$ and $h_{x,\overline{y}}$ is total then $F_{x,\overline{y}}$ is total and dominates $h_{x,\overline{y}}$

$$(\forall^\infty n)\ \left[F_{x,\overline{y}}(n) > h_{x,\overline{y}}(n)\right]$$

Proof of Lemma 6.1. (a): The strategy $\mathcal{E}(\overline{y}; s)$ extends $F_{x,\overline{y}}$ to the least element not yet in the domain of F_x. If $(\exists^\infty s)\ \mathcal{E}(\overline{y}; s)$, then $F_{x,\overline{y}}$ is total.

(b): The strategy $\mathcal{A}(\overline{y}; s)$ only extends the domain of $F_{x,\overline{y}}$ to a new element n, if n was defined on $h_{x,\overline{y}}$, and then ensures that

$$F_{x,\overline{y}}(n) > h_{x,\overline{y}}(n).$$

If $(\forall^\infty s)\ \mathcal{A}(\overline{y}; s)$ then there are only finitely many stages s on which $\mathcal{E}(\overline{y}; s)$ was played. Since this strategy only adds one element on each stage, then $F_{x,\overline{y}}$ is defined on only finitely many elements through the extend strategy. Let t be a stage larger than any stage s on which $\mathcal{E}(\overline{y}; s)$. Note that for any $n \geq t$, if $F_{x,\overline{y}}(n)$ is defined, it must occur on a stage s in which $\mathcal{A}(\overline{y}; s)$. So, for any $n \geq t$, if $h_{x,\overline{y}}(n)$ is defined at stage s, then $F_{x,\overline{y}}(n)$ will eventually be extended to n at some attack stage, $\mathcal{A}(\overline{y}; s)$, so that

$$F_{x,\overline{y}}(n) > h_{x,\overline{y}}(n).$$

Since this is true for all $n \geq t$, then if h_x is total, $F_{x,\overline{y}}$ will be total and dominate $h_{x,\overline{y}}$. □

6.2 A Characterization of the low_n Degrees

This subsection will establish

Theorem 6.2 (Characterization of low_n Degrees)**.** *For every $n \geq 1$, the following are equivalent*

(A) A is low_n

(B) A has (n-UEP).

The proof for low_1 is Theorem 2.2. The proof is slightly different in the case of low_{2n} and low_{2n+1}. I will prove each case in turn.

Proof of Theorem 6.2 for $2n+1$. The implication of $(A) \implies (B)$ is given by Theorem 5.2, case (1).

Let $n \geq 1$. Fix a Σ^A_{2n+1} set which is Σ^A_{2n+1}-complete C, such as $A^{(2n+1)}$. We will show that there is a computable k such that for any $x \in \omega$

$$x \in C \iff (\exists^\infty y_{2n-1})(\forall^\infty y_{2n-2}) \dots (\exists^\infty y_2)(\forall^\infty y_1) \left[\varphi_{k(x,y_1,\dots,y_{2n-1})} \text{ is total}\right] \qquad (\Pi_{2n+2})$$

and a computable ℓ such that for any $x \in \omega$

$$x \in C \iff (\forall^\infty y_{2n-1})(\exists^\infty y_{2n-2}) \dots (\forall^\infty y_2)(\exists^\infty y_1) \left[\operatorname{dom} \varphi_{\ell(x,y_1,\dots,y_{2n-1})} \text{ is finite }\right] \qquad (\Sigma_{2n+2})$$

It follows that $\Sigma^A_{2n+1} \subset \Delta_{2n+2}$, so by Post's Theorem (1.3), $A^{(2n+1)} \leq_T 0^{(2n+1)}$, and A is low_{2n+1}.

