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Δ^0_2-CATEGORICAL BOOLEAN ALGEBRAS

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Abstract. We show that the Δ^0_2-categorical Boolean algebra are precisely those Boolean algebras which can be decomposed into a finite direct sum of atoms, one-atoms and atomless subalgebras. As a consequence the Δ^0_2-categorical Boolean algebras are the same class as the relatively Δ^0_2-categorical Boolean algebras.

Contents

1. Introduction

A computable Boolean algebra $\mathcal{A}$ is Δ^0_n-*categorical* if for each computable copy $\mathcal{B} \cong \mathcal{A}$, there a Δ^0_n-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. A computable Boolean algebra $\mathcal{A}$ is *relatively* Δ^0_n-*categorical* if for each copy $\mathcal{B} \cong \mathcal{A}$, there is a Δ^0_n-computable (*relative to* $\mathcal{B}$) isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Clearly, relative Δ^0_n-categoricity implies Δ^0_n-categoricity, but the converse does generally hold for algebraic structures.

The Δ^0_1-categorical Boolean algebras are those with finitely many atoms and these are also the relatively Δ^0_1-categorical Boolean algebras (shown independently by [GonDz] and [Rem81]).

We will show that the following holds for the Δ^0_2-categoricity in Boolean algebras:

Theorem 1.1. *The following are equivalent for any computable Boolean algebra* $\mathcal{A}$.

(1) $\mathcal{A}$ *is* Δ^0_2-*categorical.*
(2) $\mathcal{A}$ *is relatively* Δ^0_2-*categorical.*
(3) $\mathcal{A}$ *is a finite direct sum of subalgebras* $\mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_k$ *where each* $\mathcal{A}_i$ *is either atomless, an atom or a 1-atom.*

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McCoy [McCoy03] proved the equivalence of (2) and (3) and Nikolay Bazhenov [Baz13] has proven the equivalence of (1) and (2). We offer a proof of this equivalence that takes a different approach and is possibly generalizable to $n \geq 3$.

In McCoy's previously cited paper he proved the equivalence of (1) and (2) under a strong additional premise:

> Let $\mathcal{A}$ be a computable Boolean algebra in which the predicates *Atoms* and *Atomless* are computable. Then $\mathcal{A}$ is Δ^0_2-categorical if and only if $\mathcal{A}$ is relatively Δ^0_2-categorical.

McCoy's result is greatly generalized to arbitrary structures in [AK00, §17.4] using their metasystem framework for priority arguments. (McCoy's hypothesis amounts to the requirement that the algebra $\mathcal{A}$ is 2-friendly in the terminology of [AK00].) His argument was the starting point of the construction used in this paper, which will also use the metasystem framework.

2. Preliminaries

2.1. **Boolean algebras.** We consider only *countable* Boolean algebras and use the signature $\wedge, \vee, -, 0, 1$, but otherwise follow the standard reference [Mon89]. We will also write $\dot{\vee}$ for disjoint union. In what follows Boolean algebras are denoted by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and their elements by a, b, c. We write $\mathcal{A} \oplus \mathcal{B}$ for the direct sum of $\mathcal{A}$ and $\mathcal{B}$ and $\mathcal{A} \upharpoonright a = \{b \in \mathcal{A} : b \leq a\}$ for the relative subalgebra of $\mathcal{A}$. A *partition* of an element a in a Boolean algebra $\mathcal{A}$ is a finite sequence $a_0, \ldots, a_k$ of pairwise disjoint elements (that is, $a_i \wedge a_j = 0$ for all $i \neq j$) which join to a. A partition of a Boolean algebra $\mathcal{A}$ is a partition of its unit, $1_{\mathcal{A}}$. We will write $a = a_0 \mathbin{\dot{\vee}} \ldots \mathbin{\dot{\vee}} a_k$ to mean that $a_0, \ldots, a_k$ is a partition of a. Note that if $1_{\mathcal{A}} = a_0 \mathbin{\dot{\vee}} \ldots \mathbin{\dot{\vee}} a_k$, then $\mathcal{A}$ is isomorphic to the disjoint sum $\mathcal{A} \upharpoonright a_0 \oplus \ldots \oplus \mathcal{A} \upharpoonright a_k$.

We will use the following predicates for Boolean algebras as in [AS00]:

- NON-ZERO(x): $x \neq 0$.
- ATOM(x): x is an atom – there is no partition $x = x_1 \mathbin{\dot{\vee}} x_2$, where both x_1 and x_2 are non-zero.
- INF(x): there are infinitely many elements below x.
- ATOMLESS(x): x is atomless – for each $y \leq x$, there are non-zero y_1, y_2 with $y = y_1 \mathbin{\dot{\vee}} y_2$.
- ATOMINF(x): there are infinitely many atoms below x.
- 1-ATOM(x): x is a 1-atom – INF(x) and whenever $x = x_1 \mathbin{\dot{\vee}} x_2$ one of x_1 or x_2 bounds finitely many elements.
- ATOMIC(x): x is atomic – for all $y \leq x$, there is an atom below y.
- $\sim$-INF(x): there are infinitely many $\sim$-inequivalent elements below x, where $a \sim b$ if the symmetric difference $(a - b) \vee (b - a)$ is finite.
- NOMAXATOMLESS(x): x is not a join of atomless and atomic elements.

These predicates are expressible in the language of Boolean algebras allowing *computable* infinite conjuctions and disjunctions, $L^c_{\omega_1\omega}$, and we measure their syntactic complexity by Σ^c_n and Π^c_n formulas for $n \in \omega$. (See [AK00, Chapter 7] for more.)

Π_0^c (atomic): NON-ZERO,
Π_1^c: ATOM,
Π_2^c: INF, ATOMLESS,
Π_3^c: ATOMINF, ATOMIC, 1-ATOM,
Π_4^c: NOMAXATOMLESS, $\sim$-INF,

2.2. **Ash-Knight Metatheorem.** This section describes the metatheorem of [AK00, Chapter 13]. The metatheorem provides a list of conditions guaranteeing the success of a priority construction. For a Δ_n^0 priority construction, the information needed to meet the requirements of the object of the priority construction is enumerated by a Δ_n^0 function. All other objects associated with the construction are c.e. and collected together into an *n-system*, which is a tuple of objects:

$$(L, U, \hat{\ell}, \mathcal{T}, E, (\preccurlyeq_m)_{m<n}).$$

The n-system is a means of organizing a Δ_n^0 construction. There will be a collection of requirements $\{\mathcal{R}_k\}_{k\in I}$ which the construction tries to satisfy. U is a c.e. set of objects which provide the information needed by the construction to see that the requirement is satisfied. The c.e. set L provides a collection of the possible responses by the system to extend the construction so that a particular requirement is satisfied. $\hat{\ell} \in L$ is a special object used to get the construction going.

The "call" from U and the "response" from L are organized on a non-terminal tree $\mathcal{T}$ which alternates objects from L and U:

$$\hat{\ell} u_1 \ell_1 \ldots u_n \ell_n \ldots$$

where $\ell_n \in L$ and $u_n \in U$. (We sometimes label the special object $\hat{\ell}$ as ℓ_0, but will have no corresponding u_0.)

An *instruction function* q is a function which assigns an element of U to each odd length sequence in $\mathcal{T}$ (ones ending in an element of L) so that the chosen element extends the sequence: $q(\sigma) = u$ implies $\sigma u \in \mathcal{T}$. A *run* $(\mathcal{T}, q)$ through $\mathcal{T}$ is a path

$$\hat{\ell} u_1 \ell_1 \ldots u_n \ell_n \ldots$$

in which q determines the choices of the elements of U:

$$u_{n+1} = q\big(\hat{\ell} u_1 \ell_1 \ldots u_n \ell_n\big).$$

E is a partial computable *enumeration* function on L which assigns a finite set to each element of L. The object $E(\ell)$ is a finite snapshot of the object to be constructed based on the response ℓ at a given stage of construction. If π is a run (P, q), then $E(\pi) = \cup_n E(\ell_n)$ is the object given by the priority construction.

In a Δ_n^0 construction the instruction function q is Δ_n^0, For an alternating sequence σ of length $2n+1$, $q(\sigma)$ provides the information needed to satisfy requirement $\mathcal{R}_n$ based on the construction so far (represented by σ). The goal of the construction is to produce a run π through $\mathcal{T}$ such that $E(\pi)$ satisfies all the requirements and is c.e. . Since π is unlikely to be computable, we need further conditions to guarantee that $E(\pi)$ is c.e. .

This provided by a set of c.e. relations $(\preccurlyeq_m)_{m<n}$ on L. The role these relations play in the priority construction is that they allow the construction return to earlier stages in the approximation of the instruction function q and thus "recover" to an earlier state

of the construction when the approximation to q appears to be correct. These relations satisfy four conditions:

(1) $\preccurlyeq_m$ is reflexive and transitive.
(2) For $k \leq m$, $\ell \preccurlyeq_m \ell'$ implies $\ell \preccurlyeq_k \ell'$.
(3) $\ell \preccurlyeq_0 \ell'$ implies $E(\ell) \subseteq E(\ell')$.
(4) Let $\sigma u \in P$, where σ ends in ℓ^0. Suppose that

$$\ell^0 \preccurlyeq_{n-1} \ell^1 \preccurlyeq_{n-2} \ldots \preccurlyeq_1 \ell^{n-1}.$$

Then there exists an ℓ^* such that $\sigma u \ell^* \in \mathcal{T}$ and

$$\ell^0 \preccurlyeq_{n-1} \ell^*, \ell^1 \preccurlyeq_{n-2} \ell^*, \ldots, \ell^{n-1} \preccurlyeq_0 \ell^*.$$

Here is the metatheorem for n-systems from [AK00, Theorem 13.2]:

Theorem 2.1. *Let $(L, U, \hat{\ell}, \mathcal{T}, E, (\preccurlyeq_m)_{m<n})$ be an n-system and q a Δ^0_n instruction function for $\mathcal{T}$. Then there is a Δ^0_n run π of $(\mathcal{T}, q)$ such that $E(\pi)$ is c.e. .*

We will use the metatheorem for the case $n = 3$ in this paper.

3. Tools for the construction

3.1. **Vaught's Theorem.** In the study of countable Boolean algebras one typically uses a back-and-forth construction to prove two algebras are isomorphic. The most useful such tool is Vaught's Isomorphism Theorem. (See [Pie89, Theorem 1.1.3]).

