March 4, 2010

ISOMORPHISM TYPES OF BOOLEAN ALGEBRAS IN THE BACK-AND-FORTH HIERARCHY

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Abstract. We investigate the back-and-forth types which isomorphism types. The main result is that the indecomposable isomorphism types which arise at the finite levels of the back-and-forth hierarchy are the same the finitary isomorphism types in Heindorff.

Contents

1. Introduction

In this note we investigate the *isomorphism types* which arise in the back-and-forth hierarchy at the finite levels. A back-and-forth indecomposable type $\alpha \in \mathbf{BF}_n$ is an *isomorphism type* if any two Boolean algebras with bf-type α are isomorphic. We show that any bf-type $\alpha \in \mathbf{BF}_n$ which has only one *descendant* at level $n+1$ (that is, there is only one $\beta \in \mathbf{BF}_{n+1}$ for which $(\beta)_n = \alpha$) is an isomorphism type (Theorem 3.6.) Furthermore, any level n bf-type which is $\leq_n$-maximal is an isomorphism type (Theorem 2.4); in addition, each bf-type at level n has a $\leq_{n+1}$-maximal descendant. We also exhibit a bf-type which is not $\leq_7$ maximal, but which is an isomorphism type.

The main result in this paper is that the isomorphism types which arise at finite levels of the back-and-forth hierarchy are exactly the finitary isomorphism types investigated by Hanf and Pierce in the early seventies. (See Section 4 for more details, especially Theorem 4.10.)

[HM] is a prerequisite for this paper.

2000 *Mathematics Subject Classification.* 03D80.

Key words and phrases. Boolean algebra, back-and-forth, low, approximation.

2. Maximal and minimal bf-types

The purpose of this section is to provide a deeper structural description of the back-and-forth relation on indecomposable bf-types. We will show that for any n-indecomposable bf-type $\alpha \in \mathbf{BF}_n$ there exists a maximum child $\overline{\alpha}$ and a minimum child $\underline{\alpha}$. This means, in particular, that the following conditions hold.

(1) $\overline{\alpha}, \underline{\alpha} \in \mathbf{BF}_{n+1}$,
(2) $(\overline{\alpha})_n = \alpha = (\underline{\alpha})_n$,
(3) For all $\gamma \in \mathbf{BF}_{n+1}$ such that $(\gamma)_n = \alpha$, $\underline{\alpha} \leq_{n+1} \gamma \leq_{n+1} \overline{\alpha}$.

This result is Theorem 2.4.

2.1. Review of realizable bf-types.

We will write $\mathbf{BF}_n$ for the *realizable n-indecomposable types*. A bf-type $\alpha \in \mathbf{BF}_n$ is *realizable* if there is an n-indecomposable Boolean algebra $\mathcal{A}$ with $T_n(\mathcal{A}) = \alpha$. The following combinatorial characterization of the realizable n-indecomposable types is [HM, Theorem 5.2].

Theorem 2.1. *An antichain $\alpha \subseteq \boldsymbol{BF}_n$ is realizable if and only if every $\gamma \in \alpha$ is realizable and*

$$\forall \delta \in \gamma \ \exists \beta \in \alpha \ (\gamma \leq_n^w \beta \ \& \ (\beta)_{n-1} = \delta).$$

We will also recall the relation $\gamma \lhd_n \delta$:

$$(\delta)_{n-1} \in^{\mathrm{w}} \gamma \ \& \ \gamma \leq_n^{\mathrm{w}} \delta;$$

and isolate some basic properties of $\lhd_n$:

Lemma 2.2. *Let $\alpha, \beta, \gamma \in \boldsymbol{BF}_n$ and $\delta \in \boldsymbol{BF}_{n-1}$.*
(a) If $\alpha \lhd_n \beta$ and $\beta \lhd_n \gamma$ then $\alpha \lhd_n \gamma$.
(b) If $\alpha \lhd_n \beta$ then $(\alpha)_{n-1} \lhd_{n-1} (\beta)_{n-1}$.
(c) If $\alpha \lhd_n \beta$ and $\beta \leq_n \gamma$ then $\alpha \lhd_n \gamma$.
(d) If $\delta \in \alpha$ then $(\alpha)_{n-1} \lhd_{n-1} \delta$.
(e) If $\alpha \lhd_n \beta$ the $\alpha \equiv_n \alpha + \beta$.

Proof. (a). Suppose $\alpha \lhd_n \beta$ and $\beta \lhd_n \gamma$. Since $\leq_n^{\mathrm{w}}$ is transitive, $\alpha \leq_n^{\mathrm{w}} \gamma$. Furthermore, $(\gamma)_{n-1} \in^{\mathrm{w}} \beta$ and $\alpha \leq_n^{\mathrm{w}} \beta$, so that $(\gamma)_{n-1} \in^{\mathrm{w}} \alpha$. Thus, $\alpha \lhd_n \gamma$.

(b). Suppose $\alpha \lhd_n \beta$. Then, $\alpha \leq_n^{\mathrm{w}} \beta$ so that $(\alpha)_{n-1} \leq_{n-1}^{\mathrm{w}} (\beta)_{n-1}$. Furthermore, $(\beta)_{n-1} \in^{\mathrm{w}} \alpha$, so that $(\beta)_{n-1} \leq_{n-1} \rho$ for some $\rho \in \alpha$; it follows that $(\beta)_{n-2} = (\rho)_{n-2} \in (\alpha)_{n-1}$. Therefore, $(\alpha)_{n-1} \lhd_{n-1} (\beta)_{n-1}$.

(c). Suppose $\alpha \lhd_n \beta$ and $\beta \leq_n \gamma$. Since $\alpha \leq_n^{\mathrm{w}} \beta$ and $\beta \leq_n \gamma$ it follows that $\alpha \leq_n^{\mathrm{w}} \gamma$. Furthermore, $(\gamma)_{n-1} = (\beta)_{n-1} \in^{\mathrm{w}} \alpha$. Thus, $\alpha \lhd_n \gamma$.

(d). Suppose $\delta \in \alpha$. Then $(\delta)_{n-2} \in^{\mathrm{w}} (\alpha)_{n-1}$. It remains to show that $(\alpha)_{n-1} \leq_{n-1}^{\mathrm{w}} \delta$. Suppose $\rho \in \delta$. Since $\delta \in \alpha$ there is a $\sigma \in \alpha$ with $(\sigma)_{n-2} = \rho$ (by Theorem 2.1); furthermore, $\rho = (\sigma)_{n-2} \in^{\mathrm{w}} (\alpha)_{n-1}$. Thus, $(\alpha)_{n-1} \leq_{n-1}^{\mathrm{w}} \delta$ and $(\delta)_{n-2} \in^{\mathrm{w}} (\alpha)_{n-1}$, so $(\alpha)_{n-1} \lhd_{n-1} \delta$.

(e). By [HM, Lemma 7.12], $\alpha \leq_n \alpha + \beta$ and so $(\alpha)_{n-1} \equiv_n (\alpha)_{n-1} + (\beta)_{n-1}$ as well. Now use [HM, Lemma 7.18.(3)] to show $\alpha + \beta \leq_n \alpha$. Condition (a) requires $\mathrm{dc}(\alpha \cup \beta) \subseteq \mathrm{dc}(\alpha)$, which is true because $\alpha \leq_n^{\mathrm{w}} \beta$. Condition (b) is simply that $(\alpha)_{n-1} \equiv_n (\alpha)_{n-1} + (\beta)_{n-1}$. □

Recall that a Boolean algebra $\mathcal{A}$ is n-indecomposable if for any partition $1_{\mathcal{A}} = a_0 \dot\vee \ldots \dot\vee a_k$, there is an $i \le k$ with $\mathcal{A} \equiv_n \mathcal{A} \restriction a_i$ [HM, Definition 3.1]. The following Lemma shows that we need only consider partitions of $\mathcal{A}$ into n-indecomposables.

Lemma 2.3. *A Boolean algebra $\mathcal{A}$ is n-indecomposable if and only if for any partition of $\mathcal{A}$ into n-indecomposables, $1_{\mathcal{A}} = a_0 \dot\vee \ldots \dot\vee a_k$, there is an $i \le k$ with $\mathcal{A} \equiv_n \mathcal{A} \restriction a_i$.*

Proof. The "only if" direction follows from the definition of n-indecomposable. For the "if" direction suppose the hypothesis holds, and let $1_{\mathcal{A}} = a_0 \dot\vee \ldots \dot\vee a_k$ be any partition. We will show that $\mathcal{A} \equiv_n \mathcal{A} \restriction a_{i_0}$ for some $i_0 \le k$. We can refine this partition to a partition of n-indecomposables $(b_{i,j})_{i \le k, j \le m_i}$ with $a_i = \sum_{j \le m_i} b_{i,j}$. For some i_0 and j_0, $\mathcal{A} \equiv_n \mathcal{A} \restriction b_{i_0,j_0}$. Thus,

$$T_n(b_{i_0,j_0}) = T_n(\mathcal{A}) = \sum_{i \le k} \sum_{j \le m_i} T_n(b_{i,j}).$$

Since b_{i_0,j_0} is n-indecomposable, it follows by [HM, Lemma 7.12] that $T_n(b_{i_0,j_0}) \triangleleft_n T_n(b_{i,j})$ whenever $i \ne i_0$ or $j \ne j_0$. Applying Lemma 2.2.e, $T_n(b_{i_0,j_0}) \equiv_n \sum_{j \le m_{i_0}} T_n(b_{i,j}) \equiv_n T_n(a_{i_0})$. Thus, $T_n(\mathcal{A}) = T_n(b_{i_0,j_0}) = T_n(a_{i_0})$; so that $\mathcal{A} \equiv_n \mathcal{A} \restriction a_{i_0}$. □

2.2. **Maximum and minimum types.** We will show that for any $\alpha \in \mathbf{BF}_n$ there are (not necessarily distinct) $\overline{\alpha}, \underline{\alpha} \in \mathbf{BF}_{n+1}$ which are maximum and minimum (respectively) among all children (descendants at level $n+1$) of α.