Let $\overline{y} = \langle y_1, \dots, y_{2n-1}\rangle$. We will suppress the variables associated with each quantifier, and write (for instance)

$$\exists^\infty\forall^\infty \dots \exists^\infty\forall^\infty \text{ for } (\exists^\infty y_{2n-1})(\forall^\infty y_{2n-2}) \dots (\exists^\infty y_2)(\forall^\infty y_1).$$

The hypothesis is that A has $2n+1$-UEP, which is the property

- Then there is a u.e. array of families of partial computable functions $\left\{h_{e,y_1,\dots,y_{2n-2}}\right\}_{e\in\omega}$ such that the following holds: For every e and u.e. family of functions $\left\{\Phi^A_{e,y_1,\dots,y_{2n-2}}\right\}$ if

$$(\exists^\infty y_{2n-1})(\forall^\infty y_{2n-2}) \dots (\exists^\infty y_2)(\forall^\infty y_1) \left[\Phi^A_{e,y_1,\dots,y_{2n-2}} \text{ total }\right]$$

  then $(\exists^\infty y_{2n-1})(\forall^\infty y_{2n-2}) \dots (\exists^\infty y_2)(\forall^\infty y_1)$ such that

  1. $h_{e,y_1,\dots,y_{2n-1}}$ is total and
  2. $h_{e,y_1,\dots,y_{2n-1}}$ escapes $\Phi^A_{e,y_1,\dots,y_{2n-1}}$

By the relativization of Corollary 4.3 to A, there is a computable f (I am suppressing the index for C) such that

$$\begin{aligned} x \in C &\implies \forall^\infty\forall \ldots \forall^\infty\forall \left[\overline{y} \in W^A_{f(x)}\right] \\ x \notin C &\implies \forall\forall^\infty \ldots \forall\forall^\infty \left[\overline{y} \notin W^A_{f(x)}\right] \end{aligned}$$

Case Π_{2n+2}:

The construction describes for each $x \in \omega$ a u.e. family of A-partial computable functions $F_{x,\overline{y}}$, which by the Relativized Kleene Fixed Point Theorem [Soa87, Theorem III.1.6], we may assume ahead has index $p(x)$. Let $\left\{h_{p(x),x,\overline{y}}\right\}$ be the u.e. family of partial computable functions given by $2n+1$-UEP. So, if

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[F_{x,\overline{y}} \text{ total }\right]$$

then

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[h_{p(x),x,\overline{y}} \text{ is total and escapes } F_{x,\overline{y}}\right].$$

Let k be the computable function which gives the index of $h_{p(x),x,\overline{y}}$, so that $\varphi_{k(x,\overline{y})} = h_{p(x),x,\overline{y}}$.

Define $F_{x,\overline{y}}$ in stages s by

- If $\overline{y} \in W^A_{f(x),s}$ then $F_{x,\overline{y}}$ is extended according to the extend strategy, $\mathcal{E}(\overline{y};s)$.
- If $\overline{y} \notin W^A_{f(x),s}$ then $F_{x,\overline{y}}$ is extended according to the attack strategy, $\mathcal{A}(\overline{y};s)$, against $\varphi_{k(x,\overline{y})}$.

For each stage s, the function $F_{x,\overline{y}}$ is extended according to exactly one of the strategies $\mathcal{E}(\overline{y};s)$ or $\mathcal{A}(\overline{y};s)$, and that

- If $\overline{y} \in W^A_{f(x)}$ then $(\forall^\infty s)\ \mathcal{E}(\overline{y};s)$.
- If $\overline{y} \notin W^A_{f(x)}$ then $(\forall s)\ \mathcal{E}(\overline{y};s)$.

so that Lemma 6.1 applies here.

Suppose that $x \in C$. Then

$$\forall^\infty\forall \ldots \forall^\infty\forall \left[\overline{y} \in W^A_{f(x)}\right],$$

so that

$$\forall^\infty\forall \ldots \forall^\infty\forall \left[(\forall^\infty s)\left[\mathcal{E}(\overline{y};s)\right]\right].$$

By Lemma 6.1a

$$\forall^\infty\forall \ldots \forall^\infty\forall \left[F_{x,\overline{y}} \text{ is total }\right].$$

Since $\forall^\infty \Rightarrow \exists^\infty$ and $\forall \Rightarrow \forall^\infty$, this implies that

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[F_{x,\overline{y}} \text{ total } \right].$$