Definition 3.1 (V-relation)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be countable Boolean algebras. A subset R of $\mathcal{A} \times \mathcal{B}$ is a *V-relation* between $\mathcal{A}$ and $\mathcal{B}$ if

(i) $1_{\mathcal{A}} R 1_B$;
(ii) $aR0_{\mathcal{B}}$ implies $a = 0_{\mathcal{A}}$ and $0_{\mathcal{A}} R b$ implies $b = 0_{\mathcal{B}}$;
(iii) $aR(b_1 \dot{\vee} \ldots \dot{\vee} b_k)$ implies $a = a_1 \dot{\vee} \ldots \dot{\vee} a_k$, where $x_i R y_i$ for each $i \leq k$; $(a_1 \dot{\vee} \ldots \dot{\vee} a_k) R y$ implies $b = b_1 \dot{\vee} \ldots \dot{\vee} b_k$, where $a_i R b_i$ for each $i \leq k$;

Theorem 3.2 (Vaught's Theorem)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be countable Boolean algebras. Then $\mathcal{A} \cong \mathcal{B}$ if and only if there exists a V-relation between $\mathcal{A}$ and $\mathcal{B}$.*

If R is a V-relation between $\mathcal{A}$ and $\mathcal{B}$ and $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism, then $b = f(a)$ implies the existence of decompositions $a = a_0 \dot{\vee} \ldots \dot{\vee} a_k$ and $b = b_0 \dot{\vee} \ldots \dot{\vee} b_k$ for which $a_i R b_i$ and $f(a_i) = b)i$ for each $i \leq k$.

We use Vaught's Theorem to prove the following due to [Rem81].

Theorem 3.3. *Let $\mathcal{A}$ be a countable Boolean algebra with infinitely many atoms and let $\mathcal{B}$ be a computable extension of $\mathcal{A}$ satisfying the following conditions:*

(1) $\mathcal{B}$ is generated by elements of $\mathcal{A}$ and new atoms lying below atoms of $\mathcal{A}$,
(2) if a is an atom in $\mathcal{A}$, then it is a finite join of atoms in $\mathcal{B}$.

Then $\mathcal{A} \cong \mathcal{B}$.

If $\mathcal{A}$ and $\mathcal{B}$ are computable, then there is a Δ^0_4-computable isomorphism $F : \mathcal{A} \cong \mathcal{B}$.

Proof. Let $0_{\mathcal{A}} R 0_B$ and $1_{\mathcal{A}} R 1_{\mathcal{B}}$ and for all other $a \in \mathcal{A}$ and $b \in \mathcal{B}$ let aRb if and only if $a \sim b$ (the symmetric difference of a and b is finite) and both a and b bound the same number of atoms. It remains to show condition (iii) of Theorem 3.2 holds. For the first half of this condition, It sufficient for this to show that if $aR(b_1 \dot\vee b_2)$ then there exists a partition $a = a_1 \dot\vee a_2$ such that a_1Rb_1 and a_2Rb_2. Let $a'_1 = a \wedge b_1$ and $a'_2 = a - a'_1$. Since $a \sim (b_1 \dot\vee b_2)$, it follows that $a'_1 \sim b_1$ and $a'_2 \sim b_2$ and this remains true if we shift finitely many atoms between a'_1 and a'_2. If a'_1 has n fewer atoms than b_1 then either a'_2 and b_2 bound infinitely many atoms or a'_2 bounds n more atoms than b_2 bounds, so let $a_1 = a'_1 \dot\vee c$ where $c \leq a'_2$ is a join of n atoms and $a_2 = a'_2 - c$, so that a_1Rb_1 and a_2Rb_2. If a'_1 has n more atoms than b_1, we can similarly shift n atoms from a'_1 to a'_2 to produce a_1 and a_2 with a_1Rb_1 and a_2Rb_2. The second half of condition (iii) is similarly established.

If follows from the proof of Theorem 3.2 that if the relation R in that theorem is Z-computable, then a witnessing isomorphism $f : \mathcal{A} \cong \mathcal{B}$ will also be Z-computable. It is sufficient for R as defined in the previous paragraph to be Z-computable that the predicates $\text{ATOM}(x)$, $\text{ATOMLESS}(x)$ and $\text{ATOMINF}(x)$ to be Z-computable on both $\mathcal{A}$ and $\mathcal{B}$. Since both $\mathcal{A}$ and $\mathcal{B}$ are computable, these predicates are Δ_4^0, and so there is a Δ_4^0-computable witnessing isomorphism $f : \mathcal{A} \cong \mathcal{B}$. □

3.2. **Back-and-forth relations.** The *standard back-and-forth relations* $\leq_\gamma$ between Boolean algebras are defined in [AK00, §15.3.4] for all $\gamma < \omega_1$, but we only need the definition for $n \in \omega$.

Definition 3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras.

(i) $\mathcal{A} \leq_0 \mathcal{B}$ if and only if either both $\mathcal{A}$ and $\mathcal{B}$ are trivial one-element Boolean algebras.

(ii) $\mathcal{A} \leq_{n+1} \mathcal{B}$ if and only if for every partition $1_\mathcal{B} = b_0 \dot\vee \ldots \dot\vee b_k$ of $\mathcal{B}$, there exists a partition $1_\mathcal{A} = a_0 \dot\vee \ldots \dot\vee a_k$ of $\mathcal{A}$ such that $\mathcal{B}\restriction b_i \leq_n \mathcal{A}\restriction a_i$ for each $i \leq k$.

The relation $\leq_n$ is transitive and reflexive for each n.

We write $\mathcal{A} \equiv_n \mathcal{B}$ if and only if $\mathcal{A} \leq_n \mathcal{B}$ and $\mathcal{B} \leq_n \mathcal{A}$.

The purpose behind the use of these relations is to identify the computable Boolean algebras which cannot be distinguished by $0^{(n)}$.

Theorem 3.5 (Karp; Ash and Knight)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be non-isomorphic Boolean algebras. The following are equivalent:*

(1) Given a Boolean algebra $\mathcal{C}$ that is isomorphic to either $\mathcal{A}$ or $\mathcal{B}$, deciding whether $\mathcal{C} \cong \mathcal{A}$ is (boldface) $\mathbf{\Sigma}_n^0$-hard.

(2) All the infinitary Σ_n sentences true in $\mathcal{B}$ are true in $\mathcal{A}$.

(3) $\mathcal{A} \leq_n \mathcal{B}$.

In [HM] we studied the family of *ordered monoids*

$$(\mathbf{BA}/\equiv_n\,,\ \leq_n\,,\ \oplus).$$

We call the equivalence classes *bf-types*, or *n-bf-types*. We showed that there is a computably presentable ordered monoid $(\mathbf{INV}_n\,,\ \leq_n\,,\ +)$ with

$$(\mathbf{BA}/\equiv_n\,, \leq_n\,, \oplus)\ \cong\ (\mathbf{INV}_n\,, \leq_n\,, +),$$

and we built a map T_n from Boolean algebras to $\mathbf{INV}_n$ such that

$$\mathcal{A} \leq_n \mathcal{B} \iff T_n(\mathcal{A}) \leq_n T_n(\mathcal{B}) \quad \text{and} \quad T_n(\mathcal{A} \oplus \mathcal{B}) = T_n(\mathcal{A}) + T_n(\mathcal{B}).$$

Here are the first four levels for the back-and-forth relations:

- $\mathcal{A} \leq_0 \mathcal{B}$ if and only if ($|\mathcal{A}| = 1 \iff |\mathcal{B}| = 1$).
- $\mathcal{A} \leq_1 \mathcal{B}$ if and only if $\mathcal{A} \leq_0 \mathcal{B}$ and $|\mathcal{A}| \geq |\mathcal{B}|$.
- $\mathcal{A} \leq_2 \mathcal{B}$ if and only if $|\mathcal{A}| = |\mathcal{B}|$ and $|\mathrm{At}(\mathcal{A})| \geq |\mathrm{At}(\mathcal{B})|$, where $\mathrm{At}(\mathcal{B})$ is the set of atoms of $\mathcal{B}$.
- $\mathcal{A} \leq_3 \mathcal{B}$ if and only if $|\mathcal{A}| = |\mathcal{B}|$, $|\mathrm{At}(\mathcal{A})| = |\mathrm{At}(\mathcal{B})|$, ($\mathcal{A}$ atomic $\Rightarrow \mathcal{B}$ atomic), and $|\mathcal{A}/F(\mathcal{A})| \geq |\mathcal{B}/F(\mathcal{B})|$, where $F(\mathcal{A})$ is the ideal of finite joins of atoms in $\mathcal{A}$.

If $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then we will write $a \leq_n b$ ($n = 0, 1, 2$) meaning $\mathcal{A} \restriction a \leq_n \mathcal{B} \restriction b$.

The key to our investigation of the n-back-and-forth types in [HM] are the n-indecomposable Boolean algebras:

Definition 3.6. A non-trivial Boolean algebra $\mathcal{A}$ is *n-indecomposable* if for every partition $1_{\mathcal{A}} = a_0 \dot{\vee} \ldots \dot{\vee} a_k$, there is some $i \leq k$ with $\mathcal{A} \equiv_n \mathcal{A} \restriction a_i$.

We list the n-indecomposable Boolean algebras for $n = 0, 1, 2, 3$ as well as provide a name for the algebras which fall into these classes:

(0) Every non-trivial Boolean algebra is 0-indecomposable (NON-ZERO).
(1) The 1-indecomposable algebras are determined by how many non-zero elements they bound:
   - two element algebras, which bound one element (ATOM), and
   - infinite algebras (INF).
(2) The 2-indecomposable algebras include ATOM together with two new algebras bounding infinitely many elements:
   - algebras bounding no atoms (ATOMLESS), and
   - algebras bounding infinitely many atoms (ATOMINF).
(3) The 3-indecomposable algebras include ATOM and ATOMLESS together with three new algebras bounding infinitely many atoms:
   - atomic algebras $\mathcal{A}$ for which $|\mathcal{A}/F(\mathcal{A})| = 1$, all of which are isomorphic to the interval algebra $\mathrm{I}(\omega)$ (1-ATOM);
   - atomic algebras $\mathcal{A}$ for which $|\mathcal{A}/F(\mathcal{A})|$ is infinite, such as the interval algebras $\mathrm{I}(\omega^2)$ and $\mathrm{I}(\omega \cdot \eta)$ (AT/$\sim$-INF); and
   - algebras which cannot be split into an atomic and an atomless part, such as the interval algebras $\mathrm{I}(\omega + \eta)$ and $\mathrm{I}(\omega^2 + \eta)$ (NOMAX).