We start by defining $\overline{\alpha}$ for $\alpha \in \mathbf{BF}_n$. Intuitively, for each $\gamma \in \alpha$, we want to include $\gamma_* \in \overline{\alpha}$ where γ_* is as small as possible, consistent with $\left(\gamma_*\right)_{n-1} = \gamma \in \alpha$. This is made explicit in Lemma 2.6. The definition of $\overline{\alpha}$ is

$$\overline{\alpha} = \max \Big\{ \gamma_* \, : \, \gamma \in \alpha \;\text{ and }\; \gamma_* = \max \big\{ \delta \, : \, \delta \in \alpha \;\&\; \gamma \triangleleft_n \delta \big\} \Big\}.$$

We will show shortly that $\left(\overline{\alpha}\right)_n = \alpha$ and $\overline{\alpha}$ is a maximal $n+1$-indecomposable type.

We now turn to defining $\underline{\alpha}$ using $\bar{\cdot}$. Intuitively, for each $\gamma \in^{\mathrm{w}} \alpha$ we want to include $\overline{\gamma}$, provided $\overline{\gamma} \subset \operatorname{dc} \alpha$:

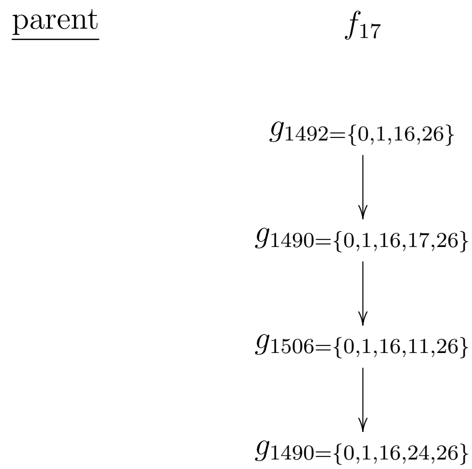
$$\underline{\alpha} = \max \Big\{ \overline{\gamma} \, : \, \alpha \triangleleft_n \overline{\gamma} \Big\}.$$

We will show shortly that $\left(\underline{\alpha}\right)_n = \alpha$ and $\underline{\alpha}$ is a minimal $n+1$-indecomposable type.

We look at an example which illustrates the conditions on $\overline{\alpha}$ and $\underline{\alpha}$. Many of the patterns arise in the first five levels of the indecomposable types, but an example from level six shows that some care is needed in these definitions. Recall, $f_{17} \in \mathbf{BF}_5$ and $f_{17} = \{e_0, e_1, e_4, e_8\}$ where e_0 is an atom, e_1 is a 1-atom and e_8 is an atomless type, and e_4 is not an isomorphism type and whose descendants at level five are $f_6 <_5 f_{16}$.

In the next diagram we omit the character 'f' and for example write $g_{1492=\{0,1,16,26\}}$ instead of $g_{1492} = \{f_0, f_1, f_{16}, f_{26}\}$. The maximal element g_{1492} is an isomorphism type as well as exclusive (and the only exclusive type):

bf-predicates for descendants of f_{17}



Using our new notation the six type $g_{1492} = \overline{f_{17}} = \{f_0, f_1, f_{16}, f_{26}\}$. Here we have

$$f_6 = \underline{e_4} <_5 \left(e_4\right)_* = f_{16}$$

but $f_6 \notin g_{1492}$; this illustrates why we could not simplify the definition of $\overline{\alpha}$ to include $\underline{\gamma}$ where $\gamma \in \alpha$. In this case $e_7 \in f_6$ but $e_7 \notin^{\mathrm{w}} f_{17}$.

We also have $g_{1490} = \underline{f_{17}} = \{f_0, f_1, f_{16}, f_{24}, f_{26}\}$, where $e_6 \in^{\mathrm{w}} f_{17}$ (since $e_6 <_4 e_4$) and $\overline{e_6} = f_{24}$. On the other hand $e_5 \in^{\mathrm{w}} f_{17}$ and $\overline{e_5} = f_{21} = \{e_0, e_3, e_8\}$, but $f_{21} \notin \underline{f_{17}}$ since $f_{17} \not\trianglelefteq_5 f_{21}$ (as $e_3 \in f_{21}$ but $e_3 \notin^{\mathrm{w}} f_{17}$).

We now prove the main structural property of the lattice of children for each realizable indecomposable type:

Theorem 2.4. *For every $\alpha \in \boldsymbol{BF}_n$ there are $\overline{\alpha}, \underline{\alpha} \in \boldsymbol{BF}_{n+1}$ such that (i) $\left(\overline{\alpha}\right)_n = \alpha = \left(\underline{\alpha}\right)_n$ and (ii) $\underline{\alpha} \leq_{n+1} \beta \leq_{n+1} \overline{\alpha}$ for every $\beta \in \boldsymbol{BF}_{n+1}$ with $\left(\beta\right)_n = \alpha$.*

Proof. Let $\alpha \in \mathbf{BF}_n$ (which implies that α is realizable.) We start by proving that $\overline{\alpha}$ is largest among all $\beta \in \mathbf{BF}_{n+1}$ with $(\beta)_n = \alpha$. We recall

$$\overline{\alpha} = \max\Big\{\gamma_* \,:\, \gamma \in \alpha \;\text{ and }\; \gamma_* = \max\big\{\delta \,:\, \delta \in \alpha \;\&\; \gamma \triangleleft_n \delta\big\}\Big\}.$$

also that $\gamma \triangleleft_n \delta$ implies two conditions: (a) $(\delta)_{n-1} \in^{\mathrm{w}} \gamma$ and (b) $\gamma \leq^{\mathrm{w}}_n \delta$.

First, $(\overline{\alpha})_n = \alpha$. For this, it is sufficient to show that $(\gamma_*)_{n-1} = \gamma$ for each $\gamma \in \alpha$. Let $\delta \in \gamma \in \alpha$, so there is a $\delta' \in \alpha$ with $\gamma \triangleleft_{n-1} \delta'$ (α is realizable), and thus, $\delta' \in \gamma_*$. So, $(\gamma_*)_{n-1} \leq^{\mathrm{w}}_{n-1} \gamma$. By the defining condition of γ_*, if $\xi \in \gamma_*$ then $(\xi)_{n-2} \in^{\mathrm{w}} \gamma$, so $\gamma \leq^{\mathrm{w}}_{n-1} (\gamma_*)_{n-1}$. Thus, $(\gamma_*)_{n-1} = \gamma$ (by [HM, Lemma 3.8]). Note that this also shows that $\gamma_* \in \overline{\alpha}$ when $\gamma \in \alpha$, since γ_* is the unique descendant meeting the conditions in the definition of $\overline{\alpha}$.

Next, let $\beta \in \mathbf{BF}_{n+1}$ (and realizable) with $(\beta)_n = \alpha$. We will show that $\beta \leq_{n+1} \overline{\alpha}$. Fix $\gamma_* \in \overline{\alpha}$. Then, $\gamma \in \alpha$, so there is a $\xi \in \beta$ with $(\xi)_{n-1} = \gamma$. We claim that $\gamma_* \leq_n \xi$. Let $\nu \in \xi \in \beta$, so that there is some $\nu' \in \beta$ with $(\nu')_{n-1} = \nu$ and $\xi \leq^{\mathrm{w}}_n \nu'$ (by the realizability of β.) We also have $\nu \in^{\mathrm{w}} \alpha$ (since $(\beta)_n = \alpha$), so let $\delta \in \alpha$ with $\delta \geq_{n-1} \nu$. We will show that $\delta \in \gamma_*$, from which it follows that $\gamma_* \leq_n \xi$. We require (i) $\delta \in \alpha$ (which has been shown already), (ii) $(\delta)_{n-2} \in^{\mathrm{w}} \gamma$ and (iii) $\gamma \leq^{\mathrm{w}}_{n-1} \delta$. Since $\nu \in \xi$ and $(\xi)_{n-1} = \gamma$, we have (ii) $(\delta)_{n-2} = (\nu)_{n-2} \in^{\mathrm{w}} \gamma$. Since $\xi \leq^{\mathrm{w}}_n \nu'$, we have $\gamma = (\xi)_{n-1} \leq^{\mathrm{w}}_{n-1} \nu$, as well as

$\nu \leq_{n-1} \delta$, so it follows that (iii) $\gamma \leq^{\mathrm{w}}_{n-1} \delta$. Thus, $\delta \in \gamma_*$, and so $\gamma_* \leq_n \xi$ as required. Therefore, $\beta \leq_{n+1} \overline{\alpha}$.

Now, we turn to showing that $\overline{\alpha}$ is realizable. We first prove a useful lemma:

Lemma 2.5. *Let $\alpha \in \boldsymbol{BF}_n$ and $\gamma \in \alpha$. Let γ_* be defined by*

$$\gamma_* = \max\left\{\delta \,:\, \delta \in \alpha \;\&\; \gamma \lhd_{n-1} \delta\right\}.$$

Then γ_* is realizable and is smallest among the realizable ξ satisfying (i) $(\xi)_{n-1} = \gamma$ and (ii) $\alpha \leq^{w}_{n-1} \xi$. That is, if $\xi \in \boldsymbol{BF}_n$ and ξ satisfies (i) and (ii), then $\gamma_* \leq_n \xi$.