Thus, by $2n+1$-UEP

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[\varphi_{k(x,\overline{y})} \text{ is total } \right].$$

Suppose that $x \notin C$. Then

$$\forall\forall^\infty \ldots \forall\forall^\infty \left[\overline{y} \notin W^A_{f(x)}\right],$$

and so

$$\forall\forall^\infty \ldots \forall\forall^\infty \left[(\forall s)\ \left[\mathcal{A}(\overline{y};s)\right]\right]. \tag{22}$$

Suppose that

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[\varphi_{k(x,\overline{y})} \text{ is total } \right] \tag{23}$$

then by Lemma (6.1b) and conditions (22) and (23)

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[F_{x,\overline{y}} \text{ is total } \right]$$

By $2n+1$-UEP

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[\varphi_{k(x,\overline{y})} \text{ is total and escapes } F_{x,\overline{y}}\right]. \tag{24}$$

Thus, the conjunction of conditions (22) and (24) hold:

$$\forall\forall^\infty \ldots \forall\forall^\infty \left[(\forall s)\ \left[\mathcal{A}(\overline{y};s)\right]\right] \text{ and}$$
$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[\varphi_{k(x,\overline{y})} \text{ is total and escapes } F_{x,\overline{y}}\right].$$

This is a conjunction of two conditions with $2n-1$ alternating quantifiers which line-up as $\{\forall, \exists^\infty\}$ and $\{\forall^\infty, \forall^\infty\}$. By Lemma 1.1(gi)

$$\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[(\forall s)\ \left[\mathcal{A}(\overline{y};s)\right]\ \&\ \left[\varphi_{k(x,\overline{y})} \text{ is total and escapes } F_{x,\overline{y}}\right]\right].$$

But by Lemma (6.1b) it is impossible to have the following three things occur for any $\overline{y}$

$$(\forall s)\ \left[\mathcal{A}(\overline{y};s)\right],$$
$$\varphi_{k(x,\overline{y})} \text{ is total and}$$
$$\varphi_{k(x,\overline{y})} \text{ escapes } F_{x,\overline{y}}.$$

Thus,

$$\neg\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty \left[\varphi_{k(x,\overline{y})} \text{ is total } \right].$$

Case Σ_{2n+2}:

The construction describes for each $x \in \omega$ a u.e. family of A partial computable functions $G_{x,\overline{y}}$, which by the Relativized Kleene Fixed Point Theorem [Soa87, Theorem III.1.6], we may assume ahead has index $p(x)$. Let $\{h_{p(x),x,\overline{y}}\}$ be the u.e. family of partial computable functions given by $2n+1$-UEP. So, if

$$\exists^\infty \forall^\infty \ldots \exists^\infty \forall^\infty \left[G_{x,\overline{y}} \text{ is total }\right]$$

then

$$\exists^\infty \forall^\infty \ldots \exists^\infty \forall^\infty \left[h_{p(x),x,\overline{y}} \text{ is total and escapes } G_{x,\overline{y}}\right]$$

Define

$$h'_{x,\overline{y}}(n) = \begin{cases} h_{p(x),x,\overline{y}}(n) & \text{if } (\forall m \leq n) \left[h_{p(x),x,\overline{y}}(m)\downarrow\right] \\ \uparrow & \text{otherwise} \end{cases}$$

Let ℓ be the computable function which gives the index of $h'_{x,\overline{y}}$. So,

$$\begin{aligned} h_{p(x),x,\overline{y}} \text{ is total} &\iff \varphi_{\ell(x,\overline{y})} \text{ is total} \\ h_{p(x),x,\overline{y}} \text{ not total} &\iff \operatorname{dom} \varphi_{\ell(x,\overline{y})} \text{ is finite} \end{aligned}$$

Define $G_{x,\overline{y}}$ in stages s by

- If $\overline{y} \in W^A_{f(x),s}$ then $G_{x,\overline{y}}$ extends according to the attack strategy, $\mathcal{A}(\overline{y}; s)$, against $\varphi_{\ell(x,\overline{y})}$.
- If $\overline{y} \notin W^A_{f(x),s}$ then $G_{x,\overline{y}}$ extends according to the extend strategy, $\mathcal{E}(\overline{y}; s)$.