These algebras will be the building blocks of our invariants:

$$\begin{aligned} \mathbf{BF}_0 &= \{\text{NON-ZERO}\} \\ \mathbf{BF}_1 &= \{\text{ATOM}, \text{INF}\} \\ \mathbf{BF}_2 &= \{\text{ATOM}, \text{ATOMLESS}, \text{ATOMINF}\} \\ \mathbf{BF}_3 &= \{\text{ATOM}, \text{ATOMLESS}, \text{1-ATOM}, \text{AT}/\sim\text{-INF}, \text{NOMAX}\} \end{aligned}$$

The key to proceeding is that every Boolean algebra can be decomposed into a finite disjoint sum of n-indecomposable subalgebras. This result is generalized to $n \in \omega$ in [HM], but we will only need it for $n \leq 3$.

Theorem 3.7. *Every non-trivial Boolean algebra can be partitioned into a finite disjoint sum of n-indecomposable subalgebras (for $n \leq 3$).*

Proof. Every non-trivial Boolean algebra is 0-indecomposable. Every infinite Boolean algebra is 1-indecomposable with type INF, so if an algebra $\mathcal{A}$ is not 1-indecomposable, it must be finite, and so it is possible to partition $\mathcal{A}$ as $\mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_k$, where each $\mathcal{A}_i$ is a two-element subalgebra of $\mathcal{A}$ and is thus 1-indecomposable with type ATOM. This establishes the theorem for $n = 1$.

Every finite algebra can be partitioned into a disjoint sum of 2-indecomposable subalgebras and every infinite algebra that bounds infinitely many atoms is 2-indecomposable with type ATOMINF. Suppose $\mathcal{A}$ is an infinite algebra which bounds finitely many atoms, then it is possible to partition $\mathcal{A}$ as $\mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_k$, where $\mathcal{A}_0$ bounds no atoms and is infinite and so is 2-indecomposable with type ATOMLESS, and each $\mathcal{A}_i$ is a two-element algebra so has type ATOM. This establishes the theorem for $n = 2$.

If $\mathcal{A}$ is finite or bounds finitely many atoms, then it can be partitioned into 3-indecomposable subalgebras each of whose type is ATOMLESS or ATOM. Let $\mathcal{A}$ be atomic and infinite, so it bounds infinitely many atoms. If $\mathcal{A}$ can only be partitioned into finitely many disjoint subalgebras which are infinite, then there is a partition of $\mathcal{A}$ as $\mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_k$ where each $\mathcal{A}_i$ is infinite and atomic, but none can be further partitioned into two such subalgebras. Thus, each $\mathcal{A}_i$ is a 1-atom and so 3-indecomposable with type 1-ATOM. If for each $k \in \omega$, there is a partition of $\mathcal{A}$ as $\mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_k$ where each $\mathcal{A}_i$ is infinite, then $|A$ is itself 3-indecomposable with type AT/$\sim$-INF. Suppose $\mathcal{A}$ is not atomic but bounds infinitely many atoms and has a maximal atomless element, then $\mathcal{A}$ can be decomposed as $\mathcal{A}_0 \oplus \mathcal{B}$, where $\mathcal{B}$ is atomless and so is 3-indecomposable with type ATOMLESS and $\mathcal{A}_0$ is infinite and atomic, so can be partitioned into 3-indecomposable subalgebras Thus, $\mathcal{A}$ can be partitioned into 3-indecomposable subalgebras as well. Finally, if $\mathcal{A}$ is not atomic, bounds infinitely many atoms and has no maximal atomless element, then it is already 3-indecomposable with type NOMAX. This establishes the theorem for $n = 3$. □

For each $n = 0, 1, 2$ we will define an ordered commutative monoid built from the indecomposable classes $\mathbf{BF}_n$, $\mathbf{INV}_n = \langle \mathbf{BF}_n, 0_n, +_n, \leq_n \rangle$. The operator $+_n$ is commutative and associative which is compatible with the ordering $\leq_n$: $x \leq_n y$ implies that $x +_n z \leq_n y +_n z$. We will drop the subscripts on $0, +, \leq, =$ when it is clear which monoid is intended. The monoid $\mathbf{INV}_3$ will not play a role in this paper, but only the 3-indecomposable classes $\mathbf{BF}_3$ (and the relations $\leq_3$ between these classes) without the additional structure. We will also define monoid morphisms T_n from $(\mathbf{BA}/\equiv_n, \oplus, \leq_n)$ so that $T_n(\mathcal{A}) = 0_n$ if and only if $\mathcal{A}$ is a trivial (one element) algebra.

The monoid $\mathbf{INV}_0$ is obtained from the free monoid generated by $\mathbf{BF}_0$ with the additional relation

$$\text{NON-ZERO} + \text{NON-ZERO} = \text{NON-ZERO}.$$

Thus, $\mathbf{INV}_0$ has only two objects: 0 and NON-ZERO. The map $T_0(\mathcal{A}) =$ NON-ZERO if and only if $\mathcal{A}$ is a non-trivial algebra.

The monoid $\mathbf{INV}_1$ is obtained from the free monoid generated by $\mathbf{BF}_1$ with the additional relations

$$\begin{aligned} \textsc{inf} &< \textsc{atom}, \\ \textsc{inf} &= \textsc{inf} + \textsc{inf}, \\ \textsc{inf} &= \textsc{inf} + \textsc{atom}, \\ \textsc{atom} + \textsc{atom} &< \textsc{atom} \end{aligned}$$

Thus, the objects of $\mathbf{INV}_1$ are 0, ATOM, INF, and finite sums of ATOM. The $\equiv_1$-equivalence classes are determined by the cardinality of the non-zero elements in the algebra, so it is clear that there is a 1-1 correspondence between these classes and the invariants. By Theorem 3.7 every algebra $\mathcal{A}$ can be partitioned into 1-indecomposable subalgebras $\mathcal{A} = \mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_k$, so we let $T_1(\mathcal{A}) = T_1(\mathcal{A}_0) + \cdots + T_1(\mathcal{A}_k)$.

The monoid $\mathbf{INV}_2$ is obtained from the free monoid generated by $\mathbf{BF}_2$ with the additional relations

$$\begin{aligned} \textsc{atominf} &\leq \textsc{atomless}, \\ \textsc{atominf} &= \textsc{atominf} + \textsc{atom}, \\ \textsc{atominf} &= \textsc{atominf} + \textsc{atomless}, \\ \textsc{atomless} &= \textsc{atomless} + \textsc{atomless}, \end{aligned}$$

Thus, the objects of $\mathbf{INV}_2$ are 0, ATOM, ATOMLESS, ATOMINF, finite sums of ATOM, and ATOMLESS plus finite sums of ATOMs. Since $\equiv_2$ is determined by the cardinality of non-zero elements in the algebra and the cardinality of the atoms, it is clear that there is a 1-1 correspondence between these classes and the invariants. By Theorem 3.7 every algebra $\mathcal{A}$ can be partitioned into 2-indecomposable subalgebras $\mathcal{A} = \mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_k$, so we let $T_2(\mathcal{A}) = T_2(\mathcal{A}_0) + \cdots + T_2(\mathcal{A}_k)$.

Finally, we point out the relations on $\mathbf{BF}_3$: ATOM and ATOMLESS are incomparable with each other and the other three 3-indecomposable types, while

$$\textsc{nomax} < \textsc{at}/{\sim}\text{-}\textsc{inf} < 1\text{-}\textsc{atom}.$$

We won't need any further details about the monoid $\mathbf{INV}_3$.

3.3. **Approximations.** All Boolean algebras considered in this paper are countable, with fixed universe ω. It will be useful to view such an algebra as a directed sum of finite subalgebras. An element c in an algebra $\mathcal{C}$ is *minimal* in $\mathcal{C}$ if the only element it bounds in $\mathcal{C}$ is the zero element. (The element c is an atom of the algebra $\mathcal{C}$, but we may be thinking about $\mathcal{C}$ as a subalgebra of a larger algebra $\mathcal{A}$ in which c is not an atom. For this reason we distinguish minimal elements in an algebra from atoms.)

Lemma 3.8. *If $\mathcal{C}$ is a finite subalgebra of a Boolean algebra $\mathcal{A}$ and $a \in \mathcal{A}$, then the subalgebra of $\mathcal{A}$ generated by $\mathcal{C} \cup \{a\}$ is finite.*

Proof. Let X consist of the elements $x \wedge a$ and $x \wedge -a$, where x is a minimal element in the finite subalgebra $\mathcal{C}$ (so X is finite). The subalgebra generated by $\mathcal{C} \cup \{a\}$ is obtained by collecting joins of the elements in X. □

Let $\mathcal{A}$ be a computable Boolean algebra. We will now explain in what sense there exists a Δ^0_{n+1}-computable *approximation* of the n-back-and-forth types of the elements of $\mathcal{A}$ (for $n = 0, 1, 2$). This section is based on material in [HM2, Section 3].

A finite n-labeled Boolean algebra is a finite algebra whose elements are labeled with n-types from $\mathbf{INV}_n$.

Definition 3.9. A *finite n-labeled Boolean algebra* is a pair $(\mathcal{C}, \mathfrak{t}_n^{\mathcal{C}})$, where $\mathcal{C}$ is a finite Boolean algebra and $\mathfrak{t}_n^{\mathcal{C}} \colon \mathcal{C} \to \mathbf{INV}_n$ is such that

(1) every minimal element of $\mathcal{C}$ is given a type in $\mathbf{BF}_n$;
(2) if $b = \dot{\bigvee}{}_{i=0}^{k} a_i$, where each a_i is a minimal element in $\mathcal{C}$, then $\mathfrak{t}_n^{\mathcal{C}}(b) = \sum_{i=0}^{k} \mathfrak{t}_n^{\mathcal{C}}(a_i)$.

Given two finite n-labeled Boolean algebras $\mathcal{C}_0$ and $\mathcal{C}_1$, we write

$$\mathcal{C}_0 \subseteq_n \mathcal{C}_1$$

if $\mathcal{C}_0$ is a subalgebra of $\mathcal{C}_1$ and for every $x \in \mathcal{C}_0$, $\mathfrak{t}_n^{\mathcal{C}_0}(x) = \mathfrak{t}_n^{\mathcal{C}_1}(x)$.

We write

$$\mathcal{C}_0 \leq_n \mathcal{C}_1$$

if $\mathcal{C}_0$ is a subalgebra of $\mathcal{C}_1$ and for every $x \in \mathcal{C}_0$, $\mathfrak{t}_n^{\mathcal{C}_0}(x) \leq_n \mathfrak{t}_n^{\mathcal{C}_1}(x)$.