Proof. First, we show that γ_* is realizable. To this end, let $\rho \in \delta \in \gamma_*$. Then, $\delta \in \alpha$, so that $\rho \in \delta \in \alpha$. Let $\rho' \in \alpha$ satisfy $(\rho')_{n-2} = \rho$ and $\delta \leq^{\mathrm{w}}_{n-1} \rho'$; so, $\gamma \leq^{\mathrm{w}}_{n-1} \rho'$ as well (as, $\gamma \leq^{\mathrm{w}}_{n-1} \delta$ from $\delta \in \gamma_*$.) Thus, $\rho' \in \gamma_*$ and so γ_* is realizable.

Let $\xi \in \mathbf{BF}_n$ satisfy the conditions (i) $(\xi)_{n-1} = \gamma$ and (ii) $\alpha \leq^{\mathrm{w}}_n \xi$, and we will show that $\gamma_* \leq_n \xi$. To this end, fix $\rho \in \xi$, so that, by (ii), there is some $\sigma \in \alpha$ with $\rho \leq_{n-1} \sigma$. We will show that $\sigma \in \gamma_*$. First, $\sigma \in \alpha$ by above. Next, $(\sigma)_{n-2} \in^{\mathrm{w}} \gamma$, since by (ii), $(\sigma)_{n-2} = (\rho)_{n-2} \in^{\mathrm{w}} (\xi)_{n-1} = \gamma$. Finally, we will show that $\gamma \leq^{\mathrm{w}}_{n-1} \sigma$, from which it will follow that $\sigma \in \gamma_*$. Let $\tau \in \rho \in \xi$, so that, by (i), there is some $\tau' \in \xi$ with $(\tau')_{n-2} = \tau \in^{\mathrm{w}} (\xi)_{n-1} = \gamma$. It follows that $\gamma \leq^{\mathrm{w}}_{n-1} \rho$, which together with $\rho \leq_{n-1} \sigma$ implies $\gamma \leq^{\mathrm{w}}_{n-1} \sigma$. Therefore, $\rho \leq_{n-1} \sigma \in \gamma_*$, and since $\rho \in \xi$ was arbitrary, $\gamma_* \leq_n \xi$. ☐

We continue the main proof, showing $\overline{\alpha}$ is realizable. Let $\delta \in \gamma_* \in \overline{\alpha}$. We will show that $\delta \in \alpha$ (so that $\delta_* \in \overline{\alpha}$), $(\delta_*)_{n-1} = \delta$, and $\gamma_* \leq^{\mathrm{w}}_n \delta_*$. This will complete the claim that $\overline{\alpha}$ is realizable. Since $\delta \in \gamma_*$, we have $\delta \in \alpha$, $(\delta)_{n-2} \in^{\mathrm{w}} \gamma$, and $\gamma \leq^{\mathrm{w}}_{n-1} \delta$. Since $\delta \in \alpha$, we have $\delta_* \in \overline{\alpha}$ and $(\delta_*)_{n-1} = \delta$ (as previously shown). We claim that $\gamma_* \leq^{\mathrm{w}}_n \delta_*$. Let $\xi \in \delta_*$, so that $\xi \in \alpha$, $(\xi)_{n-2} \in^{\mathrm{w}} \delta$ and $\delta \leq^{\mathrm{w}}_{n-1} \xi$. We will show that $\xi \in \gamma_*$; to this end, we require (i) $\xi \in \alpha$ (already shown), (ii) $(\xi)_{n-2} \in^{\mathrm{w}} \gamma$ and (iii) $\gamma \leq^{\mathrm{w}}_{n-1} \xi$. First, (iii) holds since $\gamma \leq^{\mathrm{w}}_{n-1} \delta$ and $\delta \leq^{\mathrm{w}}_{n-1} \xi$. Next, since $(\xi)_{n-2} \in^{\mathrm{w}} \delta$ and $\gamma \leq^{\mathrm{w}}_{n-1} \delta$, it follows that $(\xi)_{n-2} \in^{\mathrm{w}} \gamma$, which establishes (ii). Thus, $\xi \in \gamma_*$, and $\gamma_* \leq^{\mathrm{w}}_n \delta_*$. We have shown that from the assumption, $\delta \in \gamma_* \in \overline{\alpha}$, that $\delta_* \in \overline{\alpha}$, $(\delta_*)_{n-1} = \delta$ and $\gamma_* \leq^{\mathrm{w}}_n \delta_*$. This completes the argument that $\overline{\alpha}$ is realizable.

Let $\alpha \in \mathbf{BF}_n$ (which implies that α is realizable.) We start by proving that $\underline{\alpha}$ is smallest among all $\beta \in \mathbf{BF}_{n+1}$ with $(\beta)_n = \alpha$. We recall

$$\underline{\alpha} = \max\left\{\overline{\gamma} \,:\, \alpha \lhd_n \overline{\gamma}\right\}.$$

We have already shown that $\overline{\gamma} \geq_n \delta$ for all $\delta \in \mathbf{BF}_n$ with $(\delta)_{n-1} = \gamma$.

Let $\beta \in \mathbf{BF}_{n+1}$ (and realizable) with $(\beta)_n = \alpha$. We will show that $\underline{\alpha} \leq_{n+1} \beta$. To this end, fix $\xi \in \beta$, so that $(\xi)_{n-1} \in^{\mathrm{w}} \alpha$. Let $(\xi)_{n-1} = \gamma$, then $\overline{\gamma} \geq_n \xi$, so we must only show that $\overline{\gamma} \in \underline{\alpha}$. Since we already have $\gamma \in^{\mathrm{w}} \alpha$, we only require that $\alpha \leq^{\mathrm{w}}_n \overline{\gamma}$. Let $\delta_* \in \overline{\gamma}$ (for any $\delta \in \gamma$); since $\overline{\gamma} \geq_n \xi$, there is a $\nu \in \xi$ with $\nu \geq_n \delta_*$. Note that since $\nu \in \xi \in \beta$, there is some $\nu' \in \beta$ with $(\nu')_{n-1} = \nu$ (by realizability), and thus, $\nu \in^{\mathrm{w}} \alpha$. It follows that $\delta_* \in^{\mathrm{w}} \alpha$, as well. Since $\delta_* \in \overline{\gamma}$ was arbitrary, it follows that $\alpha \leq^{\mathrm{w}}_n \overline{\gamma}$, and thus that $\overline{\gamma} \in \underline{\alpha}$. This completes the argument that $\underline{\alpha} \leq_{n+1} \beta$.

Now, we show that $\underline{\alpha}$ is realizable. First, we have already shown that $\overline{\gamma}$ is realizable for every $\gamma \in^{\mathrm{w}} \alpha$. Let $\delta \in \overline{\gamma} \in \underline{\alpha}$, and we will show that $\overline{\delta} \in \underline{\alpha}$ and $\overline{\gamma} \leq^{\mathrm{w}}_n \overline{\delta}$. It is sufficient to show that (a) $\delta \in^{\mathrm{w}} \alpha$ and (b) $\overline{\gamma} \leq^{\mathrm{w}}_n \overline{\delta}$; from (b) and $\alpha \leq^{\mathrm{w}}_n \overline{\gamma}$ (which holds by definition of $\overline{\gamma} \in \underline{\alpha}$) it follows that $\alpha \leq^{\mathrm{w}}_n \overline{\delta}$, which together with (a) implies that $\overline{\delta} \in \underline{\alpha}$. (Note that $\overline{\gamma}$ is the unique element in $\overline{\alpha}$ whose parent is γ.)

First, (a) $\delta \in^{\mathrm{w}} \alpha$, since $\delta \in \overline{\gamma}$ and $\alpha \leq^{\mathrm{w}}_n \overline{\gamma}$ (by definition of $\overline{\gamma} \in \underline{\alpha}$.) Now, for (b) $\overline{\gamma} \leq^{\mathrm{w}}_n \overline{\delta}$ when $\delta \in \overline{\gamma}$. Let $\nu_* \in \overline{\delta}$, so that $\nu \in \delta$ (by definition), and we will produce a $\xi \in \overline{\gamma}$ with $\nu_* \leq_{n-1} \xi$. By hypothesis, $\nu \in \delta \in \overline{\gamma}$, and so by realizability, there is some $\xi \in \overline{\gamma}$ with $(\xi)_{n-2} = \nu$ and $\delta \leq^{\mathrm{w}}_{n-1} \xi$. It follows by the Lemma 2.5 that $\nu_* \leq_{n-1} \xi$. So, $\overline{\gamma} \leq^{\mathrm{w}}_n \overline{\delta}$.

□

It will be useful to highlight the following property of maximal bf-types and which follows from Lemma 2.5:

Corollary 2.6. *$\alpha \in \boldsymbol{BF}_{n+1}$ is maximal if and only if each $\gamma \in \alpha$ satisfies $(\gamma)_{n-1} \in (\alpha)_n$ and is smallest among all $\xi \in \boldsymbol{BF}_n$ with (i) $(\xi)_{n-1} = (\gamma)_{n-1}$ and (ii) $(\alpha)_n \lhd_n \xi$.*

The following is an easy consequence of Lemma 2.6.