For each s, $G_{x,\overline{y}}$ is extended according to exactly one of the strategies $\mathcal{E}(\overline{y}; s)$ or $\mathcal{A}(\overline{y}; s)$, and that

- If $\overline{y} \in W^A_{f(x)}$ then $(\forall^\infty s)\ \mathcal{A}(\overline{y}; s)$.
- If $\overline{y} \notin W^A_{f(x)}$ then $(\forall s)\ \mathcal{E}(\overline{y}; s)$.

so that Lemma 6.1 applies here.

Suppose that $x \notin C$. Then

$$\forall \forall^\infty \ldots \forall \forall^\infty \left[\overline{y} \notin W^A_{f(x)}\right]$$

so that

$$\forall \forall^\infty \ldots \forall \forall^\infty \left[(\forall s)\left[\mathcal{E}(\overline{y}; s)\right]\right].$$

By Lemma 6.1a

$$\forall^\infty \forall \ldots \forall^\infty \forall \left[G_{x,\overline{y}} \text{ is total }\right].$$

Since $\forall^\infty \Rightarrow \exists^\infty$ and $\forall \Rightarrow \forall^\infty$, this implies that

$$\exists^\infty \forall^\infty \ldots \exists^\infty \forall^\infty \left[G_{x,\overline{y}} \text{ total }\right].$$

Thus, by $2n+1$-UEP

$$\exists^\infty \forall^\infty \ldots \exists^\infty \forall^\infty \left[\varphi_{\ell(x,\overline{y})} \text{ is total }\right],$$

and so

$$\neg\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\left[\varphi_{\ell(x,\overline{y})}\text{ is finite }\right].$$

Suppose that $x \in C$. Then

$$\forall^\infty\forall\ldots\forall^\infty\forall\left[\overline{y}\in W^A_{f(x)}\right],$$

and so

$$\forall^\infty\forall\ldots\forall^\infty\forall\left[(\forall^\infty s)\ \left[\mathcal{A}(\overline{y};s)\right]\right]. \tag{25}$$

Suppose that

$$\exists^\infty\forall^\infty\ldots\exists^\infty\forall^\infty\left[\varphi_{\ell(x,\overline{y})}\text{ is total }\right] \tag{26}$$

then by Lemma (6.1b) and conditions (25) and (26)

$$\exists^\infty\forall^\infty\ldots\exists^\infty\forall^\infty\left[G_{x,\overline{y}}\text{ is total }\right].$$

By $2n+1$-UEP

$$\exists^\infty\forall^\infty\ldots\exists^\infty\forall^\infty\left[\varphi_{\ell(x,\overline{y})}\text{ is total and escapes } G_{x,\overline{y}}\right]. \tag{27}$$

Thus, the conjunction of conditions (25) and (27) holds:

$$\forall\forall^\infty\ldots\forall\forall^\infty\left[(\forall^\infty s)\ \left[\mathcal{A}(\overline{y};s)\right]\right]\text{ and}$$
$$\exists^\infty\forall^\infty\ldots\exists^\infty\forall^\infty\left[\varphi_{\ell(x,\overline{y})}\text{ is total and escapes } G_{x,\overline{y}}\right].$$

This is the conjunction of two conditions with $2n-1$ alternating quantifiers which line-up as $\{\forall,\exists^\infty\}$ and $\{\forall^\infty,\forall^\infty\}$. By Lemma 1.1(gi)

$$\exists^\infty\forall^\infty\ldots\exists^\infty\forall^\infty\left[(\forall^\infty s)\ \left[\mathcal{A}(\overline{y};s)\right]\ \ \&\ \ \left[\varphi_{\ell(x,\overline{y})}\text{ is total and escapes } G_{x,\overline{y}}\right]\right].$$

But by Lemma (6.1b) it is impossible to have the following three things occur for any $\overline{y}$

$$(\forall^\infty s)\ \left[\mathcal{A}(\overline{y};s)\right],$$
$$\varphi_{\ell(x,\overline{y})}\text{ is total and}$$
$$\varphi_{\ell(x,\overline{y})}\text{ escapes } G_{x,\overline{y}}.$$