Remark 3.10. Any finite 1-labeled algebra $(\mathcal{C}, \mathfrak{t}_1^{\mathcal{C}})$ can be also viewed as a 0-labeled algebra by defining $\mathfrak{t}_0^{\mathcal{C}}(c) = \text{NON-ZERO}$ for each non-zero $c \in \mathcal{C}$. Any finite 2-labeled algebra $(\mathcal{C}, \mathfrak{t}_2^{\mathcal{C}})$ can also viewed as a 1-labeled algebra as follows. For each minimal $c \in \mathcal{C}$, let $\mathfrak{t}_1^{C}(c) = \text{ATOM}$ if $\mathfrak{t}_2^{C}(c) = \text{ATOM}$ and $\mathfrak{t}_1^{C}(c) = \text{INF}$ otherwise. Then extend $\mathfrak{t}_1^{\mathcal{C}}$ to all non-minimal elements of $\mathcal{C}$ by extending using the definition of $+_1$.

Definition 3.11. An *n-approximation* is a sequence $\{(\mathcal{C}_k, \mathfrak{t}_n^k) : k \in \omega\}$ of finite n-labeled Boolean algebras such that each of the following hold:

For each k and each minimal b in $\mathcal{C}_k$, if $b = b_0 \dot{\vee} \dots \dot{\vee} b_m$, where each b_i is minimal in $\mathcal{C}_{k+1}$, then the following conditions hold (depending on $n = 0, 1, 2$):

(0) For $n = 0$: $\mathfrak{t}_0^{k+1} = \text{NON-ZERO}$.
(1) For $n = 1$, one of the following holds:
   (a) $\mathfrak{t}_1^k(b) = \text{ATOM} = \mathfrak{t}_1^{k+1}(b)$ and $m = 0$;
   (b) $\mathfrak{t}_1^k(b) = \text{INF} = \mathfrak{t}_1^{k+1}(b)$, $m \geq 1$ and there is some i with $\mathfrak{t}_1^{k+1}(b_i) = \text{INF}$; or
   (c) $\mathfrak{t}_1^k(b) = \text{INF} <_1 \mathfrak{t}_1^{k+1}(b)$, $m \geq 1$ and $\mathfrak{t}_1^{k+1}(b_i) = \text{ATOM}$ for each i.
(2) For $n = 2$, one of the following holds:
   (a) $\mathfrak{t}_2^k(b) = \text{ATOM} = \mathfrak{t}_2^{k+1}(b)$ and $m = 0$;
   (b) $\mathfrak{t}_2^k(b) = \text{ATOMLESS} = \mathfrak{t}_2^{k+1}(b)$, $m \geq 1$ and $\mathfrak{t}_2^{k+1}(b_i) = \text{ATOMLESS}$ for each i;
   (c) $\mathfrak{t}_2^k(b) = \text{ATOMINF} = \mathfrak{t}_2^{k+1}(b)$, $m \geq 1$ and $\mathfrak{t}_2^{k+1}(b_i) = \text{ATOMINF}$ for some i and $\mathfrak{t}_2^{k+1}(b_j) = \text{ATOM}$ for some j; or
   (d) $\mathfrak{t}_2^k(b) = \text{ATOMINF} <_2 \mathfrak{t}_2^{k+1}(b)$, $m \geq 1$ and $\mathfrak{t}_2^{k+1}(b_i) = \text{ATOMLESS}$ for some i and $\mathfrak{t}_2^{k+1}(b_j) = \text{ATOM}$ for some j.

We will say that a minimal element $b \in \mathcal{C}_k$ splits into an *n-full partition* in $\mathcal{C}_{k+1}$ if the appropriate condition (n) holds (for $n = 0, 1, 2$). Note that a sequence defined in this way satisfies $\mathcal{C}_k \leq_n \mathcal{C}_{k+1}$ for each k.

We will say an n-approximation $\{(\mathcal{A}_k, \mathfrak{t}_n^k)\}_{k\in\omega}$ is an *n-approximation for $\mathcal{A}$* if $\mathcal{A} = \cup_k \mathcal{A}_k$ and $T_n^{\mathcal{A}}(x) = \lim_{k\to\infty} \mathfrak{t}_n^k(x)$ and that a Boolean algebra $\mathcal{A}$ is *n-Z-approximable* if there is a Z-computable n-approximation for $\mathcal{A}$.

Lemma 3.12. *Every computable Boolean algebra $\mathcal{A}$ is n-Δ_n^0-approximable ($n = 0, 1, 2$).*

Proof. Let $\mathcal{A}$ be a computable Boolean algebra.

If $\mathcal{A}$ is a finite Boolean, the lemma clearly holds. So, we will assume $\mathcal{A}$ is infinite, and since $\mathcal{A}$ is computable, its domain can be computably enumerated $\langle a_0, a_1, \ldots\rangle$ where $a_0 = 0_{\mathcal{A}}$ and $a_1 = 1_{\mathcal{A}}$.

$\mathcal{A}$ is 0-Δ_1^0-approximable. Let $\mathcal{A}_0$ be the 2-element algebra generated from $1_{\mathcal{A}}$ and $0_{\mathcal{A}}$ and let $\mathfrak{t}_0^0(1_{\mathcal{A}}) = \text{NON-ZERO}$ and $\mathfrak{t}_0^0(0_{\mathcal{A}}) = 0$. Given $(\mathcal{A}_{k-1}, \mathfrak{t}_0^{k-1})$, if no new element has appeared in the enumeration of $\mathcal{A}$, let $\mathcal{A}_k = \mathcal{A}_{k-1}$. Suppose a_j is the next element to appear in the enumeration and we will extend $\mathcal{A}_{k-1}$ to include a_j. For each minimal element $b \in \mathcal{A}_{k-1}$ which splits into non-zero elements $b_0 = a_j \wedge b$ and $b_1 = b - b_0$, let $\mathfrak{t}_0^k(b_0) = \mathfrak{t}_0^k(b_1) = \text{NON-ZERO}$. We extend $\mathfrak{t}_0^k$ to all of $\mathcal{A}_k$ by setting $\mathfrak{t}_0^k(b) = \text{NON-ZERO}$ for each new non-zero $b \in \mathcal{A}_k$ (if $b \in \mathcal{A}_{k-1}$. The sequence $\{(\mathcal{A}_k, \mathfrak{t}_n^k)\}_{k\in\omega}$ defined in this way satisfies Definition 3.9 for a 0-approximation. Furthermore, $\mathcal{A} = \cup_k \mathcal{A}_k$ and for each non-zero $a \in \mathcal{A}_k$, $T_0^{\mathcal{A}}(a) = \text{NON-ZERO} = \lim_{k\to\infty} \mathfrak{t}_0^k(a)$, so that $\{(\mathcal{A}_k, \mathfrak{t}_n^k)\}_{k\in\omega}$ is a 0-approximation for $\mathcal{A}$.

$\mathcal{A}$ is 1-Δ_2^0-approximable. The key is that it is Δ_2^0 decidable whether an element of $\mathcal{A}$ is an atom. Let $\mathcal{A}_0$ be the 2-element algebra generated from $1_{\mathcal{A}}$ and $0_{\mathcal{A}}$ and let $\mathfrak{t}_1^0(1_{\mathcal{A}}) = \text{INF}$ and $\mathfrak{t}_1^0(0_{\mathcal{A}}) = 0$. Given $(\mathcal{A}_{k-1}, \mathfrak{t}_1^{k-1})$, which includes a_j for each $j < k$, if the enumeration of $\mathcal{A}$ is $\{a_0, \ldots, a_j\}$ where $j < k$, then this is Δ_2^0 decidable and each minimal element in $\mathcal{A}_{k-1}$ is really an atom of $\mathcal{A}$ and labeled ATOM in $\mathcal{A}_{k-1}$, so let $\mathcal{A}_k = \mathcal{A}_{k-1}$. Otherwise, extend $\mathcal{A}_{k-1}$ to include a_k by splitting each minimal element $b \in \mathcal{A}_{k-1}$ by $b_0 = a_k \wedge b$ and $b_1 = b - b_0$. For each minimal element $b \in \mathcal{A}_{k-1}$ which has not yet been split by a_k and for which $\mathfrak{t}_1^{k-1}(b) = \text{INF}$, there is a splitting $b = b_0 \dot\vee b_1$ in $\mathcal{A}$ since b cannot be an atom in $\mathcal{A}$, so extend $\mathcal{A}_{k-1}$ to $\mathcal{A}_k$ by also including a splitting of each such b. For each minimal element c of $\mathcal{A}_k$, set $\mathfrak{t}_1^k(c) = \text{ATOM}$ if c is an atom in $\mathcal{A}$ and $\mathfrak{t}_1^k(c) = \text{INF}$ otherwise, and extend $\mathfrak{t}_1^k$ to all of $\mathcal{A}_k$ by the definition 3.9 of a 1-labeled algebra. Let $b \in \mathcal{A}_{k-1}$ be minimal. If $\mathfrak{t}_1^{k-1}(b) = \text{ATOM}$, the b is an atom of $\mathcal{A}$, so that $\mathfrak{t}_1^k(b) = \text{ATOM}$ and b does not split in $\mathcal{A}_k$. If $\mathfrak{t}_1^{k-1}(b) = \text{INF}$, then there is a partition $b = b_0 \dot\vee b_1$ into minimal elements of $\mathcal{A}_k$. If both b_0 and b_1 are atoms of $\mathcal{A}$, then $\mathfrak{t}_1^k(b_0) = \mathfrak{t}_1^k(b_0) = \text{ATOM}$ and $\mathfrak{t}_1^k(b) = \text{ATOM} + \text{ATOM} \geq_1 \text{INF} = \mathfrak{t}_1^{k-1}(b)$; otherwise, some b_i (for $i = 0, 1$) is not an atom, so $\mathfrak{t}_1^k(b_i) = \text{INF}$ and $\mathfrak{t}_1^k(b) = \text{INF} = \mathfrak{t}_1^{k-1}(b)$. Thus, the sequence $\{(\mathcal{A}_k, \mathfrak{t}_n^k)\}_{k\in\omega}$ defined in this way satisfies Definition 3.9 for a 1-approximation. Furthermore, $\mathcal{A} = \cup_k \mathcal{A}_k$ and for each element $a \in \mathcal{A}$, $T_1^{\mathcal{A}}(a) = \lim_{k\to\infty} \mathfrak{t}_1^k(a)$, because the type assigned to a can change at most once.