Lemma 2.7. *Let $\alpha \in \boldsymbol{BF}_{n+1}$ be maximal. For any $\gamma \in \boldsymbol{BF}_n$, if $(\alpha)_n \lhd_n \gamma$ and $\gamma \in^w \alpha$ then $\gamma \in \alpha$.*

Proof. Let $\alpha \in \mathbf{BF}_{n+1}$ be maximal. Suppose $\gamma \in \mathbf{BF}_n$, $(\alpha)_n \lhd_n \gamma$ and $\gamma \in^{\mathrm{w}} \alpha$. Let $\delta \in \alpha$ such that $\gamma \leq_n \delta$. Then, $(\alpha)_n \lhd_n \delta$ (by Lemma 2.2.d) and $(\gamma)_{n-1} = (\delta)_{n-1} \in (\alpha)_n$ (membership is by Lemma 2.6). Since $(\gamma)_{n-1} \in (\alpha)_n$ and $(\alpha)_n \lhd_n \gamma$, it follows that $\gamma = \delta$ by the minimality of δ from Lemma 2.6. Thus, $\gamma \in \alpha$. □

3. Isomorphism types

Definition 3.1. A realizable back-and-forth type $\alpha \in \mathbf{BF}_n$ is an *isomorphism type* if $\mathcal{A} \cong \mathcal{B}$ whenever $T_n(\mathcal{A}) = \alpha = T_n(\mathcal{B})$.

The goal in this section is to provide a simple structural characterization of the isomorphism types of indecomposable Boolean algebras which arise in the back-and-forth hierarchy. Theorem 3.6 states that $\alpha \in \mathbf{BF}_n$ is an isomorphism type if and only if α has exactly one descendant in $\mathbf{BF}_{n+1}$. A key step in this proof is that all maximal bf-types have exactly one descendant (Theorem 3.3), and so will be isomorphism types. The converse is not true: there are isomorphism bf-types which are not maximal (Example 3.8).

3.1. Maximal bf-types are isomorphism types.

Definition 3.2. Recall, $\beta \in \mathbf{BF}_m$ is a *descendant* of $\alpha \in \mathbf{BF}_n$ (where $m \geq n$) if $(\beta)_n = \alpha$.

A bf-type $\alpha \in \mathbf{BF}_n$ is *isolated at level m* (where $m \geq n$) if α has a unique descendant in $\mathbf{BF}_m$.

It follows from Theorem 2.4 that $\alpha \in \mathbf{BF}_n$ is isolated at $n+1$ if and only if $\overline{\alpha} = \underline{\alpha}$.

The main aim in this subsection is to prove that the $\leq_n$-maximal bf-types have a unique descendant.

Theorem 3.3. *If $\alpha \in \boldsymbol{BF}_n$ is $\leq_n$-maximal then α is isolated at $n+1$. Equivalently, $\overline{\alpha} = \underline{\alpha}$.*

Proof. Let $\alpha \in \mathbf{BF}_n$. We will show that $\overline{\alpha}$ is isolated at $n+2$. It is sufficient to show that

$$\overline{(\overline{\alpha})} \equiv_{n+2} \underline{(\overline{\alpha})}$$

Since $\underline{(\overline{\alpha})} \leq_{n+2} \overline{(\overline{\alpha})}$, it is left to show $\overline{(\overline{\alpha})} \leq_{n+2} \underline{(\overline{\alpha})}$.

First, recall

$$\underline{(\overline{\alpha})} = \left\{ \overline{\gamma} \,:\, \overline{\alpha} \lhd_{n+1} \overline{\gamma} \right\}$$

We will first show that $\overline{\gamma} \subseteq \overline{\alpha}$ whenever $\overline{\gamma} \in \underline{(\overline{\alpha})}$. Suppose $\delta \in \overline{\gamma}$; then, by realizability there is $\overline{\rho} \in \underline{(\overline{\alpha})}$, so that $\overline{\alpha} \lhd_{n+1} \overline{\rho}$ and $\rho = \delta$. By Lemma 2.2.b, $\alpha \lhd_n \delta$. So, both $\delta \in^{\mathrm{w}} \overline{\alpha}$ and $\alpha \lhd_n \delta$, which implies that $\delta \in \overline{\alpha}$, by Lemma 2.7. Therefore, $\overline{\gamma} \subseteq \overline{\alpha}$.

By Corollary 2.6 we have $\delta \in \overline{\gamma}$ if and only if $(\delta)_{n-1} \in \gamma$ and is smallest among $\xi \in \mathbf{BF}_n$ with (i) $\gamma \lhd_n \xi$ and (ii) $(\xi)_{n-1} \in \gamma$. Since $\overline{\gamma} \subseteq \overline{\alpha}$, it follows that

> $\delta \in \overline{\gamma}$ if and only if $(\delta)_{n-1} \in \gamma$ and is smallest among $\xi \in \mathbf{BF}_n$ such that $\gamma \lhd_n \xi$, $(\xi)_{n-1} \in \gamma$ and (ii) $\xi \in \overline{\alpha}$.

Suppose that $(\delta)_{n-1} \in \gamma$ and $\delta \in \overline{\alpha}$. Then $(\delta)_{n-1} \in \alpha$. Suppose $(\xi)_{n-1} = (\delta)_{n-1} \in \alpha$ and $\gamma \lhd_n \xi$. Since $\alpha \lhd_n \gamma$, it follows that $\alpha \lhd_n \xi$, so that $\delta \leq_n \xi$ by Corollary 2.6 (and the hypothesis that $\delta \in \overline{\alpha}$.) So, δ is minimal among the $\xi \in \mathbf{BF}_n$ with $(\xi)_{n-1} = (\delta)_{n-1}$ and $\gamma \lhd_n \xi$, and thus $\delta \in \overline{\gamma}$ by Lemma 2.7. So, we can simplify the above condition,

> $\delta \in \overline{\gamma}$ if and only if $\gamma \lhd_n \delta$, $(\delta)_{n-1} \in \gamma$ and $\delta \in \overline{\alpha}$.

Since $\overline{\gamma} \in \underline{(\overline{\alpha})}$, we have $\overline{\alpha} \lhd_{n+1} \overline{\gamma}$. It follows that $\alpha \lhd_n \gamma$ and $\gamma \in^{\mathrm{w}} \overline{\alpha}$, which implies that $\gamma \in \overline{\alpha}$ by Lemma 2.7.

Now, let $\gamma_* \in \overline{(\overline{\alpha})}$ where $\gamma \in \overline{\alpha}$. Since $(\gamma_*)_n = \gamma = (\overline{\gamma})_n$, we have $\gamma_* \leq_{n+1} \overline{\gamma}$; so, it is sufficient to show that $\overline{\gamma} \leq_{n+1} \gamma_*$. Suppose $\delta \in \gamma_*$. Then by definition of $\overline{(\overline{\alpha})}$,

> $\delta \in \gamma_*$ if and only if $\gamma \lhd_n \delta$ and $\delta \in \overline{\alpha}$.

It is sufficient to show that $(\delta)_{n-1} \in \gamma$ to conclude $\delta \in \overline{\gamma}$, and thus that $\overline{\gamma} \equiv_{n+1} \gamma_*$.

We have $\delta \in \overline{\alpha}$, so $(\delta)_{n-1} \in \alpha$ by definition. Since $\gamma \in \overline{\alpha}$ we have $\alpha \lhd_n \gamma$, and thus that $\alpha \leq_n^{\mathrm{w}} \gamma$. Since $\gamma \lhd_n \delta$ it follows that $(\delta)_{n-1} \in^{\mathrm{w}} \gamma$; so, let $\rho \in \gamma$ such that $(\delta)_{n-1} \leq_{n-1} \rho$, and $\sigma \in \alpha$ with $\rho \leq_{n-1} \sigma$. Since $(\delta)_{n-1} \in \alpha$, we have

$$(\delta)_{n-1} \leq_{n-1} \rho \leq_{n-1} \sigma \leq_{n-1} (\delta)_{n-1}.$$

Therefore, $(\delta)_{n-1} = \sigma \in \gamma$.

Thus, $\overline{\gamma} \equiv_{n+1} \gamma_*$ where $\gamma \in \overline{\alpha}$. It follows that $\overline{(\overline{\alpha})} \leq_{n+2} \underline{(\overline{\alpha})}$. and thus $\overline{(\overline{\alpha})} = \underline{(\overline{\alpha})}$. □

3.2. A structural characterization of the indecomposable isomorphism types.

Our goal in this section is to show that there is a very simple structural characterization of when a bf-type is an isomorphism type: $\alpha \in \mathbf{BF}_n$ is an isomorphism type if and only if α is isolated at level $n+1$.

The main tool for proving Boolean algebras are isomorphic, is Vaught's Theorem.

Definition 3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. A subset R of $\mathcal{A} \times \mathcal{B}$ is a *V-relation* between $\mathcal{A}$ and $\mathcal{B}$ if

(i) $(1_{\mathcal{A}}, 1_{\mathcal{B}}) \in R$,
(ii) $(x, 0_{\mathcal{B}}) \in R$ implies $x = 0_{\mathcal{A}}$; $(0_{\mathcal{A}}, y) \in R$ implies $y = 0_{\mathcal{B}}$

(iii) $(x, y_1 \dot\vee y_2) \in R$ implies $x = x_1 \dot\vee x_2$, where $(x_1, y_1) \in R$ and $(x_2, y_2) \in R$; $(x_1 \dot\vee x_2, y) \in R$ implies $y = y_1 \dot\vee y_2$, where $(x_1, y_1) \in R$ and $(x_2, y_2) \in R$;

The following result is central to the study of countable Boolean algebras and is found in Vaught's dissertation thesis (see [Pie89, Section 1] for the statement given here and proof),

Theorem 3.5 (Vaught's Theorem)**.** *If $\mathcal{A}$ and $\mathcal{B}$ are countable Boolean algebras and $R \subseteq \mathcal{A} \times \mathcal{B}$ is a V-relation, then $\mathcal{A}$ is isomorphic to $\mathcal{B}$.*

The following is the main result in this subsection:

Lemma 3.6. *The following are equivalent for all $\alpha \in \boldsymbol{BF}_n$:*

(1) α is an isomorphism type.
(2) $\overline{\alpha} = \underline{\alpha}$.