Thus, we must have

$$\neg\exists^\infty\forall^\infty\ldots\exists^\infty\forall^\infty\left[\varphi_{\ell(x,\overline{y})}\text{ is total }\right],$$

or equivalently,

$$\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\left[\varphi_{\ell(x,\overline{y})}\text{ is finite }\right].$$

□

The argument for low_{2n} is near identical to low_{2n+1}:

Proof of Theorem 6.2 for $2n$. The implication of $(A) \implies (B)$ is given by Theorem 5.2, case (2).

Let $n \geq 1$. Fix a Π^A_{2n} set which is Π^A_{2n}-complete C, such as $\overline{A^{(2n+1)}}$. We will show that there is a computable k such that for any $x \in \omega$

$$x \in C \iff (\forall^\infty y_{2n-2})(\exists^\infty y_{2n-3}) \dots (\forall^\infty y_3)(\exists^\infty y_2)(\forall^\infty y_1) \left[\varphi_{k(x,y_1,\dots,y_{2n-1})} \text{ is total}\right] \qquad (\Sigma_{2n+1})$$

and a computable ℓ such that for any $x \in \omega$

$$x \in C \iff (\exists^\infty y_{2n-2})(\forall^\infty y_{2n-3}) \dots (\exists^\infty y_3)(\forall^\infty y_2)(\exists^\infty y_1) \left[|\varphi_{\ell(x,y_1,\dots,y_{2n-1})} \text{ is finite }\right]. \qquad (\Pi_{2n+1})$$

It follows that $\Pi^A_{2n} \subset \Delta_{2n+2}$, so by Post's Theorem (1.3), $A^{(2n)} \leq_T 0^{(2n)}$, and A is low_{2n}.

Let $\overline{y} = \langle y_1, \dots, y_{2n-1}\rangle$. In what follows we will suppress the variables associated with each quantifier, and write (for instance)

$$\forall^\infty\exists^\infty \dots \forall^\infty\exists^\infty\forall^\infty \text{ for } (\forall^\infty y_{2n-1})(\exists^\infty y_{2n-2}) \dots (\forall^\infty y_3)(\exists^\infty y_2)(\forall^\infty y_1).$$

Our hypothesis is that A has $2n$-UEP, which is the property

- Then there is a u.e. array of families of partial computable functions $\left\{h_{e,y_1,\dots,y_{2n-2}}\right\}_{e\in\omega}$ such that the following holds: For every e and u.e. family of functions $\left\{\Phi^A_{e,y_1,\dots,y_{2n-2}}\right\}$ if

$$(\forall^\infty y_{2n-2})(\exists^\infty y_{2n-3}) \dots (\forall^\infty y_3)(\exists^\infty y_2)(\forall^\infty y_1) \left[\Phi^A_{e,y_1,\dots,y_{2n-2}} \text{ total }\right]$$

  then $(\forall^\infty y_{2n-2})(\exists^\infty y_{2n-3}) \dots (\forall^\infty y_3)(\exists^\infty y_2)(\forall^\infty y_1)$ such that

  1. $h_{e,y_1,\dots,y_{2n-2}}$ is total and
  2. $h_{e,y_1,\dots,y_{2n-2}}$ escapes $\Phi^A_{e,y_1,\dots,y_{2n-2}}$.

By the relativization of Corollary 4.3 there is a computable f (I am suppressing the index for C) such that

$$\begin{aligned} x \in C &\implies \forall\forall^\infty \dots \forall\forall^\infty\forall \left[\overline{y} \in W^A_{f(x)}\right] \\ x \notin C &\implies \forall^\infty\forall \dots \forall^\infty\forall\forall^\infty \left[\overline{y} \notin W^A_{f(x)}\right] \end{aligned}$$

Case Σ_{2n+1}:

The construction describes for each $x \in \omega$ a u.e. family of A-partial computable functions $F_{x,\overline{y}}$, which by the Relativized Kleene Fixed Point Theorem [Soa87, Theorem III.1.6], we may assume ahead has index $p(x)$. Let $\left\{h_{p(x),x,\overline{y}}\right\}$ be the u.e. family of partial computable functions given by $2n$-UEP. So, if

$$\forall^\infty\exists^\infty \dots \forall^\infty\exists^\infty\forall^\infty \left[F_{x,\overline{y}} \text{ is total }\right]$$

then

$$\forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[h_{p(x),x,\overline{y}} \text{ is total and escapes } F_{x,\overline{y}}\right]$$

Let k be the computable function which gives the index of $h_{p(x),x,\overline{y}}$, so that $\varphi_{k(x,\overline{y})} = h_{p(x),x,\overline{y}}$.

Define $F_{x,\overline{y}}$ in stages s by

- If $\overline{y} \in W^A_{f(x),s}$ then $F_{x,\overline{y}}$ is extended according to the extend strategy, $\mathcal{E}(\overline{y}; s)$.
- If $\overline{y} \notin W^A_{f(x),s}$ then $F_{x,\overline{y}}$ is extended according to the attack strategy, $\mathcal{A}(\overline{y}; s)$, against $\varphi_{k(x,\overline{y})}$.

For each stage s, the function $F_{x,\overline{y}}$is extended according to exactly one of the strategies $\mathcal{E}(\overline{y}; s)$ or $\mathcal{A}(\overline{y}; s)$, and that

- If $\overline{y} \in W^A_{f(x)}$ then $(\forall^\infty s)\ \mathcal{E}(\overline{y}; s)$.
- If $\overline{y} \notin W^A_{f(x)}$ then $(\forall s)\ \mathcal{E}(\overline{y}; s)$.

so that Lemma 6.1 applies here.

Suppose that $x \in C$. Then

$$\forall\forall^\infty \ldots \forall\forall^\infty\forall \left[\overline{y} \in W^A_{f(x)}\right],$$

so that

$$\forall\forall^\infty \ldots \forall\forall^\infty\forall \left[(\forall^\infty s)\left[\mathcal{E}(\overline{y}; s)\right]\right].$$

By Lemma 6.1a

$$\forall\forall^\infty \ldots \forall\forall^\infty\forall \left[F_{x,\overline{y}} \text{ is total }\right]$$

Since $\forall^\infty \Rightarrow \exists^\infty$ and $\forall \Rightarrow \forall^\infty$, this implies that

$$\forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[F_{x,\overline{y}}\text{is total }\right]$$

Thus, by $2n$-UEP

$$\forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[\varphi_{k(x,\overline{y})} \text{ is total }\right].$$

Suppose that $x \notin C$. Then

$$\forall^\infty\forall \ldots \forall^\infty\forall\forall^\infty \left[\overline{y} \notin W^A_{f(x)}\right],$$

and so

$$\forall^\infty\forall \ldots \forall\forall^\infty\forall\forall^\infty \left[(\forall s)\ \left[\mathcal{A}(\overline{y}; s)\right]\right]. \tag{28}$$

Suppose that

$$\forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[\varphi_{k(x,\overline{y})} \text{ is total }\right] \tag{29}$$

then by Lemma (6.1b) and conditions (28) and (29)

$$\forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[F_{x,\overline{y}} \text{ is total } \right]$$

By $2n$-UEP

$$\forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[\varphi_{k(x,\overline{y})} \text{ is total and escapes } F_{x,\overline{y}} \right]. \tag{30}$$

Thus, the conjunction of conditions (28) and (30) hold:

$$\forall^\infty \forall \ldots \forall^\infty \forall \forall^\infty \left[(\forall s) \ \left[\mathcal{A}(\overline{y}; s) \right] \right] \text{ and}$$
$$\forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[\varphi_{k(x,\overline{y})} \text{ is total and escapes } F_{x,\overline{y}} \right].$$