$\mathcal{A}$ is 2-Δ_3^0-approximable. The key is that it is Δ_3^0 decidable whether an element of $\mathcal{A}$ bounds finitely many or infinitely many elements, and in the later case whether it bounds any atoms or is atomless. We will suppose that $\mathcal{A}$ is infinite, since every finite algebra is clearly 2-Δ_3^0-approximable. Let $\mathcal{A}_0$ be the 2-element algebra generated from

$1_\mathcal{A}$ and $0_\mathcal{A}$ and let $\mathfrak{t}^0_2(1_\mathcal{A}) = $ ATOMLESS if $\mathcal{A}$ is atomless and $\mathfrak{t}^0_2(1_\mathcal{A}) = $ ATOMINF otherwise. Let $\mathfrak{t}^0_2(0_\mathcal{A}) = 0$. Given $(\mathcal{A}_{k-1}, \mathfrak{t}^{k-1}_2)$, we extend to include a_k by splitting each minimal element $b \in \mathcal{A}_{k-1}$ by $b_0 = a_k \wedge b$ and $b_1 = b - b_0$ (one of these may be $0_\mathcal{A}$). If b bounds finitely many elements of $\mathcal{A}$, then $\mathfrak{t}^{k-1}_2(b) = $ ATOM and is an atom of $\mathcal{A}$, so that one of b_0 or b_1 is $0_\mathcal{A}$, therefore let $\mathfrak{t}^k_2(b) = $ ATOM. If b bounds infinitely many elements of $\mathcal{A}$, then $\mathfrak{t}^{k-1}_2(b) = $ ATOMLESS or $\mathfrak{t}^{k-1}_2(b) = $ ATOMINF. If $\mathfrak{t}^{k-1}_2(b) = $ ATOMLESS, then there is a partition of b in $\mathcal{A}$, so we may suppose both b_0 and b_1 are non-zero (or extend $\mathcal{A}_k$ so that b is partitioned into a pair of non-zero elements). If $\mathfrak{t}^{k-1}_2(b) = $ ATOMINF, then one of b_0 or b_1 bounds an atom in $\mathcal{A}$, so suppose it is b_0 and let $b_0 = c_0 \dot\vee c_1$ in $\mathcal{A}$ where c_1 is an atom of $\mathcal{A}$ and extend $\mathcal{A}_k$ to include c_0 and c_1. Finally, if any element that is minimal in $\mathcal{A}_k$ bounds finitely many elements of $\mathcal{A}$, then extend $\mathcal{A}_k$ to include all these elements (which will be atoms of $\mathcal{A}$). For each minimal $c \in \mathcal{A}_k$, let $\mathfrak{t}^k_2(c) = $ ATOM if c is an atom of $\mathcal{A}$, let $\mathfrak{t}^k_2(c) = $ ATOMLESS if c is atomless in $\mathcal{A}$ and let $\mathfrak{t}^k_2(c) = $ ATOMINF if it bounds at least one atom. Extend $\mathfrak{t}^k_2$ from minimal elements of $\mathcal{A}_k$ to all of $\mathcal{A}_k$. Thus, the sequence $\{(\mathcal{A}_k, \mathfrak{t}^k_2)\}_{k\in\omega}$ defined in this way satisfies Definition 3.9 for a 2-approximation. Furthermore, $\mathcal{A} = \cup_k \mathcal{A}_k$ and for each element $a \in \mathcal{A}$, $T^\mathcal{A}_2(a) = \lim_{k\to\infty} \mathfrak{t}^k_2(a)$, because the type assigned to a can change at most once (if it does change between $\mathfrak{t}^k_2$ and $\mathfrak{t}^{k+1}_2$, then a bounds infinitely many elements of $\mathcal{A}$ but only finitely many atoms).

□

Lemma 3.13. *Let $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ be finite 2-labeled Boolean algebras with $\mathcal{C}_0 \leq_2 \mathcal{C}_1 \leq_1 \mathcal{C}_2$. Then, there is a finite 2-labeled Boolean algebra $\mathcal{D}$ such that $\mathcal{C}_0 \subset_2 \mathcal{D}$, $\mathcal{C}_1 \leq_1 \mathcal{D}$ and $\mathcal{C}_2 \leq_0 \mathcal{D}$.*

Proof. Let the domain of $\mathcal{D}'$ be $\mathcal{C}_3$ and take $\mathfrak{t}^{\mathcal{D}'}_0 = \mathfrak{t}^{\mathcal{C}_2}_0$ and $\mathfrak{t}^{\mathcal{D}'}_2 = \mathfrak{t}^{\mathcal{C}_1}_2$. For minimal b in $\mathcal{C}_1$ let the minimal elements in $\mathcal{C}_2$ be $b = c_0 \dot\vee \ldots \dot\vee c_k$. Since $\mathfrak{t}^{\mathcal{C}_1}_1(b) \leq_1 \mathfrak{t}^{\mathcal{C}_2}_1(b)$, if b is labeled ATOMin $\mathcal{C}_1$ then it is in $\mathcal{C}_2$ as well. Suppose $\mathfrak{t}^{\mathcal{C}_1}_1(b) = $ INF, then set $\mathfrak{t}^{\mathcal{D}'}_2(c_0) = \mathfrak{t}^{\mathcal{C}_1}_2(b)$. If $\mathfrak{t}^{\mathcal{C}_1}_2(b) = $ ATOMINF, then set $\mathfrak{t}^{\mathcal{D}'}_2(c_i) = $ ATOM for each $1 \leq i \leq k$, and otherwise set $\mathfrak{t}^{\mathcal{D}'}_2(c_i) = $ ATOMLESS for each $1 \leq i \leq k$. Thus, $\mathcal{C}_2 \leq_0 \mathcal{D}'$ and $\mathcal{C}_1 \subset_2 \mathcal{D}'$.

Since $\mathcal{C}_0 \leq_2 \mathcal{C}_1$ and $\mathcal{C}_1 \subset_2 \mathcal{D}'$, we have $\mathcal{C}_0 \leq_2 \mathcal{D}'$. Let the domain of $\mathcal{D}$ be $\mathcal{D}'$, $\mathfrak{t}^\mathcal{D}_2 = \mathfrak{t}^{\mathcal{C}_0}_2$ and $\mathfrak{t}^\mathcal{D}_1 = \mathfrak{t}^{\mathcal{D}'}_1$ (since $\mathfrak{t}^{\mathcal{C}_0}_2(b) \leq_2 \mathfrak{t}^{\mathcal{D}'}_2(b)$ for any $b \in \mathcal{C}_0$, it follows that $\mathfrak{t}^{\mathcal{C}_0}_1(b) = \mathfrak{t}^{\mathcal{D}'}_1(b)$). For minimal b in $\mathcal{C}_0$ let the minimal elements in $\mathcal{D}'$ be $b = c_0 \dot\vee \ldots \dot\vee c_k$, where $\mathfrak{t}^{\mathcal{C}_0}_1(b) = \mathfrak{t}^{\mathcal{D}'}_1(c_0)$. Extend $\mathfrak{t}^\mathcal{D}_2(c_0) = \mathfrak{t}^{\mathcal{C}_0}_2(b)$. If $\mathfrak{t}^{\mathcal{C}_0}_2(b) = $ ATOMLESS, then $\mathfrak{t}^{\mathcal{D}'}_1(b) = $ ATOMLESS, so b bounds no ATOMs in $\mathcal{D}$. Extend $\mathfrak{t}^\mathcal{D}_1(c_i) = $ ATOMLESS for each $1 \leq i \leq k$. Otherwise, $\mathfrak{t}^{\mathcal{C}_0}_2(b) = $ ATOMINF, so extend $\mathfrak{t}^\mathcal{D}_1(c_i) = $ ATOM for each $1 \leq i \leq k$. (This raises 1-types of elements in $\mathcal{D}'$ not in $\mathcal{C}'$. Thus, $\mathcal{C}_0 \subset_2 \mathcal{D}$ and $\mathcal{D}' \leq_1 \mathcal{D}$, from which it follows that $\mathcal{C}_1 \leq_1 \mathcal{D}$. □

4. Δ^0_2 CATEGORICITY

In this section we prove the equivalence of (1) and (3) of the following:

Theorem 4.1. *The following are equivalent for any computable Boolean algebra $\mathcal{A}$.*

(1) *$\mathcal{A}$ is Δ^0_2-categorical.*
(2) *$\mathcal{A}$ is relatively Δ^0_2-categorical.*
(3) *$\mathcal{A}$ is a finite direct sum of subalgebras $\mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_k$ where each $\mathcal{A}_i$ is atomless, an atom or a 1-atom.*

[McCoy03] proved the equivalence of (2) and (3), and (2) $\implies$ (1) follows trivially from the definitions. We provide a proof of (3) $\implies$ (1) since it is pretty short.

(3) $\implies$ (1). Atoms and computable atomless algebras are Δ^0_1-categorical. Computable 1-atoms are Δ^0_2-categorical because given any way of partitioning a computable 1-atom $\mathcal{A} = \mathcal{A}_0 \dot\vee \mathcal{A}_1$ one subalgebra must be a finite join of atoms and the other subalgebra is a 1-atom, so that Δ^0_2 can decide which is which.

(1) $\implies$ (3). We point out that the 3-bf-types mentioned in (3) (ATOM, ATOMLESS, 1-ATOM) are all $\leq_3$-maximal. Let $\mathcal{A}$ is a computable Boolean algebra for which (3) fails. By Theorem 3.7 there is a partition $\mathcal{A} = \mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_k$ into 3-indecomposable subalgebras, but since (3) fails there is $i \leq k$ for which $\mathcal{A}_i$ has 3-type AT/$\sim$-INF or NOMAX. These are the remaining (non-maximal) 3-bf-types

- AT/$\sim$-INF bounds infinitely many elements of 2-bf-type ATOM (whose 1-bf-type is also ATOM) and infinitely many disjoint elements of 2-bf-type ATOMINF (whose 1-bf-type is INF),
- NOMAX bounds infinitely many elements of 2-bf-type ATOM (whose 1-bf-type is also ATOM) and infinitely many disjoint elements of 2-bf-type ATOMLESS (whose 1-bf-type is INF),

In both cases, we have an algebra $\mathcal{A}$ which bounds infinitely many ATOMs and infinitely many disjoint INFs. (Note that since INF $<_1$ ATOM the problem of distinguishing elements of 1-type INF and finite joins of ATOMs in an algebra is $\mathbf{\Sigma}^0_2$-hard by Theorem 3.5.)