Proof. (1) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (1). We first derive several facts.

Lemma 3.7. *Let $\alpha \in \boldsymbol{BF}_n$.*

(a) If α is isolated (at level n) then α is isolated at level $n+1$.
(b) If α is isolated (at level n) and $\beta \in \alpha$ then β is isolated at level n.
(c) Suppose α is isolated (at level n). Whenever $\beta_0, \dots, \beta_k \in \boldsymbol{BF}_n$ and $\alpha \leq_n \sum_{i \leq k} \beta_i$, then $\alpha = \beta_i$, for some i, and $\alpha \equiv_n \sum_{i \leq k} \beta_i$.
(d) If α is isolated (at level n) and $\mathcal{A}$ a Boolean algebra with $T_n(\mathcal{A}) = \alpha$, then $\mathcal{A}$ is n-indecomposable. (This need not be the case for non-isolated bf-types. See [HM, Example 7.20].)

Proof of Lemma. (a). If α is isolated at level n, then α is maximal at level n. So, $\overline{\alpha} = \underline{\alpha}$ by Theorem 3.3. Thus, α is maximal at level $n+1$.

(b). Suppose α is isolated and $\beta \in \alpha$. Since $\alpha = \underline{(\alpha)_{n-1}}$, β is $\leq_{n-1}$-maximal. Thus, β is isolated at level n (by Theorem 3.3.)

(c). Suppose α is isolated at level n and $\beta_0, \dots, \beta_k \in \mathbf{BF}_n$ are such that $\alpha \equiv_n \sum_{i \leq k} \beta_i$. Then, by [HM, Lemmas 7.12], $\alpha \leq_n \beta_{i_0}$ for some $i_0 \leq k$; but, as α is isolated, $\alpha \equiv_n \beta_{i_0}$. Furthermore, $\beta_i \in^{\mathrm{w}} \alpha$ for all $i \neq i_0$, by [HM, Lemma 7.12]. Since each $\delta \in \alpha$ is isolated, by (b), in fact, $\beta_i \in \alpha$. But $\delta \in \alpha$ implies that $\alpha + \delta \equiv_n \alpha$; so, $\alpha \equiv_n \sum_{i \leq k} \beta_i$.

(d). Suppose α is isolated at level n and $\mathcal{A}$ is a Boolean algebra with $T_n(\mathcal{A}) = \alpha$. By Lemma 2.3, it is sufficient to show that for any partition of $\mathcal{A}$ into n-indecomposables, $1_{\mathcal{A}} = a_0 \dot\vee \dots \dot\vee a_k$, there is an $i \leq k$ with $T_n(\mathcal{A}) = T_n(\mathcal{A} \restriction a_i)$. Consider any partition of $\mathcal{A}$ into n-indecomposables, $1_{\mathcal{A}} = a_0 \dot\vee \dots \dot\vee a_k$; then $\alpha = T_n(1_{\mathcal{A}}) = \sum_{i \leq k} T_n(a_i)$, so that $T_n(\mathcal{A}) = T_n(\mathcal{A} \restriction a_i)$ for some $i \leq k$ (by (c).) Thus, $\mathcal{A}$ is n-indecomposable. □

Let $\alpha \in \mathbf{BF}_n$ be isolated at level n, and $\mathcal{A}$ and $\mathcal{B}$ be any Boolean algebras with $T_n(\mathcal{A}) = \alpha = T_n(\mathcal{B})$. Since α is isolated at level n, it is isolated at level $n+1$ (by Lemma 3.7.a), so that $T_{n+1}(\mathcal{A}) = T_{n+1}(\mathcal{B})$. Define the relation R on $\mathcal{A} \times \mathcal{B}$ by $(x, y) \in R$ if and only if $T_{n+1}(x) = T_{n+1}(y)$. We will show that R is a V-relation between $\mathcal{A}$ and $\mathcal{B}$ (Definition 3.4) from which it follows that $\mathcal{A} \cong \mathcal{B}$ (by Theorem 3.5.)

$(1_{\mathcal{A}}, 1_{\mathcal{B}}) \in R$ by hypothesis. Clearly, $(x, 0_{\mathcal{B}}) \in R$ implies $x = 0_{\mathcal{A}}$, and $(0_{\mathcal{A}}, y) \in R$ implies $y = 0_{\mathcal{B}}$. For the final condition, it is sufficient to show the following:

($*$) Suppose $(x, y) \in R$. For any partition $x = x_0 \dot\vee \ldots x_k$ into $(n+1)$-indecomposables, there is a partition $y = y_0 \dot\vee \ldots \dot\vee y_k$ into $(n+1)$-indecomposables such that $(x_i, y_i) \in R$. Similarly, for any partition $y = y_0 \dot\vee \ldots \dot\vee y_k$ into $(n+1)$-indecomposables, there is a partition $x = x_0 \dot\vee \ldots x_k$ into $(n+1)$-indecomposables such that $(x_i, y_i) \in R$

The third condition for V-relations follows from ($*$).

To prove ($*$): Suppose $(x, y) \in R$. Let $x = x_0 \dot\vee \ldots x_k$ be a partition into $(n+1)$-indecomposables. By Lemma 3.7.b and the assumption that α is isolated at level n, $T_n(x_i)$ is isolated at level n for each i. Since $T_{n+1}(x) = T_{n+1}(y)$, there is a partition $y = y_0 \dot\vee \ldots \dot\vee y_k$ with $T_n(x_i) \leq_n T_n(y_i)$. By Lemma 3.7.c we have $T_n(x_i) = T_n(y_i)$; so, by Lemma 3.7.d, we have that y_i is n-indecomposable and thus isolated at level n (by Lemma 3.7.b.) Finally, $T_n(x_i)$ is isolated at level $n+1$ (by Lemma 3.7.a), so that $T_{n+1}(x_i) = T_{n+1}(y_i)$. The third sentence of ($*$) is proved similarly.

It follows that R is a V-relation, so that $\mathcal{A} \cong \mathcal{B}$ by Theorem 3.5.

□

All isomorphism types through level 6 are *maximal types*; however, this is not generally the case, as the next example shows.

Example 3.8. Our example will construct a seven type which is not maximal, but is an isomorphism type; it is based upon the following six types:

- Children of f_0: $g_0 = \{\}$. This is the *atom* type.
- Children of f_1: $g_1 = \{f_0\}$. This is the *1-atom* type.
- Children of f_2: $g_2 = \{f_0, f_1\}$. This is the *2-atom* type.
- Children of f_3: $g_3 <_6 g_4$ whose members are as follows:
  - $g_4 = \{f_0, f_1, f_3\}$. This is the isomorphism type, $\mathrm{I}(\omega^2 \cdot \eta)$ (the *2-atomless* type.)
  - $g_3 = \{f_0, f_1, f_2\}$. This is *not* an isomorphism type; it is the type of such algebras as $\mathrm{I}(\omega^n)$ for (n-*atom*) and $\mathrm{I}(\omega^n \cdot \eta)$ (n-*atomless*), for $n \geq 3$.

  Note that $f_3 <_5 f_2$, which is important in what follows.
- Children of f_{23}: $g_{1574} <_6 g_{1573}$ whose members are as follows:
  - $g_{1573} = \{f_0, f_1, f_3, f_{26}\}$. This is the isomorphism type, $\mathrm{I}(\omega^2 \cdot \eta + \eta)$.
  - $g_{1574} = \{f_0, f_1, f_2, f_{26}\}$. This is *not* an isomorphism type; it is the type of such algebras as $\mathrm{I}(\omega^n + \eta)$ and $\mathrm{I}(\omega^n \cdot \eta + \eta)$, for $n \geq 3$.
- Children of f_{24}: $g_{1575} = \{f_0, f_1, f_{26}\}$. This is the isomorphism type of $\mathrm{I}(\omega^2 + \eta)$. Note that $f_{24} \leq_5 f_{23}$.
- Children of f_{26}: $g_{1577} = \{f_{26}\}$. This is the *atomless* type.
- Children of f_{12}: $g_{1166} <_6 g_{1558}$, there are many others but these two are most relevant in what follows. Their members are as follows:
  - $g_{1558} = \{f_0, f_1, f_3, f_{12}, f_{26}\}$. This is the maximal descendant of f_{12} and the isomorphism type, $\mathrm{I}((\omega^2 \cdot \eta + 1 + \eta) \cdot \eta)$.
  - $g_{1166} = \{f_0, f_1, f_3, f_{24}, f_{26}\}$. This is *not* an isomorphism type.

- Children of g_{1166}: $h_2 <_7 h_1$, there are many others but these two are most relevant in what follows. Their members are as follows:
  - $h_1 = \{g_0, g_1, g_4, g_{1575}, g_{1577}\}$. This is the maximal descendant of g_{1166} and the isomorphism type, $\mathrm{I}\big((\omega^2 \cdot \eta + \omega^2 + \eta) \cdot \omega\big)$.
  - $h_2 = \{g_0, g_1, g_4, g_{1573}, g_{1575}, g_{1577}\}$. This is the counterexample: it is not maximal but it is the isomorphism type of $\mathrm{I}\big(((\omega^2 \cdot \eta) + (\omega^2 + \eta) + (\omega^2 \cdot \eta + \eta)) \cdot \omega\big)$.