This is the conjunction of two propositions with $2n-2$ alternating quantifiers which line-up as $\{\forall, \exists^\infty\}$ and $\{\forall^\infty, \forall^\infty\}$. By Lemma 1.1(gi) we have

$$\forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[(\forall s) \ \left[\mathcal{A}(\overline{y}; s) \right] \ \& \ \left[\varphi_{k(x,\overline{y})} \text{ is total and escapes } F_{x,\overline{y}} \right] \right].$$

But by Lemma (6.1b) it is impossible to have the following three things occur for any $\overline{y}$

$$\begin{aligned} &(\forall s) \ \left[\mathcal{A}(\overline{y}; s) \right], \\ &\varphi_{k(x,\overline{y})} \text{ is total and} \\ &\varphi_{k(x,\overline{y})} \text{ escapes } F_{x,\overline{y}}. \end{aligned}$$

Thus,

$$\neg \forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[\varphi_{k(x,\overline{y})} \text{ is total } \right]$$

Case Π_{2n+1}:

The construction describes for each $x \in \omega$ a u.e. family of A partial computable functions $G_{x,\overline{y}}$, which by the Relativized Kleene Fixed Point Theorem [Soa87, Theorem III.1.6], we may assume ahead has index $p(x)$. Let $\left\{ h_{p(x),x,\overline{y}} \right\}$ be the u.e. family of partial computable functions given by $2n$-UEP. So, if

$$\forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[G_{x,\overline{y}} \text{ is total } \right]$$

then

$$\forall^\infty \exists^\infty \ldots \forall^\infty \exists^\infty \forall^\infty \left[h_{p(x),x,\overline{y}} \text{ is total and escapes } G_{x,\overline{y}} \right].$$

Define

$$h'_{x,\overline{y}}(n) = \begin{cases} h_{p(x),x,\overline{y}}(n) & \text{if } (\forall m \le n) \ \left[h_{p(x),x,\overline{y}}(m)\downarrow \ \right] \\ \uparrow & \text{otherwise} \end{cases}$$

Let ℓ be the computable function which gives the index of $h'_{x,\overline{y}}$. So,

$$\begin{aligned} h_{p(x),x,\overline{y}} \text{ is total} &\iff \varphi_{\ell(x,\overline{y})} \text{ is total} \\ h_{p(x),x,\overline{y}} \text{ not total} &\iff \operatorname{dom} \varphi_{\ell(x,\overline{y})} \text{ is finite} \end{aligned}$$

Define $G_{x,\overline{y}}$ in stages s by

- If $\overline{y} \in W^A_{f(x),s}$ then $G_{x,\overline{y}}$ extends according to the attack strategy, $\mathcal{A}(\overline{y}; s)$, against $\varphi_{\ell(x,\overline{y})}$.
- If $\overline{y} \notin W^A_{f(x),s}$ then $G_{x,\overline{y}}$ extends according to the extend strategy, $\mathcal{E}(\overline{y}; s)$.

For each s, $G_{x,\overline{y}}$ is extended according to exactly one of the strategies $\mathcal{E}(\overline{y}; s)$ or $\mathcal{A}(\overline{y}; s)$, and that

- If $\overline{y} \in W^A_{f(x)}$ then $(\forall^\infty s)\ \mathcal{A}(\overline{y}; s)$.
- If $\overline{y} \notin W^A_{f(x)}$ then $(\forall s)\ \mathcal{E}(\overline{y}; s)$.

so that Lemma 6.1 applies here.

Suppose that $x \notin C$. Then

$$\forall^\infty\forall \ldots \forall^\infty\forall\forall^\infty \left[\overline{y} \notin W^A_{f(x)}\right]$$

so that

$$\forall^\infty\forall \ldots \forall^\infty\forall\forall^\infty \left[(\forall s)\left[\mathcal{E}(\overline{y}; s)\right]\right].$$

By Lemma 6.1a

$$\forall^\infty\forall \ldots \forall^\infty\forall\forall^\infty \left[G_{x,\overline{y}} \text{ is total }\right].$$

Since $\forall \Rightarrow \exists^\infty$, this implies that

$$\forall^\infty\exists^\infty \ldots \forall^\infty\exists^\infty\forall^\infty \left[G_{x,\overline{y}} \text{ is total }\right].$$