The construction produces a computable Boolean algebra $\mathcal{B}$ which is isomorphic to $\mathcal{A}$ but not via any Δ^0_2-computable isomorphism. We are not actually going to build the witnessing isomorphism, instead we will build an embedding of $\mathcal{A}$ into $\mathcal{B}$ which ensures that they are isomorphic using Theorem 3.3. With this in mind, we prove the following lemma:

Lemma 4.2. *Let $\mathcal{A}$ be a computable Boolean algebra which cannot be decomposed into a finite join of atoms, 1-atoms and atomless subalgebras. Then there is a Δ^0_3-embedding f of $\mathcal{A}$ into a computable Boolean algebra $\mathcal{B}$ such that*

(1) $\mathcal{B}$ is generated by $\operatorname{ran} f$ and new atoms, each lying below an atom $f(a)$,
(2) if a is an atom in $\mathcal{A}$, then $f(a)$ is a finite join of atoms in $\mathcal{B}$,

but no Δ^0_2-isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

It follows from Theorem 3.3 that $\mathcal{B}$ is isomorphic to $\mathcal{A}$ via a Δ^0_4-computable isomorphism and so $\mathcal{A}$ is not Δ^0_2-categorical.

We suppose that the domain of $\mathcal{A}$ is ω and that $1_\mathcal{A} = 1$ and $0_\mathcal{A} = 0$. (Any computable algebra $\mathcal{A}'$ is computably isomorphic to an algebra $\mathcal{A}$ with these properties.) We are going to build a computable Boolean algebra $\mathcal{B}$ whose domain is also ω and an embedding $f : \mathcal{A} \to \mathcal{B}$ that satisfies the two requirements of Lemma 4.2. The construction will be guided by a 3-system

$$(L, U, \hat{\ell}, \mathcal{T}, E, (\preccurlyeq_n)_{n<3})$$

and we will show that there exists a Δ^0_3-instruction function q so that any run π determines a Boolean algebra $\mathcal{B}$ and embedding $f:\mathcal{A}\to\mathcal{B}$ that satisfies the following set of requirements:

- $\mathcal{R}_e$: $\Phi_e^{\Delta^0_2}$ is not a an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Then, we can use Theorem 2.1 to produce a run π of $(\mathcal{T},q)$, where $E(\pi)$ is a computable algebra $\mathcal{B}$ and $f:\mathcal{A}\to\mathcal{B}$ is an embedding satisfying $\mathcal{R}_{\langle k,0\rangle}$, which implies conditions (1) and (2) of Lemma 4.2 (ensuring $\mathcal{B}$ is isomorphic to $\mathcal{A}$), while $\mathcal{R}_{\langle e,0\rangle}$ ensures that there is no Δ^0_2-computable isomorphism.

The elements of L consist of triples $\ell=\langle p,\mathcal{C},\mathcal{D}\rangle$ where $\mathcal{C}$ and $\mathcal{D}$ are finite 2-labeled algebras, and $p:\mathcal{C}\to\mathcal{D}$ is an embedding satisfying the following requirements:

(i) $p:\mathcal{C}\to\mathcal{D}$ is a Boolean algebra homomorphism
(ii) if $a\in\mathcal{C}$ and $\mathfrak{t}_1^{\mathcal{C}}(a)=\text{INF}$, then $\mathfrak{t}_2^{\mathcal{C}}(a)=\mathfrak{t}_2^{\mathcal{D}}(p(a))$,
(iii) if $a\in\mathcal{C}$ and $\mathfrak{t}_2^{\mathcal{C}}(a)=\text{ATOM}$, then $\mathfrak{t}_2^{\mathcal{D}}(p(a))$ is a join of ATOMs.

We will also define the *extended range* for $\langle p,\mathcal{C},\mathcal{D}\rangle$,

- $\overline{\operatorname{ran}}\,p$ is the subalgebra of $\mathcal{D}$ generated by $\operatorname{ran}p$ together with elements labeled ATOM below the image of an ATOM of $\mathcal{C}$.

The map E is defined on L by $E(\langle p,\mathcal{C},\mathcal{D}\rangle)=\mathcal{D}$.

For the special object $\hat{\ell}$, let $1_{\mathcal{B}}=1$ and $0_{\mathcal{B}}=0$ and let $\mathcal{D}$ be the two element 2-labeled algebra generated from $1_{\mathcal{B}}$ and $0_{\mathcal{B}}$ with the label $\mathfrak{t}_2^{\mathcal{D}}(1_{\mathcal{B}})=\text{ATOMINF}$, let $\mathcal{C}$ be the two element algebra generated from $1_{\mathcal{A}}$ and $0_{\mathcal{A}}$ with label $\mathfrak{t}_2^{\mathcal{C}}(1_{\mathcal{A}})=\text{ATOMINF}$, and define p by $p(1_{\mathcal{A}})=1_{\mathcal{B}}$ and $p(0_{\mathcal{A}})=0_{\mathcal{B}}$.

The relations $(\preccurlyeq_n)_{n<3}$ are as follows: for objects of ℓ_0,ℓ_1 of L with $\ell_0=\langle p_0,\mathcal{C}_0,\mathcal{D}_0\rangle$ and $\ell_1=\langle p_1,\mathcal{C}_1,\mathcal{D}_1\rangle$,

- $\ell_0\preccurlyeq_0\ell_1$ if $\mathcal{D}_0\leq_0\mathcal{D}_1$,
- $\ell_0\preccurlyeq_1\ell_1$ if $\mathcal{D}_0\leq_1\mathcal{D}_1$,
- $\ell_0\preccurlyeq_2\ell_1$ if $\ell_0\leq_1\ell_1$, $p_0\subseteq p_1$ and $\overline{\operatorname{ran}}\,p_0\leq_2\overline{\operatorname{ran}}\,p_1$.

These relations are reflexive and transitive, and $\preccurlyeq_{n+1}$ implies $\preccurlyeq_n$ for $n=0,1$.

The elements of U consists of pairs $\langle\mathcal{G},\Gamma\rangle$, where $\mathcal{G}$ is a finite 2-labeled Boolean algebra and Γ is a finite set of ordered pairs satisfying the following conditions

(i) $(a,b)\in\Gamma$ implies $a\in\mathcal{G}$,
(ii) $(a,b)\in\Gamma$ and $(a,b')\in\Gamma$ implies $b=b'$, and
(iii) $(a,b)\in\Gamma$ and $(a',b)\in\Gamma$ implies $a=a'$.

These conditions on Γ imply that Γ is a partial 1-1 function whose domain is a subset of $\mathcal{G}$. Write $\Gamma(a)=b$ when $(a,b)\in\Gamma$ and $\Gamma(a)\uparrow$ when there is no b with $(a,b)\in\Gamma$.

An object $\langle p,\mathcal{C},\mathcal{D}\rangle\in L$ *completes* $\langle\mathcal{G},\Gamma\rangle\in U$ if one of the following occurs:

(i) $\Gamma(x)\uparrow$ for some $x\in\mathcal{G}$,
(ii) Γ is not an injective function on $\mathcal{G}$.
(iii) $\Gamma(1_{\mathcal{A}})\neq 1_{\mathcal{B}}$,
(iv) $\operatorname{ran}\Gamma$ is not a subalgebra of $\mathcal{D}$, or
(v) $\mathfrak{t}_1^{\mathcal{G}}(x)\neq\mathfrak{t}_1^{\mathcal{D}}(\Gamma(x))$ for some $x\in\mathcal{G}$ and $\Gamma(x)\in\overline{\operatorname{ran}}\,p$.

An object $\langle \mathcal{G}, \Gamma\rangle \in U$ *guides* $\langle p, \mathcal{C}, \mathcal{D}\rangle \in L$ if one of the following hold:

(i) $\langle p, \mathcal{C}, \mathcal{D}\rangle$ completes $\langle \mathcal{G}, \Gamma\rangle$; or

(ii) There is an $x \in \mathcal{G}$ labeled ATOMINF for which $\Gamma(x)$ is minimal in $\overline{\operatorname{ran}}\, p$ and a pair of elements $y_1, y_2 < x$ labeled INF in $\mathcal{G}$.

The alternating tree $\mathcal{T}$ consists of all finite sequences $\hat{\ell}, u_1, \ell_1, \ldots$ (recall, $\hat{\ell} = \ell_0$) satisfying the following conditions (writing $\ell_n = \langle p, \mathcal{C}_n, \mathcal{D}_n\rangle$ and $u_n = \langle \mathcal{G}_n, \Gamma_n\rangle$):

(T1) $\ell_n \preccurlyeq_2 \ell_{n+1}$.

(T2) $\mathcal{G}_n \leq_2 \mathcal{G}_{n+1}$ and each minimal element of $\mathcal{G}_n$ splits into a 2-full partition in $\mathcal{G}_{n+1}$ (see Definition 3.11).

(T3) $\mathcal{C}_n = \mathcal{G}_n$.

(T4) $n \in \overline{\operatorname{ran}}\, p, \mathcal{C}_n$.

(T5) u_{n+1} guides ℓ_n.

(T6) ℓ_{n+1} completes u_{n+1}.

$\mathcal{T}$ is clearly a *c.e.* tree. We still need to show that the relations $(\preccurlyeq_i)_{i<3}$ satisfy the final property (4) for Theorem 2.1 ([AK00, Theorem 13.2]), which will also establish that $\mathcal{T}$ is extendible.

- Let $\sigma u \in \mathcal{T}$, where σ ends in ℓ^0. Suppose that

$$\ell^0 \preccurlyeq_2 \ell^1 \preccurlyeq_1 \ell^2.$$

Show. There exists an ℓ^* such that $\sigma u \ell^* \in \mathcal{T}$ and $\ell^i \preccurlyeq_{2-i} \ell^*$ for all $i = 0, 1, 2$.

Write $\ell^i = \langle p^i, \mathcal{C}^i, \mathcal{D}^i\rangle$ for $i = 0, 1, 2$. By Lemma 3.13 there is a 2-labeled algebra $\mathcal{E}$ such that $\mathcal{D}^2 \leq_0 \mathcal{E}$, $\mathcal{D}^1 \leq_1 \mathcal{E}$ and $\mathcal{D}^0 \subset_2 \mathcal{E}$.

Let $u = \langle \mathcal{G}, \Gamma\rangle$, so that u guides ℓ^0 by (T5). The construction proceeds in two steps to construct $\ell^* = \langle p^*, \mathcal{G}, \mathcal{E}^*\rangle$ by extending the 2-labeled algebra $\mathcal{E}$ to $\mathcal{E}^*$ modifying labels and adding new elements. Step (A) ensures that Γ does not embed $\mathcal{G}$ isomorphically into $\mathcal{E}^*$, while step (B) extends the map p^0 to p^* on $\mathcal{G}$, after which we will verify that ℓ^* completes u.