So, $h_2 <_7 h_1$; we will show that there is exactly one realizable descendant of h_2, that is $\underline{h_2} = \overline{h_2}$. Note that each of the members of h_2 are already isomorphism types (they are either isolated or maximal); so, for any bf-type $\alpha \in \mathbf{BF}_7$ such that $\alpha \in \underline{h_2} - \overline{h_2}$, there is a $\beta \in h_2$ with $(\alpha)_6 <_6 \beta$. There are only two possibilities for such a β, g_4 or g_{1573}, and thus $(\alpha)_6$ must be either g_3 or g_{1574}. Note that $f_2 \in g_3, g_{1574}$ and f_2 has an only child g_2; so, we must have $f_2 \in (\alpha)_6$ and $g_2 \in \alpha$. Since $\alpha \in \underline{h_2}$, we have $h_2 \lhd_7 \alpha$ (by definition), and thus, $g_2 \in h_2$ (since g_2 is isolated). Since this is not the case, $\underline{h_2} = \overline{h_2}$.

4. Finitary isomorphism types

The aim in this section is to show that the isomorphism types which arise in the finite levels of the back-and-forth hierarchy are exactly the finitary isomorphism types introduced independently by Palyutin and Pierce (see [Pal71] and [Pie89, Section 3.13]). We begin by presenting two equivalent charactizations of the finitary isomorphism types, and use these to establish the correspondence. The main sources for the discussion in this section are [Pie89] and [Hei92].

We begin with some remarks about typography. We will only treat countable Boolean algebras. If $\mathcal{A}$ is a Boolean algebra then we will write $[\mathcal{A}]$ for the isomorphism type of $\mathcal{A}$, that is the set $\{\mathcal{B} : \mathcal{B} \cong \mathcal{A}\}$. We will write $[\mathcal{A}] + [\mathcal{B}]$ for $[\mathcal{A} \times \mathcal{B}]$. For $a \in \mathcal{A}$ we will write $[a]$ for $[\mathcal{A} \upharpoonright a]$ when the algebra $\mathcal{A}$ is clear from the context. We will also use the following typographic conventions:

- $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$: countable Boolean algebras,
- a, b, c : elements of Boolean algebras,
- $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$: isomorphism types of Boolean algebras,
- $\mathbf{C}$, $\mathbf{PI}$, $\mathbf{P}$: sets of isomorphism types

4.1. Pseudo-indecomposable and primitive isomorphism types.

We will write $\mathbf{BA}$ for the set of isomorphism types of all countable Boolean algebras. This set has some nice structural properties which we isolate here. First, $(\mathbf{BA}, +, \mathbf{0})$ is a commutative monoid with unique zero element (the one element trivial Boolean algebra $\mathbf{0}$):

- (Associativity) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$, for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{BA}$,
- (Commutativity) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, for all $\mathbf{a}, \mathbf{b} \in \mathbf{BA}$,
- (Identity) $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ for all $\mathbf{a} \in \mathbf{BA}$.

There is a natural ordering $\leq$ in the monoid given by

$$\mathbf{a} \leq \mathbf{b} \quad \Longleftrightarrow \quad \mathbf{b} = \mathbf{a} + \mathbf{c} \qquad \text{for some } \mathbf{c} \in \mathbf{BA},$$

which satisfies the following easily verified properties

- (Reflexivity) $\mathbf{a} \leq \mathbf{a}$ for any $\mathbf{a} \in \mathbf{BA}$,
- (Transitivity) If $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{b} \leq \mathbf{c}$ then $\mathbf{a} \leq \mathbf{c}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{BA}$,
- (Monotonicity) If $\mathbf{a} \leq \mathbf{b}$ then $\mathbf{a} + \mathbf{c} \leq \mathbf{b} + \mathbf{c}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{BA}$.

However, it is not generally true that $\leq$ is a partial order since *antisymmetry*, $\mathbf{a} \leq \mathbf{b} \leq \mathbf{a} \implies \mathbf{a} = \mathbf{b}$, can fail. (See the discussion in [Pie89, Section 2.5].)

There is an additional property on this structure which will be important in what follows: the *refinement property*:

(Refinement) For all $\mathbf{a}_i, \mathbf{b}_j \in \mathbf{BA}$ ($i < m$ and $j < n$), if $\sum_{i<m} \mathbf{a}_i = \sum_{j<n} \mathbf{b}_j$ then there exists $\mathbf{c}_{ij} \in \mathbf{BA}$ (for $i < m$ and $j < n$) such that $\mathbf{a}_i = \sum_{j<n} \mathbf{c}_{ij}$ and $\mathbf{b}_j = \sum_{i<m} \mathbf{c}_{ij}$.

This property is easily verified on products of Boolean algebras. The refinement property leads to a useful fact concerning the natural ordering (see [Pie89, Lemma 2.2.2]):

If $\mathbf{a} \leq \sum_{j<m} \mathbf{b}_j$ then there are $\mathbf{c}_j \leq \mathbf{b}_j$ with $\mathbf{a} = \sum_{j<m} \mathbf{c}_j$.

Remark 4.1. [Pie89] calls commutative monoids with a unique zero element *m-monoids*, for *measure monoids* (see [Pie89, Section 1.12]); an m-monoid is called a *r-monoid*, for *refinement monoid*, if it satisfies the refinement property as well (see [Pie89, Section 2.2].) A type $\mathbf{a}$ which satisfies $\forall \mathbf{b} \big(\mathbf{a} \leq \mathbf{b} \leq \mathbf{a} \implies \mathbf{a} = \mathbf{b}\big)$ is said to have the *Schröder-Bernstein property*, or *S-B property* (see [Pie89, Section 2.2].)

Definition 4.2.
- $\mathcal{B}$ is *pseudo-indecomposable* if for all $b \in \mathcal{B}$, either $\mathcal{B} \cong \mathcal{B} \restriction b$ or $\mathcal{B} \cong \mathcal{B} \restriction -b$. We will say an element $b \in \mathcal{B}$ is pseudo-indecomposable if $\mathcal{B} \restriction b$ is. An isomorphism type $\mathbf{a}$ is pseudo-indecomposable when $\mathbf{a} = \mathbf{b} + \mathbf{c}$ implies $\mathbf{a} = \mathbf{b}$ or $\mathbf{a} = \mathbf{c}$ for any $\mathbf{b}, \mathbf{c} \in \mathbf{BA}$.
- $\mathcal{B}$ is *primitive* if every element of $\mathcal{B}$ is a disjoint sum of finitely many pseudo-indecomposable elements. We will similarly say an element $b \in \mathcal{B}$ is primitive when $\mathcal{B} \restriction b$ is. An isomorphism type $\mathbf{a}$ is primitive when for any $\mathbf{b} \leq \mathbf{a}$ there are pseudo-indecomposable $\mathbf{c}_1, \ldots, \mathbf{c}_k$ with $\mathbf{b} = \sum_{i \leq k} \mathbf{c}_i$.
- $\mathbf{PI}$ and $\mathbf{P}$ denote the set of pseudo-indecomposable and primitive isomorphism types (respectively).

$\mathbf{P}$ is a *hereditary submonoid* of $\mathbf{BA}$: that is, if $\mathbf{a}, \mathbf{b} \in \mathbf{P}$, then so is $\mathbf{a} + \mathbf{b}$ as well as $\mathbf{c}$ for any $\mathbf{c} \leq \mathbf{a}$. (This follows directly from the definition of $\mathbf{P}$.)

Remark 4.3. It is not true that every pseudo-indecomposable Boolean algebra is primitive. For example, let $\mathcal{B} \cong \sum_{n \in \omega} \mathrm{I}\big(\omega^n + \eta\big)$. Then $\mathcal{B}$ is not pseudo-indecomposable or primitive; however, $\mathcal{C} = \mathrm{I}\big(\mathcal{B} \cdot \omega\big)$ is pseudo-indecomposable but not primitive.

On the other hand, the isomorphism types in $\bigcup_n \mathbf{BF}_n$ are both pseudo-indecomposable and primitive (this follows from the compactness theorem [HM, Theorem 3.11].)

A key structural property of $\mathbf{PI} \cup \mathbf{P}$ is that $\leq$ is a partial order: that is, every $\mathbf{a} \in \mathbf{PI} \cup \mathbf{P}$ has the S-B property: $\mathbf{a} \leq \mathbf{b} \leq \mathbf{a} \implies \mathbf{a} = \mathbf{b}$ for every $\mathbf{b} \in \mathbf{BA}$. This property is not obvious and its proof crucially uses Vaught's Isomorphism Theorem. The proof that every element of $\mathbf{PI}$ has the S-B property is given in [Pie89, Lemma 2.24.2] and the proof that every element of $\mathbf{P}$ has the S-B property is given in [Pie89, Lemma 3.3.1].

4.2. Finitary Boolean algebras.

Definition 4.4. For any $\mathbf{a} \in \mathbf{BA}$, define $\mathscr{D}(\mathbf{a}) = \{\mathbf{b} \in \mathbf{PI} : \mathbf{b} \leq \mathbf{a}\}$, the *diagram* of $\mathbf{a}$. We write $\mathscr{D}(\mathcal{A})$ for $\mathscr{D}([\mathcal{A}])$ (see [Pie89, Sections 3.3, 3.8] for use of this terminology).