Thus, by $2n$-UEP

$$\forall^\infty\exists^\infty \ldots \forall^\infty\exists^\infty\forall^\infty \left[h_{x,\overline{y}} \text{ total }\right]$$

so that

$$\forall^\infty\exists^\infty \ldots \forall^\infty\exists^\infty\forall^\infty \left[\varphi_{\ell(x,\overline{y})} \text{ is total }\right],$$

and thus

$$\neg\exists^\infty\forall^\infty \ldots \exists^\infty\forall^\infty\exists^\infty \left[\varphi_{\ell(x,\overline{y})} \text{ is finite }\right].$$

Suppose that $x \in C$. Then

$$\forall\forall^\infty \ldots \forall\forall^\infty\forall \left[\overline{y} \in W^A_{f(x)}\right]$$

and so

$$\forall\forall^\infty \ldots \forall\forall^\infty\forall \left[(\forall^\infty s)\ \left[\mathcal{A}(\overline{y}; s)\right]\right]. \tag{31}$$

Suppose that

$$\forall^\infty\exists^\infty \ldots \forall^\infty\exists^\infty\forall^\infty \left[\varphi_{\ell(x,\overline{y})} \text{ is total }\right] \tag{32}$$

then by Lemma (6.1b) and conditions (31) and (32)

$$\forall^\infty\exists^\infty \ldots \forall^\infty\exists^\infty\forall^\infty \left[G_{x,\overline{y}} \text{ is total }\right].$$

By $2n$-UEP

$$\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\forall^\infty\left[\varphi_{\ell(x,\overline{y})}\text{ is total and escapes }G_{x,\overline{y}}\right]. \tag{33}$$

Thus, he conjunction of conditionss (31) and (33) holds:

$$\forall\forall^\infty\ldots\forall\forall^\infty\forall\left[(\forall^\infty s)\ \left[\mathcal{A}(\overline{y};s)\right]\right]$$
$$\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\forall^\infty\left[\varphi_{\ell(x,\overline{y})}\text{ is total and escapes }G_{x,\overline{y}}\right].$$

This is the conjunction of two propositions with $2n-1$ alternating quantifiers which line-up as $\left\{\forall,\forall^\infty\right\}$ and $\left\{\forall^\infty,\exists^\infty\right\}$. By Lemma 1.1(fh)

$$\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\forall^\infty\left[(\forall^\infty s)\ \left[\mathcal{A}(\overline{y};s)\right]\ \&\ \left[\varphi_{\ell(x,\overline{y})}\text{ is total and escapes }G_{x,\overline{y}}\right]\right].$$

But by Lemma (6.1b) it is impossible to have the following three things occur for any $\overline{y}$

$$(\forall^\infty s)\ \left[\mathcal{A}(\overline{y};s)\right],$$
$$\varphi_{\ell(x,\overline{y})}\text{ is total and}$$
$$\varphi_{\ell(x,\overline{y})}\text{ escapes }G_{x,\overline{y}}.$$

Thus, we must have

$$\neg\forall^\infty\exists^\infty\ldots\forall^\infty\exists^\infty\forall^\infty\left[\varphi_{\ell(x,\overline{y})}\text{ is total }\right],$$

or equivalently,

$$\exists^\infty\forall^\infty\ldots\exists^\infty\forall^\infty\exists^\infty\left[\varphi_{\ell(x,\overline{y})}\text{ is finite }\right].$$

□

References

[CHKS04] B. Csima, D. Hirschfeldt, J.F. Knight, and R.I. Soare. Bounding prime models. *J. Symbolic Logic*, 69(4):1117–42, 2004.

[Rog87] Hartley Rogers. *Theory of Recursive Functions and Effective Computability.* The MIT Press, 1987.

[Soa87] R.I. Soare. *Recursively Enumerable Sets and Degrees.* Perspectives in Mathematics. Springer-Verlag, 1987.

[Soa05] R.I. Soare. Computability theory and applications. Draft at url: http://www.people.cs.uchicago.edu/~soare/newbook/draft.pdf, 2005.