Step (A). The modifications in Step (A) will not change the 2-labels of elements in $\overline{\operatorname{ran}}\, p$, while the 1-labels for all other elements in $\mathcal{E}$ will be non-decreasing. (The reason this is important is elaborated on in Step (B).)

If u guides ℓ^0 because ℓ^0 already completes u, then proceed to step (B). Otherwise, the second condition holds:

- There is an $x \in \mathcal{G}$ labeled ATOMINF for which $\Gamma(x)$ is minimal in $\overline{\operatorname{ran}}\, p^0$ and a pair of elements $y_1, y_2 < x$ labeled INF in $\mathcal{G}$.

The construction will modify $\mathcal{E}$ by (possibly) adding new elements below $\Gamma(x)$ and changing 2-labels of elements below $\Gamma(x)$ in such a way that the 1-labels on either y_1 and $\Gamma(y_1)$ or y_2 and $\Gamma(y_2)$ are different. In addition, after completion of step (A) there will be some minimal element below $\Gamma(x)$ which has the 2-label ATOMINF, so that the 2-label on $\Gamma(x)$ remains ATOMINF.

If $\Gamma(y_1) \notin \mathcal{E}$, then let $z \leq \Gamma(x)$ be minimal in $\mathcal{E}$ with 2-label ATOMINF and extend $\mathcal{E}$ by partitioning $z = \Gamma(y_1) \dot\vee z_1$ and extend the 2-labels by $\mathfrak{t}_2^{\mathcal{E}^*}(\Gamma(y_1)) =$ ATOM and $\mathfrak{t}_2^{\mathcal{E}^*}(z_1) =$ ATOMINF, then proceed to step (B). Note that it is still the case that $\mathfrak{t}_2^{\mathcal{E}^*}(\Gamma(x)) =$ ATOMINF, but now the 1-labels on y_1 and $\Gamma(y_1)$ are different. If $\Gamma(y_2) \notin \mathcal{E}$

and $\Gamma(y_1) \in \mathcal{E}$, then take the same action with $\Gamma(y_2)$. Otherwise, both $\Gamma(y_1) \in \mathcal{E}$ and $\Gamma(y_2) \in \mathcal{E}$. If one of these elements has a 1-label other than INF in $\mathcal{E}$, then there is nothing more to be done, so proceed to step (B). Suppose both $\Gamma(y_1)$ and $\Gamma(y_2)$ have 1-labels INF in $\mathcal{E}$. Modify the 2-labels on $\mathcal{E}$ so that each minimal element in $\mathcal{E}$ below $\Gamma(y_1)$ is labeled ATOM. Now the 1-label on y_1 in $\mathcal{G}$ and $\Gamma(y_1)$ in $\mathcal{E}^*$ are different. Since the 1-label on $\Gamma(y_2)$ is INF, modify the 2-label on $\mathcal{E}$ so that the $\mathfrak{t}_2^{\mathcal{E}^*}(\Gamma(y_2)) =$ ATOMINF by making sure that some minimal element below $\Gamma(y_2)$ with 1-label INF now has 2-label ATOMINF. Note that we have only increased 1-labels below $\Gamma(x)$ without changing the 2-label of $\Gamma(x)$, so that no 2-labels on elements of $\overline{\operatorname{ran}}^0$ have been changed in $\mathcal{E}^*$.

Step (B). In this step we will extend the map p^0 to p^* so that $\operatorname{dom} p^* = \mathcal{G}$, $n \in \overline{\operatorname{ran}}\, p^*$, and every element labeled ATOM, including those generated from step (A), is in $\overline{\operatorname{ran}}\, p^*$. The 1-labels on all elements of $\mathcal{E}$ will be non decreasing in $\mathcal{E}^*$, which (together with the same fact from Step (A)) ensures that $\mathcal{D}^1 \leq_1 \mathcal{E}^*$ and so $\ell^1 \preccurlyeq_1 \ell^*$. Because the 2-labels of elements in $\overline{\operatorname{ran}}\, p$ were not modified in Step (A), we will be able to extend the map p to $p^* : \mathcal{G} \to \mathcal{E}^*$, while only increasing the 2-labels of elements in $\overline{\operatorname{ran}}\, p^0$ – this is because $\mathcal{D}^0 \subset_2 \mathcal{E}$ and $\mathcal{C}^0 \leq_2 \mathcal{G}$ (by (T2) and (T3), as $\sigma u \in \mathcal{T}$) . This ensures that $\overline{\operatorname{ran}}\, p^0 \leq_2 \overline{\operatorname{ran}}\, p^*$ and thus, $\ell^0 \preccurlyeq_2 \ell^*$.

If $n \notin \mathcal{E}$, then it can always be added by partitioning a minimal element b of $\mathcal{E}$ labeled ATOMINF, $b = n \dot\vee (b - n)$, and setting $\mathfrak{t}_2^{\mathcal{E}^*}(n) =$ ATOMINF and $\mathfrak{t}_2^{\mathcal{E}^*}(b - n) =$ ATOM. In what follows, we suppose $n \in \mathcal{E}^*$. Let $b \in \mathcal{E}^*$ be minimal in $\operatorname{ran} p^0$ and $a \in \mathcal{C}^0$ with $p^0(a) = b$. The action we take depends on $\mathfrak{t}_2^{\mathcal{C}^0}(a)$ and $\mathfrak{t}_2^{\mathcal{G}}(a)$, since it is possible that $\mathfrak{t}_2^{\mathcal{C}^0}(a) <_2 \mathfrak{t}_2^{\mathcal{G}}(a)$. If $\mathfrak{t}_2^{\mathcal{C}_0}(a) =$ ATOM, then there is no new element in $\mathcal{G}$ below a, $\mathfrak{t}_2^{\mathcal{E}^*}(b)$ is finite join of ATOMs, and every element below b is in $\overline{\operatorname{ran}}\, p^0$, so there is nothing further that needs to be done. Otherwise, let $b = c \dot\vee d$, where $c = b \wedge n$ and $d = b - n$ (one of these elements might be $0_{\mathcal{B}}$). The cases that follow are based on Definition 3.11.

(1) If $\mathfrak{t}_2^{\mathcal{C}^0}(a) =$ ATOMLESS, then $\mathfrak{t}_2^{\mathcal{G}}(a) =$ ATOMLESS and there are elements $a = a_0 \dot\vee a_1$ in $\mathcal{G}$ with $\mathfrak{t}_2^{\mathcal{G}}(a_i) =$ ATOMLESS for each $i = 0, 1$. Furthermore $\mathfrak{t}_2^{\mathcal{E}^*}(b) =$ ATOMLESS and all elements below b in $\mathcal{E}^*$ are labeled ATOMLESS, so without loss we assume c and d are both labeled ATOMLESS(if one is $0_{\mathcal{B}}$ we can split the other) and extend p^* by $a_0 \mapsto c$ and $a_1 \mapsto d$. By further splitting c and d into ATOMLESS elements, we can extend p^* to the minimal elements below a_0 and a_1.
(2) If $\mathfrak{t}_2^{\mathcal{C}^0}(a) =$ ATOMINF $= \mathfrak{t}_2^{\mathcal{G}}(a)$, then $\mathfrak{t}_2^{\mathcal{E}^*}(b) =$ ATOMINF and the action we take depends on how a splits in $\mathcal{G}$. There must be some minimal element in $\mathcal{E}^*$ below b which is labeled ATOMINF, for concreteness we suppose $c = c_0 \dot\vee c_1$, where c_0 is minimal in $\mathcal{E}^*$ and labeled ATOMINF. By increasing 1-types of minimal elements with 1-label INF in $\mathcal{E}^*$ to ATOM we can make c_1 and d a join of one or more ATOMs. Since a bounds at least one ATOM in $\mathcal{G}$, we have a partition of a into minimal elements in $\mathcal{G}$ by $a = a_0 \dot\vee a_1 \dot\vee \ldots \dot\vee a_k$ where $\mathfrak{t}_2^{\mathcal{G}}(a_0) =$ ATOMINF, $\mathfrak{t}_2^{\mathcal{G}}(a_1) =$ ATOM and each a_i has 2-label ATOM, ATOMLESS or ATOMINF, so extend $\mathcal{E}^*$ by adding additional elements and splitting $c_0 = c_0' \dot\vee c_2 \dot\vee \ldots \dot\vee c_k$ and adding 2-labels so that $\mathfrak{t}_2^{\mathcal{E}^*}(c_0') =$ ATOMINF, $\mathfrak{t}_2^{\mathcal{E}^*}(c_i) = \mathfrak{t}_2^{\mathcal{G}}(a_i)$ for $i \geq 2$. Finally, extend p^* by $a_0 \mapsto c_0'$, $a_1 \mapsto c_1 \dot\vee d$ and $a_i \mapsto c_i$ for $i \geq 2$.

(3) If $\mathfrak{t}_2^{\mathcal{C}^0}(a) <_2 \mathfrak{t}_2^{\mathcal{G}}(a)$, then there are elements $a = a_0 \dot\vee a_1 \dot\vee a_2 \dot\vee \ldots \dot\vee a_k$ in $\mathcal{G}$ with $\mathfrak{t}_2^{\mathcal{G}}(a_0) =$ ATOMLESS, $\mathfrak{t}_2^{\mathcal{G}}(a_1) =$ ATOM and each a_i labeled ATOM or ATOMLESS for $2 \le i \le k$. There must be some minimal element in $\mathcal{E}^*$ below b which is labeled ATOMINF, for concreteness we suppose $c = c_0 \dot\vee c_1$ where c_0 is minimal in $\mathcal{E}^*$ and labeled ATOMINF. By increasing 1-labels of minimal elements with 1-type INF in $\mathcal{E}^*$ to ATOM we can make c_1 and d a join of one or more ATOMs, and by adding additional elements below c_0 in $\mathcal{E}^*$, we can partition c_0 as $c_0 = c_0' \dot\vee \ \dot\vee c_2 \dot\vee \ldots c_k$ with 2-labels $\mathfrak{t}_2^{\mathcal{E}^*}(c_0') =$ ATOMLESS and $\mathfrak{t}_2^{\mathcal{E}^*}(c_i) = \mathfrak{t}_2^{\mathcal{G}}(a_i)$ for $2 \le i$. Extend p^* by $a_0 \mapsto c_0'$, $a_1 \mapsto c_1 \dot\vee d$, and $a_i \mapsto c_i$ for $2 \le i$. In the process we have *increased* the 2-labels on c_0 and b.