A Boolean algebra is *finitary* if it is primitive and has a finite diagram. We denote the set of finitary isomorphism types by $\mathbf{F}$(see [Pie89, Section 3.13]).

We record some simple facts about diagrams

(1) $\mathscr{D}(\mathbf{a}_1 + \ldots + \mathbf{a}_k) = \mathscr{D}(\mathbf{a}_1) \cup \ldots \cup \mathscr{D}(\mathbf{a}_1)$. (This follows from the fact that for $\mathbf{b} \in \mathbf{PI}$, if $\mathbf{b} \leq \mathbf{a}_1 + \ldots + \mathbf{a}_k$ then $\mathbf{b} \leq \mathbf{a}_i$ for some $i \leq k$.
(2) $\mathbf{a} \leq \mathbf{b}$ implies $\mathscr{D}(\mathbf{a}) \subseteq \mathscr{D}(\mathbf{b})$

Diagrams are posets under $\leq$, and in fact are invariant for primitive pseudo-indecomposable algebras: for $\mathbf{a}, \mathbf{b} \in \mathbf{PI} \cap \mathbf{P}$, $\mathbf{a} = \mathbf{b}$ if and only if $\mathscr{D}(\mathbf{a}) = \mathscr{D}(\mathbf{b})$ (see [Pie89, Corollary 3.8.3].)

Finitary Boolean algebras were introduced independently by Palyutin and Pierce (see [Pal71] and [Pie89, Section 3.13]). Pierce considered the class of "compact zero-dimensional metric spaces of finite type", which are the spaces whose Boolean algebras of clopen subsets are the finitary Boolean algebras. Palyutin provided a constructive categorization of the countable Boolean algebras which have a countably categorical weak second-order theory. More than that, he showed that for each such algebra $\mathcal{A}$ there is a single sentence $\phi_{\mathcal{A}}$ in the weak second-order language for Boolean algebras such that for any countable Boolean algebra $\mathcal{B}$:

$$\mathcal{B} \models \phi_{\mathcal{A}} \iff \mathcal{B} \cong \mathcal{A}.$$

It was Heindorff (see [Hei92]) who showed that the characterizations of Palyutin and Pierce coincided with the finitary Boolean algebras.

We will be interested in another characterization of the finitary Boolean algebras formulated by Heindorff using two operations on $\mathbf{BA}$ (see [Hei92, Section 2 and 3].)

Definition 4.5. Let I be an ideal of the Boolean algebra $\mathcal{A}$ and let $\mathbf{B}$ be a non-empty set of isomorphism types. We say that I *witness* $\mathcal{A} \in T(\mathbf{B})$ if the following conditions are satisfied:

(C) For each $a \in I$ there are finitely many $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \mathbf{B}$ such that $[\mathcal{A} \restriction a] \leq \mathbf{b}_1 + \ldots + \mathbf{b}_k$.
(D) For each $a \notin I$ and each $\mathbf{b} \in \mathbf{B}$ there is some $b \leq a$ such that $b \in I$ and $[\mathcal{A} \restriction b] = \mathbf{b}$.
(T) I is proper and A/I is atomless.

For any non-empty set $\mathbf{B}$ we write

$$T(\mathbf{B}) = \big\{\mathcal{A} : \text{some ideal } I \subseteq \mathcal{A} \text{ witnesses } \mathcal{A} \in T(\mathbf{B})\big\}.$$

The following summarizes the key properties about the classes $T(\mathbf{B})$ (see [Hei92, Section 2]):

Lemma 4.6. *Let* $\mathbf{B}$ *be a non-empty class of isomorphism types.*

(a) $T(\mathbf{B})$ *is an isomorphism type. (That is,* $T(\mathbf{B})$ *is non-empty, and all Boolean algebras in this set are isomorphic.)*
(b) $T(\mathbf{B})$ *is pseudo-indecomposable. In fact, if* $\mathcal{A} \in T(\mathbf{B})$ *as witnessed by* I*, then* $\mathcal{A} \cong \mathcal{A} \restriction a$ *for every* $a \notin I$.
(c) $\mathscr{D}(T(\mathbf{B})) = \{T(\mathbf{B})\} \cup \bigcup_{\mathbf{b} \in \mathbf{B}} \mathscr{D}(\mathbf{b})$. *Thus, if* $\mathscr{D}(B)$ *is finite then so is* $\mathscr{D}(T(\mathbf{B}))$.
(d) If $\mathbf{B} \subset \boldsymbol{F}$ *then* $T(\mathbf{B}) \in \boldsymbol{F}$. *(This follows from (b) and (c).)*

Definition 4.7. Let I be an ideal of the Boolean algebra $\mathcal{A}$ and let $\mathbf{B}$ be a non-empty set of isomorphism types. We say that I *witness* $\mathcal{A} \in F(\mathbf{B})$ iff conditions (C) and (D) of Definition 4.7 together with the following condition:

(F) I is a prime ideal of $\mathcal{A}$.

For any non-empty set $\mathbf{B}$ we write

$$F(\mathbf{B}) = \big\{\mathcal{A} : \text{some ideal } I \subseteq \mathcal{A} \text{ witnesses } \mathcal{A} \in F(\mathbf{B})\big\}.$$

The following summarizes the key properties about the classes $F(\mathbf{B})$ (see [Hei92, Section 2]):

Lemma 4.8. *Let $\mathbf{B}$ be a non-empty class of isomorphism types.*
(a) $F(\mathbf{B})$ is an isomorphism type.
(b) $F(\mathbf{B})$ is pseudo-indecomposable. In fact, if $\mathcal{A} \in F(\mathbf{B})$ as witnessed by I, then $\mathcal{A} \cong \mathcal{A}\restriction a$ for every $a \notin I$.
(c) $\mathscr{D}(F(\mathbf{B})) = \{F(\mathbf{B})\} \cup \bigcup_{\mathbf{b}\in\mathbf{B}} \mathscr{D}(\mathbf{b})$. Thus, if $\mathscr{D}(B)$ is finite then so is $\mathscr{D}(F(\mathbf{B}))$.
(d) If $\mathbf{B} \subset \boldsymbol{F}$ then $F(\mathbf{B}) \in \boldsymbol{F}$. (This follows from (b) and (c).)

We can now state the main result of [Hei92, Corollary 3.4]. Let $\mathbf{S}$ be the smallest subset of $\mathbf{BA}$ such that the trivial Boolean algebra $\mathbf{0} \in \mathbf{S}$ and for all non-empty finite subsets $\mathbf{B} \subseteq \mathbf{S}$, both $T(\mathbf{B}) \in \mathbf{S}$ and $F(\mathbf{B}) \in \mathbf{S}$. Let $\mathbf{K}$ be the finite sums of elements of $\mathbf{S}$.

Theorem 4.9. $\mathbf{S} = \boldsymbol{F} \cap \boldsymbol{PI}$ *and* $\mathbf{K} = \boldsymbol{F}$.

We now come to the main result of this section. For each $\alpha \in \mathbf{BF}_n$ and $n \in \omega$ write

$$[\alpha] = \big\{\mathcal{A} : \mathcal{A} \textit{ is } n\textit{-indecomposable and } t_n(\mathcal{A}) = \alpha\big\}.$$

Of course, α is an isomorphism type if and only if $\mathcal{A} \cong \mathcal{B}$ for all $\mathcal{A}, \mathcal{B} \in [\alpha]$. We write $\mathbf{BF}$ for the set $\{[\alpha] : \alpha \in \cup_n \mathbf{BF}_n \text{ an isomorphism type }\}$, and let $\mathbf{INV}$ denote the set of finite sums of elements from $\mathbf{BF}$. Then

Theorem 4.10. $\mathbf{BF} = \boldsymbol{F} \cap \boldsymbol{PI}$ *and* $\mathbf{INV} = \boldsymbol{F}$.

Proof. We will show that (1) $\mathbf{BF} \subseteq \mathbf{F} \cap \mathbf{PI} = \mathbf{S}$ and (2) $\mathbf{S} \subseteq \mathbf{BF}$.

(1). $\mathbf{BF} \subseteq \mathbf{PI}$ follows from the definition of indecomposable. We show $\mathbf{BF} \subseteq \mathbf{F}$ by induction. Suppose $\alpha \in \mathbf{BF}_n$ and $\alpha = \{\beta_0, \dots, \beta_k\}$ is isolated at level n. Then each β_i is isolated, by Lemma 3.7.b, so an isomorphism type. Thus, each $[\beta_i] \in \mathbf{F}$ by the induction hypothesis. Let $\mathcal{A}$ be any Boolean algebra with $T_n(\mathcal{A}) = \alpha$ (a slight abuse of notation, since $T_n(\mathcal{A})$ is the set of $\sigma \in \overline{\mathbf{INV}}_n$ with $\alpha \equiv_n \sigma$). Then, $\mathcal{A}$ is n-indecomposable by Lemma 3.7.d, and for any decomposition of $1_\mathcal{A} = a_0 \dot\vee \dots \dot\vee a_m$ into n-indecomposables (we may assume each a_i is n-indecomposable without loss of generality), we have $T_n(a_i) \in \alpha$. Thus, $\mathscr{D}([\alpha]) = \{[\beta_0], \dots, [\beta_k]\}$, and so $\mathbf{BF} \subseteq \mathbf{F}$.