Let $\ell^* = \langle p^*, \mathcal{G}, \mathcal{E}^* \rangle$. In step (B) we have ensured that $\operatorname{dom} p^* = \mathcal{G}$ and $n \in \overline{\operatorname{ran}}\, p^*$, so that (T3) and (T4) hold. Since $p \subset p^*$ and the 2-labels on elements of $\overline{\operatorname{ran}}^0$ and the 1-labels on elements of $\mathcal{E}$ are non-decreasing in $\mathcal{E}^*$, thus $\ell^0 \preccurlyeq_2 \ell^*$, $\ell^1 \preccurlyeq_1 \ell^*$ and $\ell^2 \preccurlyeq_0 \ell^*$ and (T1) holds. Finally, all elements labeled ATOM in $\mathcal{E}^*$ are in $\overline{\operatorname{ran}}\, p^*$, which together with step (A) ensures that condition (T6) holds, so that $\sigma u \ell^* \in \mathcal{T}$.

We have now shown that $(L, U, \hat\ell, \mathcal{T}, E, (\preccurlyeq_n)_{n<3})$ is a 3-system, so the final step of the construction is to show that there exists a Δ_3^0 instruction function q, with the property that any run hrough $\mathcal{T}$ determines an embedding $f : \mathcal{A} \to \mathcal{B}$ satisfying the two conditions of Lemma 4.2 but no Δ_2^0-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Lemma 4.3. *There is a Δ_3^0-computable instruction function q such that a run $(\mathcal{T}, q)$ determines an embedding $f : \mathcal{A} \to \mathcal{B}$ satisfying the two conditions of Lemma 4.2 but no Δ_2^0-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

Proof. Let $\{\mathcal{A}_n\}_{n\in\omega}$ be a Δ_3^0-computable 2-approximation of $\mathcal{A}$, where $\mathcal{A}_0$ is the two-element subalgebra of $\mathcal{A}$ and we will suppose that $n \in \mathcal{A}_n$. Let k be least such that the subalgebra $\mathcal{A}_k$ has two minimal elements labeled ATOMINF (which must eventually occur because $\mathcal{A}$ bounds infinitely many disjoint elements whose 2-bf-type is ATOMINF). Let $\Gamma = \Phi_0^{\Delta_2^0} \upharpoonright \mathcal{A}_k$ and $q(\hat\ell) = \langle \mathcal{A}_k, \Gamma \rangle$. Now, $q(\hat\ell)$ guides $\hat\ell$ because $1_{\mathcal{A}}$ splits into a pair of elements with 2-bf-type ATOMINF (and so having 1-bf-type INF). If $\Gamma(1_{\mathcal{A}}) \ne 1_{\mathcal{B}}$ or $\Gamma(0_{\mathcal{A}}) \ne 0_{\mathcal{B}}$ or Γ is not defined on $1_{\mathcal{A}}$ or $0_{\mathcal{A}}$, then case (i) of the definition of *guides* is fullfilled, otherwise case (ii) is fulfilled.

Let $\sigma = \hat\ell u_1 \ell_1 \ldots u_n \ell_n \in \mathcal{T}$, where $u_n = \langle \mathcal{C}_n, \Gamma \rangle$ and $\ell_n = \langle p, \mathcal{C}_n, \mathcal{D} \rangle$. It is Δ_3^0-computable to decide whether the sequence $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ can be extended to a 2-approximation of $\mathcal{A}$. Let $\mathcal{A}_k$ contain all elements of $\mathcal{C}_n$, then if $\mathcal{C}_n$ is inconsistent as a Boolean subalgebra of $\mathcal{A}_k$, or there is a minimal $x \in \mathcal{C}_n$ with $\mathfrak{t}_2^{\mathcal{A}_k}(x) <_2 \mathfrak{t}_2^{\mathcal{C}_n}$ or $\mathfrak{t}_2^{\mathcal{C}_n} <_2 \mathfrak{t}_2^{\mathcal{A}_k}(x)$ and $\mathfrak{t}_2^{\mathcal{A}_k}(x) =$ ATOMLESS, then $\mathcal{C}_n$ cannot be consistently extended to a 2-approximation of $\mathcal{A}$. (This follows straightforwardly from the definition of 2-approximation–in particular, case (2d) of Definition 3.11.) If $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ cannot be extended to a 2-approximation of $\mathcal{A}$, then let $\mathcal{G}$ be any extension of $\mathcal{C}_n$ satisfying the condition (T2) and $\Gamma = \emptyset$ (this choice satisfies (T5) by satisfying condition (1) of *guides*). So, suppose that $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ can be extended to a 2-approximation of $\mathcal{A}$. Use $0''$ to enumerate $\{\mathcal{A}_n\}_{n\in\omega}$, until an $\mathcal{A}_k$ is found so that $\mathcal{C}_n \le_2 \mathcal{A}_k$, each minimal element of $\mathcal{C}_n$ has a 2-full partition in $\mathcal{A}_k$ and one of the following conditions is satisfied:

(1) $\overline{\operatorname{ran}}\, p \not\subset \operatorname{ran} \Phi_n^{\Delta_2^0}$.

(2) $\Phi_n^{\Delta^0_2}$ is not a total injective function on $\mathcal{A}_k$.
(3) $\overline{\operatorname{ran}}\, p$ and the pre-image of $\overline{\operatorname{ran}}\, p$ under $\Phi_n^{\Delta^0_2}$ are not isomorphic as Boolean algebras.
(4) There is an $x \in \mathcal{A}_k$ with $\Phi_n^{\Delta^0_n}(x) \in \mathcal{D}$ but $\mathfrak{t}_1^{\mathcal{A}_k}(x) \neq \mathfrak{t}_1^{\mathcal{D}}(\Phi_n^{\Delta^0_2}(x))$.
(5) There is an $x \in \mathcal{A}_k$ labeled ATOMINF for which $\Phi_n^{\Delta^0_2}(x)$ is minimal in $\mathcal{D}$ and a pair of elements $y_1, y_2 < x$ with 1-labels INF in $\mathcal{A}_k$.

If (1) holds, then let $q(\sigma) = \langle \mathcal{A}_k, \emptyset\rangle$, which guides σ by condition (i) of the definition of *guides*. (In particular, $\Gamma = \emptyset$ is not total on $\mathcal{A}_k$!) If (2), (3) or (4) hold, then there is a subalgebra $\mathcal{A}_k$ of $\mathcal{A}$ and a finite graph Γ of $\Phi_n^{\Delta^0_2}$ which witnesses this, so let $q(\sigma) = \langle \mathcal{A}_k, \Gamma\rangle$. In each case, $q(\sigma)$ guides σ because condition (i) of the definition of *guides* is satisfied. If none of these previous cases arise, then there must be an x and y_1, y_2 meeting the final condition because $\Phi_e^{\Delta^0_2}$ must map some subalgebra $\mathcal{C}$ of $\mathcal{A}$ onto $\overline{\operatorname{ran}}\, p$ with some minimal element $x \in \mathcal{C}$ having 2-bf-type ATOMINF and bounding infinitely many disjoint elements of 1-bf-type INF in $\mathcal{A}$, so that for some sufficiently large k, there is an $x \in \mathcal{A}_k$ with $\Phi_e^{\Delta^0_2}(x)$ minimal in $\mathcal{D}$ and bounding a pair of elements y_1, y_2 whose 1-bf-type is INF in $\mathcal{A}_k$. Let q pick out the finite 2-labeled algebra $\mathcal{A}_k$ and let Γ be the graph of $\Phi_n^{\Delta^0_2} \restriction \mathcal{A}_k$. Then $q(\sigma) = \langle \mathcal{A}_k, \Gamma\rangle$ guides σ by condition (ii) in the definition of *guides*. Thus, (T2) and (T5) hold, so that $q(\sigma)$ is an extension of σ on $\mathcal{T}$.

If $\pi = (\mathcal{T}, q)$ is a run through $\mathcal{T}$, then $f = \bigcup_s p_s$ is well defined by (T1), since this implies $p_n \subseteq p_{n+1}$, and an embedding of $\mathcal{A}$ into $\mathcal{B}$ by (T3) and (T4). $\mathcal{B}$ is generated from $\operatorname{ran} f$ and atoms in the image of f by (T4). If $a \in \mathcal{A}$ is an atom, then for some $\ell_n = \langle p_n, \mathcal{C}_n, \mathcal{D}_n\rangle$ we have $a \in \mathcal{C}_n$ and $p_n(a) = f(a)$ is a finite join of ATOMs, so that $\mathfrak{t}_2^{\mathcal{D}_n}(f(a)) = \mathfrak{t}_2^{\mathcal{D}_m}(f(a))$ for all $m \geq n$ by (T1), and so $f(a)$ is a finite join of atoms of $\mathcal{B}$. Thus, the two conditions of Lemma 4.2 hold. Finally,we show that each of the requirements $\mathcal{R}_e$ is satisfied. Let σ be the initial subsequence of π of length $2e+3$. Then u_{e+1} was chosen to guide ℓ_e using a finite subgraph for the function $\Phi_e^{\Delta^0_2}$ and ℓ_{e+1} completes u_{e+1}. It could be because $\Phi_e^{\Delta^0_2}(x)\uparrow$ for some $x \in \mathcal{A}$ or fails to be an injective function or fails to map onto ω, so that nothing in the construction is relevant to the failure of $\Phi_e^{\Delta^0_2}$ to be a Boolean isomorphism. Otherwise, the condition (T1) ensures that all Boolean relations are unchanged throughout the construction (preserving case (iii) and (iv) of the definition of *completes*) and it also ensures that all 1-labels on elements of $\overline{\operatorname{ran}}\, p_{e+1}$ of ℓ_{e+1} are unchanged throughout the construction (preserving case (v) of the definition of *completes*)). Thus, $\Phi_e^{\Delta^0_2}$ cannot be an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ for any e. □

By the Ash-Knight metatheorem 2.1 there exists a Δ^0_3 run π of $(\mathcal{T}, q)$ such that $E(\pi)$ is c.e. . Let $\mathcal{B} = E(\pi)$, so that $\mathcal{B}$ is computable by by condition (T4). By Lemma 4.3 there is an embedding $f : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{B}$ is generated by $\operatorname{ran} f$ and atoms below the image of an atom in $\mathcal{A}$, and for each atom $a \in \mathcal{A}$, $f(a)$ is a finite join of atoms in $\mathcal{B}$, so that $\mathcal{A} \cong \mathcal{B}$ by Theorem 3.3, but there is no Δ^0_2 isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

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