(2). Note that $\mathbf{0} \in \mathbf{BF}$ (it is an isomorphism type at level 0.) We will show that $\mathbf{BF}$ is closed under Heindorff's two operations, F and T. Suppose $\mathbf{B} = \{[\beta_1], \dots, [\beta_k]\}$ where each $[\beta_i] \in \mathbf{BF}$. WLOG we may assume that each $\beta_i \in \mathbf{BF}_n$ and n is sufficiently large so that each β_i is already isolated, and that for each each $i \leq k$ and $\delta \in \beta_i$ there is a $j \leq k$ with $(\beta_j)_{n-1} = \delta$ and $\beta_i \leq^{\mathrm{w}}_n \beta_j$. Note that this is possible, since $\delta \in \beta_i$ implies that δ is an isomorphism type. Let $\alpha = \{\beta_1, \dots, \beta_k\}$, so that $\alpha \in \mathbf{BF}_{n+1}$.

Claim 1. $F(\mathbf{B}) = [\{\beta_1, \ldots, \beta_k\}] = [\alpha]$.

Proof of Claim. Let $\mathcal{A} \in [\alpha]$, so that $\mathcal{A}$ is $n+1$-indecomposable by, Lemma 3.7.d. Let U be an ultrafilter which witnesses $\mathcal{A}$ is $n+1$-indecomposable, so that $t_{n+1}(U) = \alpha$; let $I = \mathcal{A} - U$, so I is a prime ideal and (T) holds. We claim that I witnesses $\mathcal{A} \in F(\mathbf{B})$. For (C), if $a \in I$ then $-a \in U$ so that $t_{n+1}(-a) = \alpha$. By compactness, $a = a_1 \dot{\vee} \ldots \dot{\vee} a_m$ where each of a_i are n-indecomposable, so that $t_n(a_i) \in^{\mathrm{w}} \alpha$ which implies $t_n(a_i) \in \alpha$ (since each β_i is isolated). Thus, $[a] \leq [a_1] + \ldots + [a_m]$ where $[a_i] \in \alpha$ for each $i \leq m$. For (D), if $a \notin I$ then $a \in U$, so that there is a $c \leq a$ with $\mathcal{A} \restriction c$ $n+1$-indecomposable and $t_n(c) = t_n(U) = \alpha$. For any $\beta_i \in \alpha$ there is a $b \leq c$ such that b is n-indecomposable and $t_n(b) = \beta_i$. Thus, for each $\beta_i \in \alpha$ there is a $b \leq a$ with $[b] \in [\beta_i]$. Therefore, I witnesses $\mathcal{A} \in F(\mathbf{B})$.

Conversely, suppose $\mathcal{A} \in F(\mathbf{B})$ as witnessed by $I \subseteq \mathcal{A}$. Then, $\mathcal{A}$ is m-indecomposable for every m, and $\mathcal{A} \cong \mathcal{A} \restriction a$ for every $a \notin I$, by Lemma 4.8.b. Condition (C) guarantees $T_{n+1}(\mathcal{A}) \subseteq \alpha \cup \{T_n(\mathcal{A})\}$, and condition (F) (that I is a prime ideal) guarantees that $T_{n+1}(\mathcal{A}) \subseteq \alpha$. Condition (D) guarantees that $\alpha \subseteq T_{n+1}(\mathcal{A})$. □

Now, we construct $T(\mathbf{B})$. We make one further assumption about α: that not only are each β_i isolated, but that $(\beta_i)_{n-1}$ are isolated as well. For each $m \leq n+1$ we will define $\gamma_m \in \mathbf{BF}_m$, by recursion. Let $\gamma_0 = *$, and let $\gamma_m = (\alpha)_m \cup \{\gamma_{m-1}\}$ for $m > 0$.

Claim 2. γ_m is realizable and $(\gamma_m)_{m-1} = \gamma_{m-1}$, for all $m \leq n+1$.

Proof of Claim. The proof is by induction and holds for $m \in \{0, 1\}$. Suppose γ_m is realizable and $(\gamma_m)_{m-1} = \gamma_{m-1}$. Then

$$\left(\gamma_{m+1}\right)_m = \left((\alpha)_{m+1} \cup \{\gamma_m\}\right)_m = (\alpha)_m \cup \{(\gamma_m)_{m-1}\} = (\alpha)_m \cup \{\gamma_{m-1}\} = \gamma_m.$$

If $\gamma_m \notin \gamma_{m+1}$, then $\gamma_{m+1} = (\alpha)_{m+1}$ is realizable. Let $\delta \in \gamma_m \in \gamma_{m+1}$. Then, either $\delta \in (\alpha)_m$ or $\delta = \gamma_{m-1}$. If $\delta \in (\alpha)_m$, then there is a $(\beta_i)_m \in (\alpha)_{m+1} \subseteq \gamma_{m+1}$ with $(\beta_i)_{m-1} = \delta$. If $\delta = \gamma_{m-1}$, then $(\gamma_m)_{m-1} = \gamma_{m-1}$ by the inductive hypothesis. Since $(\alpha)_m$ is realizable and γ_m is also realizable (by the inductive hypothesis), it follows that γ_{m+1} is realizable as well. □

Note that $\gamma_{n+1} = \alpha \cup \{\gamma_n\}$ is realizable and $(\gamma_{n+1})_n = \gamma_n$.

Claim 3. γ_{n+1} is an isomorphism type.

Proof of Claim. If $\gamma_n \in \alpha$, then γ_{n+1} is already isolated. Suppose $\gamma_n \notin \alpha$. It is sufficient to show that γ_{n+1} is maximal among the $\eta \in \mathbf{BF}_{n+1}$ with $(\eta)_n = \gamma_n = (\gamma_{n+1})_n$. Let $\delta \in \gamma_{n+1}$, and we will show that there is a $\zeta \in \eta$ with $\delta \leq_n \zeta$. If $\delta \in \alpha \subseteq \gamma_{n+1}$ then $\delta = \beta_i$ for some i, and $(\delta)_{n-1} \in \gamma_n$ (each $(\beta_i)_{n-1} \in \gamma_n$ since $(\beta_i)_{n-1}$ is maximal by hypothesis); since $(\eta)_n = \gamma_n$ we must have $\delta = \beta_i \in \eta$. If $\delta = \gamma_n \in \gamma_{n+1}$ then $(\delta)_{n-1} = (\gamma_n)_{n-1} \in \gamma_n$ (here we use the fact that each $(\beta_i)_{n-1}$ is isolated so $\leq_n$-incomparable with $(\gamma_n)_{n-1}$), and as η is realizable, there is a $\zeta \in \eta$ with $(\zeta)_{n-1} = (\gamma_n)_{n-1}$ and $\gamma_n \leq_n^{\mathrm{w}} \zeta$, and so $\gamma_n \leq_n \zeta$. It thereby follows that $\eta \leq_{n+1} \gamma_{n+1}$. So, γ_{n+1} is maximal, and therefore an isomorphism type. □

Claim 4. $T(\mathbf{B}) = [\gamma_{n+1}]$.

Proof of Claim. Let $\mathcal{A} \in [\gamma_{n+1}]$, so that $\mathcal{A}$ is $(n+1)$-indecomposable by, Lemma 3.7.d. Let I be the ideal be the elements $a \in \mathcal{A}$ such that $[a] \leq [\beta_i]$ for some $\beta_i \in \alpha - \{\gamma_n\}$

(it is possible that $\gamma_n \in \alpha$). We will show that I witnesses $\mathcal{A} \in T(\mathbf{B})$. Condition (C) is satisfied by definition of I. For (D), if $a \notin I$, then let $a = a_1 \dot\vee \ldots \dot\vee a_k$ be a partition into n-indecomposables, then it must be that $[a_i] \in \gamma_{n+1}$ for each i. In fact, for each $a \notin I$ there is a partition $a = a_1 \dot\vee \ldots \dot\vee a_k$ ($k \geq 2$) into n-indecomposables with $[a_1] = [a_2] = \gamma_n$, and so $a_1, a_2 \notin I$ and $a_1 \not\equiv a_2 \mod I$. Thus, A/I is atomless, and condition (T) holds as well.

Conversely, suppose $\mathcal{A} \in T(\mathbf{B})$ as witnessed by $I \subseteq \mathcal{A}$. Then, $\mathcal{A}$ is m-indecomposable for every m, and $\mathcal{A} \cong \mathcal{A} \restriction a$ for every $a \notin I$, by Lemma 4.8.b. We will show that $T_m(\mathcal{A}) = \gamma_m$ for all $m \leq n+1$, by induction on m. For $m = 0$ this is trivially true. Suppose that $T_m(\mathcal{A}) = \gamma_m$. Condition (C) guarantees $T_{m+1}(\mathcal{A}) \subseteq (\alpha)_{m+1} \cup \{T_n(\mathcal{A})\}$. Condition (D) guarantees that $(\alpha)_{m+1} \subseteq T_{m+1}(\mathcal{A})$, and condition (T) (that A/I atomless) guarantees that $T_m(\mathcal{A}) \in T_{m+1}(\mathcal{A})$. Since $T_m(\mathcal{A}) = \gamma_m$, by the inductive hypothesis, we conclude that $T_{m+1}(\mathcal{A}) = (\alpha)_{m+1} \cup \{\gamma_m\} = \gamma_{m+1}$. □

Thus, $\mathbf{0} \in \mathbf{BF}$ and for all non-empty finite subsets $\mathbf{B} \subseteq \mathbf{BF}$, both $T(\mathbf{B}) \in \mathbf{BF}$ and $F(\mathbf{B}) \in \mathbf{BF}$. So, $\mathbf{S} \subseteq \mathbf{BF}$.

Since $\mathbf{INV}$ is the set of finite sums of $\mathbf{BF} = \mathbf{S}$, it follows that $\mathbf{INV} = \mathbf{K}$. □

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